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Monotonic convolution and monotonic Lévy-Hinčin formula

Abstract. Based on the notion of "monotonic independence" for random variables in a C^*-probability space, the "monotonic convolution" for probability measures on the real line is introduced. It describes the probability distribution for addition of monotonically independent random variables. A monotonic analogue of Lévy-Hinčin formula is given in terms of continuous one-parameter monotonic convolution semigroups of probability measures. In particular, the class of infinitely divisible distributions with compact supports is characterized. Also a monotonic analogue of compound Poisson distribution is given with its limit theorem.

0. Introduction

In quantum probability theory (= noncommutative probability theory [Par], [Mey], [AcO]), it is an interesting problem to find another notions of "independence" different from the classical notion of "independence" (= commutative independence) and to develop another kind of probabilistic notions based on such non-classical independences ([GvW], [vWa], [Voi], [Spe], [BLS], [AcB]).

The famous example is the "free independence" (="freeness") introduced by D. Voiculescu [Voi]. It is a kind of "independence" for noncommutative random variables in a C^*-probability space. The free analogue of various classical probabilistic notions such as convolution, Gaussian distribution, Brownian motion, Poisson process, stochastic calculus, entropy and others have been developed in the setting of "free probability theory" based on the free independence ([VDN], [HiP], [Spe], [Maa], [KuS], [BiS]). Various generalizations and deformations of the notion of the free independence are studied by several authors ([BoS], [BKS], [BLS], [Bia], [AHO], [AcB], [BoW]).

In a recent paper [Mu3], the author introduced the notion of "monotonic independence" for noncommutative random variables in a C^*-probability space. It is an algebraic abstraction of the structure which must have been hidden in the discussion in the previous works([Mu1], [Mu2], [Lu]). We proved in [Mu3] the "monotonic central limit theorem" and the "monotonic law of small numbers," where the monotonic Gaussian distribution (= the arcsine law) and the monotonic Poisson distribution (described by the product log function, a special function) were calculated, respectively.

This paper is a continuation of the previous work [Mu3]. We introduce in this paper a monotonic analogue of convolution for probability measures on the real line,

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which we call the "monotonic convolution." It describes the probability distribution for addition of monotonically independent random variables in a C^*-probability space. We also prove a monotonic analogue of Lévy-Hinčin formula for continuous one-parameter monotonic convolution semigroups of probability measures.

The contents of the paper is organized as follows. In Section 1, the definition of the notion of "monotonic independence" for random variables in a C^*-probability space is given. In Section 2, a construction of monotonically independent random variables with prescribed probability distributions is given with the help of the monotone product of C^*-probability spaces. In Section 3, the monotonic convolution for probability measures on the real line is introduced. It is shown that the probability distribution for addition of monotonically independent random variables in a C^*-probability space can be described by the monotonic convolution of probability measures. In Section 4, the infinite divisibility of probability measures in the sense of monotonic convolution is investigated in terms of continuous one-parameter monotonic convolution semigroups of probability measures. A monotonic Lévy-Hinčin formula is proved for these continuous one-parameter semigroups. In Section 5, a monotonic Lévy-Hinčin formula in terms of infinitely divisible distributions is discussed. The class of infinitely divisible distributions with compact supports is characterized. Besides, a monotonic analogue of compound Poisson distribution is given with its limit theorem. Section 6 is the Appendix where some auxiliary lemmas needed in the preceding sections are collected.

Throughout the paper, we use the following notations:

$$\begin{aligned}
&\mathbf{N} = \text{the natural numbers } (\geq 0); \quad \mathbf{N}^* = \mathbf{N} \setminus \{0\};\\
&\mathbf{Q} = \text{the rational numbers}; \quad \mathbf{Q}_+ = \{r \in \mathbf{Q} | r \geq 0\}; \quad \mathbf{Q}_+^* = \{r \in \mathbf{Q} | r > 0\};\\
&\mathbf{R} = \text{the real numbers}; \quad \mathbf{R}_+ = \{x \in \mathbf{R} | x \geq 0\}; \quad \mathbf{R}_+^* = \{x \in \mathbf{R} | x > 0\};\\
&\mathbf{C} = \text{the complex numbers}; \quad \mathbf{C}^+ = \{z \in \mathbf{C} | \Im z > 0\}; \quad \mathbf{C}^- = \{z \in \mathbf{C} | \Im z < 0\}.
\end{aligned}$$

Here $\Im z$ (resp. $\Re z$) denotes the imaginary part (resp. real part) of a complex number z. We use $\#A$ to denote the cardinality of a finite set A.

1. Monotonically independent random variables

Let $(\mathcal{A}, \phi)$ be a C^*-probability space consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ over $\mathcal{A}$. Each element $X \in \mathcal{A}$ is interpreted as a (bounded) random variable on a C^*-probability space $(\mathcal{A}, \phi)$. We often use $\langle X \rangle$ for short to denote the expectation $\phi(X)$ of a random variable $X \in \mathcal{A}$.

Definition 1.1. *[Mu3] A family $\{X_i\}_{i \in I} \subset \mathcal{A}$ of random variables on $(\mathcal{A}, \phi)$ with totally ordered index set I is said to be monotonically independent w.r.t. a state ϕ if the following two conditions are satisfied.*

$$(a) \qquad X_i X_j^p X_k = \phi(X_j^p) X_i X_k \quad \textit{whenever} \quad i < j > k.$$

$$\begin{aligned}
(b) \qquad &\phi(X_{i_m}^{p_m} \cdots X_{i_2}^{p_2} X_{i_1}^{p_1} X_i^p X_{j_1}^{q_1} X_{j_2}^{q_2} \cdots X_{j_n}^{q_n})\\
&\quad = \phi(X_{i_m}^{p_m}) \cdots \phi(X_{i_2}^{p_2}) \phi(X_{i_1}^{p_1}) \phi(X_i^p) \phi(X_{j_1}^{q_1}) \phi(X_{j_2}^{q_2}) \cdots \phi(X_{j_n}^{q_n})\\
&\qquad \textit{whenever} \quad i_m > \cdots > i_2 > i_1 > i < j_1 < j_2 < \cdots < j_n.
\end{aligned}$$

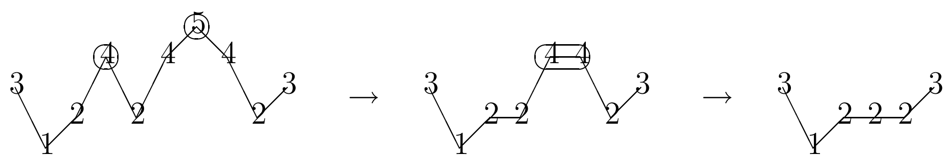
Here p's and q's are arbitrary natural numbers in $\mathbf{N}$. The notation $i < j > k$ is understood as $i < j$ and $j > k$ (there is no assumption on the order relation between i and k). The notation $i_m > \cdots > i_2 > i_1 > i < j_1 < j_2 < \cdots < j_n$ is understood as $i_m > \cdots > i_2 > i_1 > i$ and $i < j_1 < j_2 < \cdots < j_n$. Of course, the case of $m = 0$ (resp. $n = 0$) in the condition (b) is understood in the natural way.

The above two conditions (a) and (b) can be viewed as the decomposition rules for expectations $\phi(X_{i_r} \cdots X_{i_2} X_{i_1})$ of monomials $X_{i_r} \cdots X_{i_2} X_{i_1}$ in X's, as explained below.

Assume that monotonically independent random variables $\{X_i\}_{i \in I}$ are given. Let ${}_I\Pi_r \cong \mathrm{Map}(\{r, r-1, \cdots, 2, 1\}, I)$ be the set of all repeated permutations $f = (i_r \cdots i_2 i_1)$ choosing r elements from I. Of course $\#({}_I\Pi_r) = {}_{\#I}\Pi_r$ (= the number of all repeated permutations) if I is a finite set. The set $\{r, r-1, \cdots, 2, 1\}$ is interpreted as a set of *sites*. Each element $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r$ is a *configuration* of indices from I. A configuration $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r$ distribute indices ($\in I$) to the sites $\{r, r-1, \cdots, 2, 1\}$. For each configuration $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r$, let us associate to it an operator $X_{i_p} \cdots X_{i_2} X_{i_1}$, and write its expectation $\langle X_{i_r} \cdots X_{i_2} X_{i_1} \rangle$ by $\langle i_r \cdots i_2 i_1 \rangle$ for short. Then the expectation $\langle i_r \cdots i_2 i_1 \rangle$ can be uniquely decomposed based on the following procedure. We explain it by an example. Take a configuration (3124245423) which distribute, to the sites $\{r, r-1, \cdots, 2, 1\}$ ($r = 10$), indices from the set $I = \{1, 2, 3, 4, 5\}$. At first, by the repeated use of rule (a), we have

$$\begin{aligned} \langle\, 3124245423 \,\rangle &= \langle\, 4 \,\rangle \langle\, 5 \,\rangle \langle\, 31224423 \,\rangle \\ &= \langle\, 4 \,\rangle \langle\, 5 \,\rangle \langle\, 44 \,\rangle \langle\, 312223 \,\rangle. \end{aligned}$$

This process can be visualized as



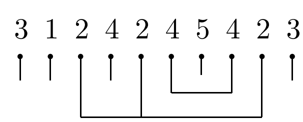
We see that once use of rule (a) means to take a "top" off the "mountains." After the maximal use of rule (a), we get a factor $\langle\, 312223 \,\rangle$ which has a form of "valley." But this final factor can be decomposed further by the use of rule (b). After all we obtain the final decomposition:

$$\langle\, 3124245423 \,\rangle = \langle 4 \rangle \langle 5 \rangle \langle 44 \rangle \langle 3 \rangle \langle 1 \rangle \langle 222 \rangle \langle 3 \rangle.$$

Of course this procedure works well for general configurations $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r$, and it uniquely defines the natural decomposition of $\langle X_{i_r} \cdots X_{i_2} X_{i_1} \rangle$.

Now, to each configuration $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r$, let us associate a unique partition $\mathcal{P}$ of the sites $\{r, r-1, \cdots, 2, 1\}$. We put two sites q, q' ($\in \{r, \cdots, 2, 1\}$) into a same equivalence class if and only if both of the corresponding X_{i_q} and $X_{i_{q'}}$ appear in the same factor in the final decomposition of $\langle X_{i_r} \cdots X_{i_2} X_{i_1} \rangle$. The resulting equivalence classes defines a noncrossing partition $\mathcal{P}$ of the set $\{r, r-1, \cdots, 2, 1\}$. An equivalence class $v \in \mathcal{P}$ is called a *block*. Each noncrossing partition $\mathcal{P}$ can be visualized by a non crossing diagram g. For example,

$$\begin{aligned} f &= (i_r \cdots i_2 i_1) \\ &= (3124245423) \end{aligned} \quad \longrightarrow$$



Here the corresponding noncrossing partition $\mathcal{P}$ consists of seven blocks $v_1 = \{4\}$, $v_2 = \{5, 3\}$, $v_3 = \{7\}$, $v_4 = \{8, 6, 2\}$, $v_5 = \{1\}$, $v_6 = \{9\}$, $v_7 = \{10\}$. We denote by $|v|$ the number of points in a block v (for example $|v_4| = 3$). The blocks v_4, v_5, v_6, v_7 are the *outer blocks* of a diagram g. We denote by $\mathrm{out}(g)$ the number of all outer blocks in a diagram g (hence $\mathrm{out}(g) = 4$ in this example). A block v_1 (and also v_2

and v_3) is in the *inner side* of a block v_4. A block v_1 is in the inner side of a block v_2.

There is the natural bijective correspondence between the set $\mathrm{NCP}(r)$ of all noncrossing partitions $\mathcal{P}$ over $\{r, \cdots, 2, 1\}$ and the set $\mathrm{NCD}(r)$ of all noncrossing diagrams g with r points. We always identify a noncrossing partition $\mathcal{P}$ with a noncrossing diagram g. So we write $\mathcal{P} = \mathcal{P}(g)$ and $g = g(\mathcal{P})$. After all, the decomposition procedure of $\langle i_r \cdots i_2 i_1 \rangle$ based on the monotonic independence yields the natural correspondence

$$\pi_{r,I} : \ {}_I\Pi_r \ni f \ \mapsto \ g \in \mathrm{NCD}(r)$$

from configurations f to diagrams g as described above. A configuration $f \in {}_I\Pi_r$ is said to be *admissible w.r.t.* g if $\pi_{r,I}(f) = g$. The following fact is basic.

Lemma 1.2. *Let $g \in \mathrm{NCD}(r)$, and let $f = (i_r \cdots i_2 i_1) \in {}_I\Pi_r \cong \mathrm{Map}(\{r, \cdots, 2, 1\}, I)$ be an admissible configuration w.r.t. g. Then we have*

(i) The function $f : \{r, \cdots, 2, 1\} \to I$ takes a constant value on each block $v \in \mathcal{P}(g)$, i.e. $f(q) = f(q')$ for $q, q' \in v$. (We denote this value by $f(v)$).
(ii) $f(v) > f(w)$ whenever a block v is in the inner side of a block w.

We identify a g-admissible configuration $f = (i_r \cdots i_2 i_1)$ with its associated function of a block variable: $\mathcal{P}(g) \ni v \to f(v) \in I$. We denote by ${}_I\mathrm{M}_g$ the set of all g-admissible configurations to suggest that f distribute, to the blocks $\mathcal{P}(g)$, indices from I so that the final decomposition of the configuration produces a diagram g based on the monotonic independence. These admissible configurations will be used in Section 3 to derivate the "monotonic convolution" for probability measures.

2. Realization

In this section, we give a construction of monotonically independent random variables with prescribed probability distributions. It is achieved based on the monotone product of C^*-probability spaces which is defined as follows. The construction of monotone product is analogous to that of the free product [Voi].

Let us start with the setting of Hilbert spaces with (fixed) unit vectors. Let $(H_i, \xi_i)_{i \in I}$ be a family of (right linear) complex Hilbert spaces H_i with unit vectors $\xi_i \in H_i$, which is indexed by some totally ordered set I. Then the *monotone product* (H, ξ) of $(H_i, \xi_i)_{i \in I}$ is defined by

$$H \quad \equiv \quad \mathbf{C}\xi \oplus \bigoplus_{r \geq 1} \bigoplus_{\sigma \in {}_I\mathrm{M}_r} H_{i_r}^\circ \otimes \cdots \otimes H_{i_1}^\circ,$$

and denoted by $(H, \xi) = \triangleright_{i \in I}(H_i, \xi_i)_{i \in I}$. Here ξ is a (formal) unit vector. ${}_I\mathrm{M}_r$ denotes the set of all monotone sequences $\sigma = (i_r {>} \cdots {>} i_2 {>} i_1)$ of length r from I. (Note that we must distinguish this notation ${}_I\mathrm{M}_r$ from the notation ${}_I\mathrm{M}_g$ in Section 1. They are different objects.) H_i° denotes the orthogonal complement $H_i^\circ \equiv H_i \ominus \mathbf{C}\xi_i$ of $\mathbf{C}\xi_i$ relative to H_i.

Let $B(H)$ (resp. $B(H_i)$) be the *-algebra of all bounded linear operators on H (resp. H_i). Let us define a natural (non-unital) *-representation $\lambda_i : B(H_i) \to B(H)$ of the algebra $B(H_i)$ on the Hilbert space H as follows. For each index $i \in I$, we put

$$H_{(=i)}^\circ \quad \equiv \quad \bigoplus_{r \geq 1} \bigoplus_{\substack{\sigma \in {}_I\mathrm{M}_r \\ i_r = i}} H_{i_r}^\circ \otimes \cdots \otimes H_{i_1}^\circ,$$

$$\begin{aligned} H^\circ_{(<i)} &\equiv \bigoplus_{r\geq 1} \bigoplus_{\substack{\sigma\in {}_I\mathrm{M}_r \\ i_r<i}} H^\circ_{i_r}\otimes\cdots\otimes H^\circ_{i_1}, \\ H^\circ_{(>i)} &\equiv \bigoplus_{r\geq 1} \bigoplus_{\substack{\sigma\in {}_I\mathrm{M}_r \\ i_r>i}} H^\circ_{i_r}\otimes\cdots\otimes H^\circ_{i_1}. \end{aligned}$$

Referring to the decomposition $H_i = \mathbf{C}\xi_i \oplus H_i^\circ$, we write

$$\begin{aligned} H_{(=i)} &= \mathbf{C}\xi_{(=i)} \oplus H^\circ_{(=i)}, \\ H_{(<i)} &= \mathbf{C}\xi_{(<i)} \oplus H^\circ_{(<i)}, \\ H_{(>i)} &= \mathbf{C}\xi_{(>i)} \oplus H^\circ_{(>i)}. \end{aligned}$$

Then, by the identification up to isomorphism, the Hilbert space H is rewritten as

$$\begin{aligned} H &= \mathbf{C}\xi \oplus \bigoplus_{r\geq 1}\bigoplus_{\sigma\in_I\mathrm{M}_r} H^\circ_{i_r}\otimes\cdots\otimes H^\circ_{i_1} \\ &= \mathbf{C}\xi \oplus H^\circ_{(=i)} \oplus H^\circ_{(<i)} \oplus H^\circ_{(>i)} \\ &= (\mathbf{C}\xi \oplus H_i^\circ \oplus H_i^\circ \otimes H^\circ_{(<i)} \oplus H^\circ_{(<i)}) \oplus H^\circ_{(>i)} \\ &= (\mathbf{C}\xi_i \oplus H_i^\circ)\otimes(\mathbf{C}\xi_{(<i)} \oplus H^\circ_{(<i)}) \oplus H^\circ_{(>i)} \\ &= H_i \otimes H_{(<i)} \oplus H^\circ_{(>i)}. \end{aligned}$$

So we get the natural identification

$$H = H_i \otimes H_{(<i)} \oplus H^\circ_{(>i)} \quad \text{and} \quad \xi = \xi_i \otimes \xi_{(<i)}.$$

Based on this decomposition, let us define the (non-unital) *-representation $\lambda_i : B(H_i) \to B(H)$ of the algebra $B(H_i)$ on the space H by

$$\lambda_i(A) \equiv A\otimes I \oplus O, \qquad A \in B(H_i).$$

Here I denotes the identity operator on $H_{(<i)}$ and O denotes the zero operator on $H^\circ_{(>i)}$. Note that this *-homomorphism λ_i is faithful but non-unital in general: $B(H_i) \ni I \mapsto \lambda_i(I) \neq I \in B(H)$.

The monotone product of Hilbert spaces with unit vectors $(H,\xi) = \rhd_{i\in I}(H_i,\xi_i)$ has the following properties. In the following, the monotonic independence for (non-unital) subalgebras is understood in the natural sense.

Theorem 2.1. *Let $(H,\xi) = \rhd_{i\in I}(H_i,\xi_i)$ be any monotone product of Hilbert spaces with unit vectors, equipped with the natural *-homomorphisms $\lambda_i : B(H_i) \to B(H)$. Put $\phi(\cdot) \equiv \langle\xi|\cdot\xi\rangle$ and $\phi_i(\cdot) \equiv \langle\xi_i|\cdot\xi_i\rangle$. Then the followings hold.*

(1) The family $\{\lambda_i(B(H_i))\}_{i\in I}$ of subalgebras of $B(H)$ is monotonically independent w.r.t. the state ϕ.

(2) λ_i preserves the expectation, i.e. $\phi(\lambda_i(A)) = \phi_i(A)$ for all $A \in B(H_i)$.

Proof. *(1)* We must show that the two conditions (a) and (b) of the monotonic independence are satisfied.

(a) Let i,j,k be arbitrary indices from I such that $i<j>k$, and let $A_i \in B(H_i)$, $A_j \in B(H_j)$, $A_k \in B(H_k)$ be arbitrary operators. Put $X_i \equiv \lambda_i(A_i)$, $X_j \equiv \lambda_j(A_j)$, $X_k \equiv \lambda_k(A_k)$. We must show that

$$X_iX_jX_k = \phi(X_j)X_iX_k.$$

For each vector of the form $h_{i_r} \otimes \cdots \otimes h_{i_1} \in H^\circ \equiv H \ominus \mathbf{C}\xi$ with $\sigma = (i_r{>} \cdots {>} i_2{>} i_1) \in {}_I\mathrm{M}_r$, let us calculate the actions of two operators $X_i X_j X_k$ and $\phi(X_j) X_i X_k$ on them. Consider the three cases : i) $k < i_r$; ii) $k = i_r$; iii) $k > i_r$.

i) Case of $k < i_r$. By the definition of the representation λ_k, we have $X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}) = 0$, and hence

$$X_i X_j X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}) = \phi(X_j) X_i X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}).$$

ii) Case of $k = i_r$. We have

$$\begin{aligned} X_i X_j X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}) &= X_i X_j((A_k h_{i_r}) \otimes (I(h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}))) \\ &= X_i X_j(\xi_j \otimes (A_k h_{i_r}) \otimes (h_{i_{r-1}} \otimes \cdots \otimes h_{i_1})) \\ &= X_i((A_j \xi_j) \otimes I((A_k h_{i_r}) \otimes (h_{i_{r-1}} \otimes \cdots \otimes h_{i_1})). \end{aligned}$$

Here the second equality comes from the identification $H^\circ_{(=k)} \cong \mathbf{C}\xi_j \otimes H^\circ_{(=k)}$. Decompose $A_j\xi_j$ as $A_j\xi_j = a_j\xi_j \oplus \eta_j$ $(a_j\xi_j \in \mathbf{C}\xi_j, \eta_j \in H^\circ_j)$, and continue the above calculation, then we have, from the definition of λ_j,

$$\begin{aligned} & X_i((A_j\xi_j) \otimes (A_k h_{i_r}) \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}) \\ &\quad = X_i((a_j\xi_j \oplus \eta_j) \otimes (A_k h_{i_r}) \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}) \\ &\quad = X_i((a_j\xi_j) \otimes (A_k h_{i_r}) \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}) \\ &\quad = a_j X_i(\xi_j \otimes (A_k h_{i_r}) \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}) \\ &\quad = a_j X_i((A_k h_{i_r}) \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}) \\ &\quad = a_j X_i X_k(h_{i_r} \otimes h_{i_{r-1}} \otimes \cdots \otimes h_{i_1}). \end{aligned}$$

Here the fourth equality comes from the identification $\mathbf{C}\xi_j \otimes H^\circ_{(=k)} \cong H^\circ_{(=k)}$. The last equality was obtained by reversing the procedure of calculation. By the way, $a_j = \langle \xi_j | A_j \xi_j \rangle = \langle \xi | X_j \xi \rangle$ since $A_j\xi_j = a_j\xi_j \oplus \eta_j$. Therefore we obtain

$$X_i X_j X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}) = \phi(X_j) X_i X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}).$$

iii) Case of $k > i_r$. Repeat the similar discussion as in the case ii).

After all, in any cases of i), ii) and iii), we get

$$X_i X_j X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}) = \phi(X_j) X_i X_k(h_{i_r} \otimes \cdots \otimes h_{i_1}).$$

Besides we have $X_i X_j X_k \xi = \phi(X_j) X_i X_k \xi$ for the "vacuum" $\xi \in H$ through the similar calculation as above. Therefore the condition (a) of the monotonic independence is satisfied.

(b) Let $i_1, \cdots, i_m, i, j_1, \cdots, j_n$ be arbitrary indices from I such that $i_m > \cdots > i_1 > i < j_1 < \cdots < j_n$, and let $A_{i_1} \in B(H_{i_1})$, $\cdots$, $A_{i_m} \in B(H_{i_m})$, $A_i \in B(H_i)$, $A_{j_1} \in B(H_{j_1})$, $\cdots$, $A_{j_n} \in B(H_{j_n})$ be arbitrary operators. Put $X_{i_1} = \lambda_{i_1}(A_{i_1})$, $\cdots$, $X_{i_m} = \lambda_{i_m}(A_{i_m})$, $X_i = \lambda_i(A_i)$, $X_{j_1} = \lambda_{j_1}(A_{j_1})$, $\cdots$, $X_{j_n} = \lambda_{j_n}(A_{j_n})$. We must show that

$$\langle X_{i_m} \cdots X_{i_1} X_i X_{j_1} \cdots X_{j_n} \rangle = \langle X_{i_m} \rangle \cdots \langle X_{i_1} \rangle \langle X_i \rangle \langle X_{j_1} \rangle \cdots \langle X_{j_n} \rangle.$$

At first we have

$$\begin{aligned} X_i X_{j_1} \cdots X_{j_n} \xi &= X_i X_{j_1} \cdots X_{j_n}(\xi_{j_n} \otimes \xi_{(<j_n)}) \\ &= X_i X_{j_1} \cdots X_{j_{n-1}}((A_{j_n}\xi_{j_n}) \otimes \xi_{(<j_n)}) \\ &= X_i X_{j_1} \cdots X_{j_{n-1}}((a_{j_n}\xi_{j_n} \oplus \eta_{j_n}) \otimes \xi_{(<j_n)}) \\ &= X_i X_{j_1} \cdots X_{j_{n-1}}((a_{j_n}\xi_{j_n}) \otimes \xi_{(<j_n)}) \\ &= a_{j_n} X_i X_{j_1} \cdots X_{j_{n-1}} \xi. \end{aligned}$$

Here we used the decomposition $A_{j_n}\xi_{j_n} = a_{j_n}\xi_{j_n} \oplus \eta_{j_n}$ ($a_{j_n}\xi_{j_n} \in \mathbf{C}\xi_{j_n}$, $\eta_{j_n} \in H^\circ_{j_n}$) and the definition of the representation λ_{j_n}. Since $i < j_1 < \cdots < j_n$, we get

$$X_i X_{j_1} \cdots X_{j_n}\xi \;=\; a_{j_1}a_{j_2}\cdots a_{j_n} X_i \xi.$$

by the repetition of the above calculation. Note that $a_{j_1} = \langle \xi_{j_1} | A_{j_1}\xi_{j_1}\rangle = \phi(X_{j_1}), \cdots$, $a_{j_n} = \langle \xi_{j_n} | A_{j_n}\xi_{j_n}\rangle = \phi(X_{j_n})$. Using the adjoints A^*'s of A's, define the numbers b's by $A^*_{i_1}\xi_{i_1} = b_{i_1}\xi_{i_1} \oplus \zeta_{i_1}$ ($b_{i_1}\xi_{i_1} \in \mathbf{C}\xi_{i_1}$, $\zeta_{i_1} \in H^\circ_{i_1}$), $\cdots$, $A^*_{i_m}\xi_{i_m} = b_{i_m}\xi_{i_m} \oplus \zeta_{i_m}$ ($b_{i_m}\xi_{i_m} \in \mathbf{C}\xi_{i_m}$, $\zeta_{i_m} \in H^\circ_{i_m}$). Since $i < i_1 < i_2 < \cdots < i_m$, we get, by the adjointness,

$$\begin{aligned}
\phi(X_{i_m}\cdots X_{i_1} X_i X_{j_1}\cdots X_{j_n}) &= \langle \xi | X_{i_m}\cdots X_{i_1} X_i X_{j_1}\cdots X_{j_n}\xi\rangle \\
&= \langle \xi | X_{i_m}\cdots X_{i_1} a_{j_1}\cdots a_{j_n} X_i\xi\rangle \\
&= a_{j_1}\cdots a_{j_n}\langle \xi | X_{i_m}\cdots X_{i_1} X_i\xi\rangle \\
&= a_{j_1}\cdots a_{j_n}\langle X_i^* X_{i_1}^*\cdots X_{i_m}^*\xi | \xi\rangle \\
&= a_{j_1}\cdots a_{j_n}\langle b_{i_1}\cdots b_{i_m} X_i^*\xi | \xi\rangle \\
&= a_{j_1}\cdots a_{j_n}\overline{b_{i_1}\cdots b_{i_m}}\langle X_i^*\xi | \xi\rangle \\
&= \phi(X_{j_1})\cdots\phi(X_{j_n})\overline{\phi(X^*_{i_1})\cdots\phi(X^*_{i_m})\phi(X^*_i)} \\
&= \phi(X_{i_m})\cdots\phi(X_{i_1})\phi(X_i)\phi(X_{j_1})\cdots\phi(X_{j_n}).
\end{aligned}$$

Therfore the second condition (b) of monotonic independence is satisfied.

(2) Using the decomposition $H = H_i \otimes H_{(<i)} \oplus H^\circ_{(>i)}$ and the definition of λ_i, we get

$$\lambda_i(A)\xi \;=\; \lambda_i(A)(\xi_i \otimes \xi_{(<i)}) \;=\; (A\xi_i)\otimes \xi_{(<i)},$$

and hence

$$\langle \xi | \lambda_i(A)\xi\rangle \;=\; \langle \xi_i\otimes\xi_{(<i)} | (A\xi_i)\otimes\xi_{(<i)}\rangle \;=\; \langle \xi_i | A\xi_i\rangle\langle \xi_{(<i)}|\xi_{(<i)}\rangle \;=\; \langle \xi_i | A\xi_i\rangle. \quad \square$$

Using the monotone product of Hilbert spaces with unit vectors $(H,\xi) = \triangleright_{i\in I}(H_i,\xi_i)$ equipped with the natural *-homomorphisms $\lambda_i : B(H_i) \to B(H)$, $i \in I$, we can construct monotonically independent random variables with prescribed probability distributions as follows.

Corollary 2.2. *Let $\{\mu_i\}_{i\in I}$ be a family of compactly supported probability measures on $\mathbf{R}$ indexed by a totally ordered set I. Then there exists a Hilbert space with unit vector (H,ξ) and a family of self-adjoint random variables $\{X_i\}_{i\in I} \subset B(H)$ such that X's are monotonically independent w.r.t. $\phi(\cdot) \equiv \langle \xi | \cdot\,\xi\rangle$ and that the probability distribution μ_{X_i} of X_i under ϕ coincides with μ_i.*

Proof. Let (H_i,ξ_i) be a Hilbert space with unit vector defined by $H_i \equiv \mathrm{L}^2(\mathbf{R},\mu_i)$ and $\xi_i \equiv 1$, and let A_i be the multiplication operator $(A_i f)(x) = xf(x)$. Since μ_i is compactly supported, A_i is a bounded self-adjoint operator. The probability distribution of A_i under $\langle \xi_i | \cdot\,\xi_i\rangle$ coincides with μ_i. Make the monotone product $(H,\xi) = \triangleright_{i\in I}(H_i,\xi_i)$ with the natural *-homomorphisms $\lambda_i : B(H_i) \to B(H)$. Then the random variables $X_i \equiv \lambda_i(A_i)$, $i \in I$, are monotonically independent w.r.t. ϕ and the probability distribution of X_i coincides with that of A_i because the both moments coincide, i.e. $\phi(X_i^p) = \phi_i(A_i^p)$ for all $p \in \mathbf{N}^*$. $\square$

Finally let us define the monotone product in the setting of C^*-probability spaces. Let $(\mathcal{A}_i,\varphi_i)_{i\in I}$ be a family of C^*-probability spaces with totally ordered index set I, where A_i's are assumed to be unital. Then the monotone product $(\mathcal{A},\varphi)$ of C^*-probability spaces $(\mathcal{A}_i,\varphi_i)_{i\in I}$ is defined as follows.

For each $i \in I$, let (π_i, H_i, ξ_i) be the GNS representation of $\mathcal{A}_i$ associated to the state φ_i. Construct the monotone product of Hilbert spaces with unit vectors $(H, \xi) = \triangleright_{i\in I}(H_i, \xi_i)$ and its associated natural *-homomorphisms $\lambda_i : B(H_i) \to B(H)$, $i \in I$, as above. Let ρ_i be the composition of the GNS representation π_i and the *-homomorphism λ_i :

$$\rho_i \;\equiv\; \lambda_i \circ \pi_i \;:\; \mathcal{A}_i \to H.$$

Denote by $\mathcal{A}$ the C^*-algebra generated by the family $\{\rho_i(\mathcal{A}_i)\}_{i\in I}$ of subalgebras of $B(H)$ and the identity $I \in B(H)$, and let φ be the state over $\mathcal{A}$ given by $\varphi(\cdot) = \langle \xi | \cdot \xi \rangle$. The C^*-probability space $(\mathcal{A}, \varphi)$ defined in this way is called the *monotone product* of C^*-probability spaces $(\mathcal{A}_i, \varphi_i)_{i\in I}$, and denoted by $(\mathcal{A}, \varphi) = \triangleright_{i\in I}(\mathcal{A}_i, \varphi_i)$. It is naturally equipped with the family of (non-unital) *-homomorphisms $\rho_i : \mathcal{A}_i \to \mathcal{A}$, $i \in I$.

In this setting, Theorem 2.1 can be rewritten in the following form.

Theorem 2.3. *Let $(\mathcal{A}, \varphi) = \triangleright_{i\in I}(\mathcal{A}_i, \varphi_i)$ be any monotone product of C^*-probability spaces, equipped with the natural *-homomorphisms $\rho_i : \mathcal{A}_i \to \mathcal{A}$. Then the followings hold.*

(1) The family $\{\rho_i(\mathcal{A}_i)\}_{i\in I}$ of subalgebras of $\mathcal{A}$ is monotonically independent w.r.t. φ.

(2) ρ_i preserves the expectation, i.e. $\varphi(\rho_i(x)) = \varphi_i(x)$ for all $x \in \mathcal{A}_i$.

3. Monotonic Convolution

In this section, we introduce a kind of convolution (= "monotonic convolution") for probability measures on the real line. It describes the probability distribution for addition of monotonically independent random variables on a C^*-probability space.

Let μ be a probability measure on the real line $\mathbf{R}$. Then the *Cauchy transform* $G_\mu(z)$ of μ is defined by

$$G_\mu(z) \;=\; \int_{-\infty}^{+\infty} \frac{1}{z-x} d\mu(x), \qquad z \in \mathbf{C}^+.$$

The *reciprocal Cauchy transform* $H_\mu(z)$ of μ is defined by

$$H_\mu(z) \;=\; \frac{1}{G_\mu(z)}, \qquad z \in \mathbf{C}^+$$

(see [Maa]). $H_\mu(z)$ satisfies $H_\mu(\mathbf{C}^+) \subset \mathbf{C}^+$.

The following theorem is the main result in this section. It saids that the reciprocal Cauchy transform $H_\mu(z)$ plays in "monotonic probability" a role analogous to that played by the Fourier transform in "classical probability" and also to that played by the Voiculescu R-transform in "free probability" [VDN].

Let $(\mathcal{A}, \phi)$ be a C^*-probability space consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ over $\mathcal{A}$. For each self-adjoint random variable $X \in \mathcal{A}$, denote by $H_X(z)$ the reciprocal Cauchy transform of the probability distribution μ_X of X under ϕ.

Theorem 3.1. *Let $X_1, X_2, \cdots, X_n \in \mathcal{A}$ be monotonically independent self-adjoint random variables on $(\mathcal{A}, \phi)$, in the natural order of $\{1, 2, \cdots, n\}$. Then*

$$H_{X_1+X_2+\cdots+X_n}(z) \;=\; H_{X_1}(H_{X_2}(\cdots(H_{X_n}(z))\cdots)).$$

Proof. The pth moment of monotonically independent sum $X_1 + X_2 + \cdots + X_n$ is given by

$$m_p \;=\; \langle\, (X_1 + X_2 + \cdots + X_n)^p \,\rangle$$

$$
\begin{aligned}
&= \sum_{\substack{i_1, i_2, \cdots, i_p \\ \in \{1,2,\cdots,n\}}} \langle\, X_{i_p} \cdots X_{i_2} X_{i_1} \,\rangle \\
&= \sum_{g\in \mathrm{NCD}(p)} \sum_{\substack{f = (i_p \cdots i_2 i_1) \\ \in\, {}_{\{1,2,\cdots,n\}}\mathrm{M}_g}} \langle\, X_{i_p} \cdots X_{i_2} X_{i_1} \,\rangle \\
&= \sum_{g\in \mathrm{NCD}(p)} \sum_{\substack{f = (i_p \cdots i_2 i_1) \\ \in\, {}_{\{1,2,\cdots,n\}}\mathrm{M}_g}} \prod_{v\in\mathcal{P}(g)} \langle\, X_{f(v)}^{|v|} \,\rangle.
\end{aligned}
$$

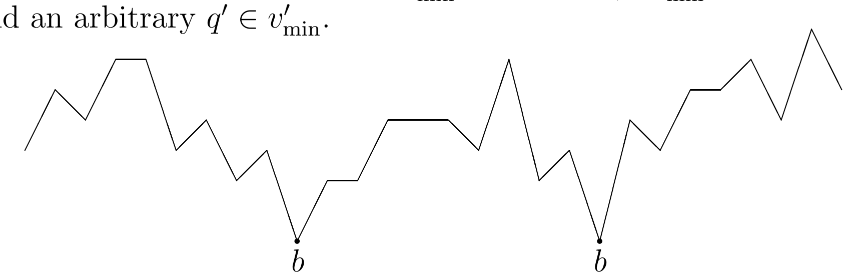
That is

$$
m_p \;=\; \sum_{g\in \mathrm{NCD}(p)} V_{X_1X_2\cdots X_n}(g), \tag{3.1}
$$

where $V(g)$ is defined by

$$
V_{X_1X_2\cdots X_n}(g) \;=\; \sum_{\substack{f = (i_p \cdots i_2 i_1) \\ \in\, {}_{\{1,2,\cdots,n\}}\mathrm{M}_g}} \prod_{v\in\mathcal{P}(g)} \langle\, X_{f(v)}^{|v|} \,\rangle. \tag{3.2}
$$

Let us obtain the recurrence relations for $V(g)$. Let $g \in \mathrm{NCD}(p)$ be a fixed arbitrary noncrossing diagram. Since any admissible configuration $f = (i_p \cdots i_2 i_1) \in {}_{\{1,2,\cdots,n\}}\mathrm{M}_g$ takes constant value on each block $v \in \mathcal{P}(g)$, we identify it with a function of a block variable: $\mathcal{P}(g) \ni v \mapsto f(v) \in I$. Then there exists in $\mathcal{P}(g)$ a unique block $v_{\min}$ such that $v_{\min}$ attains the minimal index $b \equiv \min\{f(v)|v \in \mathcal{P}(g)\}$. The uniqueness of minimizer $v_{\min}$ of $f(v)$ comes from the following reason. Let us suppose that we have two minimizers $v_{\min}$ and $v'_{\min}$ with $v_{\min} \neq v'_{\min}$. Take an arbitrary $q \in v_{\min}$ and an arbitrary $q' \in v'_{\min}$.



Since both of q and q' attain the "bottom" $b = \min\{f(v)|v \in \mathcal{P}(g)\}$ of the configuration $f = (i_p \cdots i_2 i_1)$, two sites q and q' must go into a same block v after the repeated use of the rule (a) of the monotonic independence. This contradicts to the assumption that $v_{\min} \neq v'_{\min}$. Therefore the uniqueness of minimizer of $f(v)$ must hold. Note that this unique minimizer $v_{\min}$ is an outer block of a diagram g.

Now, let

$$
g \;=\; h_1 h_2 \cdots h_l
$$

be the natural decomposition of a given diagram $g \in \mathrm{NCD}(p)$, as a concatenation, so that $h_1, h_2, \cdots, h_l \in \mathrm{NCD}^* \equiv \cup_{r=1}^{\infty}\mathrm{NCD}(r)$ and $\mathrm{out}(h_1) = \cdots = \mathrm{out}(h_l) = 1$. Suppose that the minimizer $v_{\min}$ is given by the outer block $\mathrm{ext}(h_m)$ of h_m for some unique m. Here $\mathrm{ext}(h)$ denotes the unique outer block of a diagram $h \in \mathrm{NCD}^*$ with $\mathrm{out}(h) = 1$. For each $f \in {}_{\{1,2,\cdots,n\}}\mathrm{M}_g$, this number m is uniquely determined. Let us cut the diagram g at the both sides of h_m, so we have

$$
\begin{aligned}
g &= (h_1 \cdots h_{m-1}) h_m (h_{m+1} \cdots h_l), \\
h_m &= \bullet\; g_1 \;\bullet\; g_2 \;\bullet \cdots \bullet\; g_k \;\bullet\,.
\end{aligned}
$$

Here $g_1, g_2, \cdots, g_k \in \mathrm{NCD} \equiv \cup_{r=0}^{\infty}\mathrm{NCD}(r)$ with $\mathrm{NCD}(0) \equiv \{\Lambda = \text{the empty diagram}\}$. We note that the restrictions $f|_{h_1\cdots h_{m-1}}$, $f|_{h_{m+1}\cdots h_l}$, $f|_{g_1}$, $f|_{g_2}$, $\cdots$, $f|_{g_k}$ of a given admissible configuration $f : \mathcal{P}(g) \ni v \mapsto f(v) \in I$ to "subdiagrams" $h_1 \cdots h_{m-1}$, $h_{m+1} \cdots h_l$, g_1, g_2, $\cdots$, g_k of g are always admissible configurations. That is

$$
\begin{array}{lcl}
f|_{h_1\cdots h_{m-1}} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{h_1\cdots h_{m-1}}, \\
f|_{h_{m+1}\cdots h_l} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{h_{m+1}\cdots h_l}, \\
f|_{g_1} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{g_1}, \\
 & \vdots & \\
f|_{g_k} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{g_k}.
\end{array}
$$

Conversely, when we choose arbitrary admissible configurations

$$
\begin{array}{lcl}
\varphi_{h_1\cdots h_{m-1}} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{h_1\cdots h_{m-1}}, \\
\varphi_{h_{m+1}\cdots h_l} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{h_{m+1}\cdots h_l}, \\
\varphi_{g_1} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{g_1}, \\
 & \vdots & \\
\varphi_{g_k} & \in & {}_{\{b+1,\cdots,n\}}\mathrm{M}_{g_k}
\end{array}
$$

from "subdiagrams" $h_1 \cdots h_{m-1}$, $h_{m+1} \cdots h_l$, g_1, g_2, $\cdots$, g_k, respectively, and assign the value b to the block $\mathrm{ext}(h_m)$, then, by the natural composition, we get a "global" admissible configuration φ in ${}_{\{1,2,\cdots,n\}}\mathrm{M}_g$ such that its minimal index $\min\{\varphi(v)|v \in \mathcal{P}(g)\}$ is attained at the block $\mathrm{ext}(h_m)$ and its value is b. Therefore all the admissible configurations $f \in {}_{\{1,2,\cdots,n\}}\mathrm{M}_g$ can be classified by the pair (b, i) of a possible value b of minimal index $\min\{f(v)|v \in \mathcal{P}(g)\}$ and a possible position i (in the list $\{1, 2, \cdots, l\}$) of minimizer $v_{\min}$.

Since the quantity $V_{X_1X_2\cdots X_n}(g)$ is defined by (3.2) as the "product sum" over all admissible configurations $f \in {}_{\{1,2,\cdots,n\}}\mathrm{M}_g$, it must satisfies the following recurrence relations.

Recurrence relations:

i) $\quad V_{X_1\cdots X_n}(\,\overbrace{\sqcup\!\sqcup\!\sqcup \cdots \sqcup\!\sqcup}^{r}\,) = \sum_{i=1}^{n} \langle\, X_i^r \,\rangle \qquad$ (for a single block);

ii) $\quad V_{X_1\cdots X_n}(g)$

$$
= \sum_{b=1}^{n}\sum_{i=1}^{l} V_{X_{b+1}\cdots X_n}(h_1 \cdots h_{i-1})\, V^{\circ}_{X_{b+1}\cdots X_n}(h_i)\, \langle X_b^{|\mathrm{ext}(h_i)|}\rangle\, V_{X_{b+1}\cdots X_n}(h_{i+1} \cdots h_l)
$$

$$
(\text{for} \quad g = h_1h_2 \cdots h_l \;\; \text{with} \;\; \mathrm{out}(h_1) = \mathrm{out}(h_2) = \cdots = \mathrm{out}(h_l) = 1).
$$

Here the quantity $V^{\circ}_{X_{b+1}\cdots X_n}(h)$ is defined for $h \in \{h \in \mathrm{NCD}^* \,|\mathrm{out}(h) = 1\}$ by

$$
V^{\circ}_{X_{b+1}\cdots X_n}(\,\underline{|\, g_1 \,|\, g_2 \,|\, \cdots \,|\, g_k \,|}\,) = V_{X_{b+1}\cdots X_n}(g_1)V_{X_{b+1}\cdots X_n}(g_2) \cdots V_{X_{b+1}\cdots X_n}(g_k).
$$

Also we made a convention that $V_{X_1\cdots X_n}(\Lambda) \equiv 1$ and $V_{\emptyset}(g) \equiv 0$ (if $g \neq \Lambda$), $\equiv 1$ (if $g = \Lambda$).

Let us calculate the moment generating function

$$
f_{X_1\cdots X_n}(s) = \sum_{p=0}^{\infty} m_p\, s^p
$$

with $m_p = m_p(X_1, \cdots, X_n) = \langle (X_1 + \cdots + X_n)^p \rangle$, in a formal way. Here we make a convention that $f_{\emptyset}(s) \equiv 1$. Denote by $p(g)$ the number $\#\{\text{points in } g\}$. Using the recurrence relation ii), we get

$$\begin{aligned}
f_{X_1 \cdots X_n}(s) \;&=\; \sum_{g \in \mathrm{NCD}} V_{X_1 \cdots X_n}(g)\; s^{p(g)} \\
&=\; 1 \;+\; \sum_{b=1}^{n} \sum_{l=1}^{\infty} \sum_{\substack{g \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(g) \,=\, l}} \sum_{i=1}^{l} V_{X_{b+1} \cdots X_n}(h_1 \cdots h_{i-1})\; s^{p(h_1 \cdots h_{i-1})} \\
&\qquad \cdot\, V^{\circ}_{X_{b+1} \cdots X_n}(h_i)\; \langle\, X_b^{|\mathrm{ext}(h_i)|} \,\rangle\; s^{p(h_i)} \;\cdot\; V_{X_{b+1} \cdots X_n}(h_{i+1} \cdots h_l) s^{p(h_{i+1} \cdots h_l)} \\
&=\; 1 \;+\; \sum_{b=1}^{n} f_{X_{b+1} \cdots X_n}(s) \left(\sum_{\substack{h \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(h) \,=\, 1}} V^{\circ}_{X_{b+1} \cdots X_n}(h)\; \langle\, X_b^{|\mathrm{ext}(h)|} \,\rangle\; s^{p(h)} \right) f_{X_{b+1} \cdots X_n}(s).
\end{aligned}$$

That is

$$f_{X_1 \cdots X_n}(s) \;=\; 1 \;+\; \sum_{b=1}^{n} f_{X_{b+1} \cdots X_n}(s)^2\; g_{X_{b+1} \cdots X_n}(s), \tag{3.3}$$

where $g_{X_{b+1} \cdots X_n}(s)$ is defined by

$$g_{X_{b+1} \cdots X_n}(s) \;=\; \sum_{\substack{h \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(h) \,=\, 1}} V^{\circ}_{X_{b+1} \cdots X_n}(h)\; \langle\, X_b^{|\mathrm{ext}(h)|} \,\rangle\; s^{p(h)}.$$

This $g_{X_{b+1} \cdots X_n}(s)$ is rewritten as

$$\begin{aligned}
g_{X_{b+1} \cdots X_n}(s) \;&=\; \sum_{\substack{h \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(h) \,=\, 1}} V^{\circ}_{X_{b+1} \cdots X_n}(h)\; \langle\, X_b^{|\mathrm{ext}(h)|} \,\rangle\; s^{p(h)} \\
&\qquad\qquad (\text{Put} \quad h = \bullet\, g_1 \,\bullet\, g_2 \,\bullet \cdots \bullet\, g_k \,\bullet\,) \\
&=\; \sum_{k=0}^{\infty} \sum_{\substack{g_1, g_2, \cdots, g_k \\ \in\, \mathrm{NCD}}} \Big\{ V_{X_{b+1} \cdots X_n}(g_1)\; s^{p(g_1)} \\
&\qquad\qquad \cdots\; V_{X_{b+1} \cdots X_n}(g_k)\; s^{p(g_k)} \Big\}\; \langle X_b^{k+1} \rangle\; s^{k+1} \\
&=\; \sum_{k=0}^{\infty} \Big(\sum_{g \in \mathrm{NCD}} V_{X_{b+1} \cdots X_n}(g)\; s^{p(g)} \Big)^k\; \langle X_b^{k+1} \rangle\; s^{k+1} \\
&=\; \sum_{k=0}^{\infty} \Big(s\, f_{X_{b+1} \cdots X_n}(s) \Big)^k\; \langle X_b^{k+1} \rangle\; s,
\end{aligned}$$

that is

$$g_{X_{b+1} \cdots X_n}(s) \;=\; \sum_{k=0}^{\infty} \langle X_b^{k+1} \rangle \Big(s\, f_{X_{b+1} \cdots X_n}(s) \Big)^k\; s.$$

From this expression, we get

$$1 \;+\; f_{X_{b+1} \cdots X_n}(s)\; g_{X_{b+1} \cdots X_n}(s) \;=\; 1 \;+\; \sum_{k=0}^{\infty} \langle X_b^{k+1} \rangle \Big(s\, f_{X_{b+1} \cdots X_n}(s) \Big)^{k+1}$$

$$
\begin{aligned}
&= \sum_{l=0}^{\infty} \langle X_b^l \rangle \Big(s\, f_{X_{b+1}\cdots X_n}(s) \Big)^l \\
&= f_{X_b}\Big(s\, f_{X_{b+1}\cdots X_n}(s) \Big),
\end{aligned}
$$

that is

$$
1 \;+\; f_{X_{b+1}\cdots X_n}(s)\, g_{X_{b+1}\cdots X_n}(s) \;=\; f_{X_b}\Big(s\, f_{X_{b+1}\cdots X_n}(s) \Big). \tag{3.4}
$$

Using (3.4), the expression (3.3) is rewritten as

$$
\begin{aligned}
f_{X_1\cdots X_n}(s) \quad &= \; 1 \;+\; \sum_{b=1}^{n} f_{X_{b+1}\cdots X_n}(s)^2\, g_{X_{b+1}\cdots X_n}(s) \\
&= \; 1 \;+\; \sum_{b=1}^{n} f_{X_{b+1}\cdots X_n}(s) \left(\Big(1 \;+\; f_{X_{b+1}\cdots X_n}(s)\, g_{X_{b+1}\cdots X_n}(s)\Big) - 1 \right) \\
&= \; 1 \;+\; \sum_{b=1}^{n} f_{X_{b+1}\cdots X_n}(s) \Big(f_{X_b}(s\, f_{X_{b+1}\cdots X_n}(s)) - 1 \Big) \\
&= \; 1 \;+\; \sum_{b=2}^{n} f_{X_{b+1}\cdots X_n}(s) \Big(f_{X_b}(s\, f_{X_{b+1}\cdots X_n}(s)) - 1 \Big) \\
&\qquad +\; f_{X_2\cdots X_n}(s) \Big(f_{X_1}(s\, f_{X_2\cdots X_n}(s)) - 1 \Big) \\
&= \; f_{X_2\cdots X_n}(s) \;+\; f_{X_2\cdots X_n}(s) \Big(f_{X_1}(s\, f_{X_2\cdots X_n}(s)) - 1 \Big) \\
&= \; f_{X_2\cdots X_n}(s)\; f_{X_1}(s\, f_{X_2\cdots X_n}(s)).
\end{aligned}
$$

Hence we get

$$
f_{X_1\cdots X_n}(s) \;=\; f_{X_2\cdots X_n}(s)\; f_{X_1}(s\, f_{X_2\cdots X_n}(s)). \tag{3.5}
$$

Multiply (3.5) by s, and put $s := \frac{1}{z}$, then we get

$$
\begin{aligned}
G_{X_1+\cdots+X_n}(z) \;&=\; G_{X_2+\cdots+X_n}(z)\; f_{X_1}(G_{X_2+\cdots+X_n}(z)) \\
&=\; G_{X_1}\Big(\frac{1}{G_{X_2+\cdots+X_n}(z)} \Big).
\end{aligned}
$$

So we have

$$
H_{X_1+\cdots+X_n}(z) \;=\; H_{X_1}(H_{X_2+\cdots+X_n}(z)).
$$

Repeating this formula, we obtain

$$
H_{X_1+\cdots+X_n}(z) \;=\; H_{X_1}(H_{X_2}(\cdots(H_{X_n}(z))\cdots)). \tag{3.6}
$$

Although we get this equality (3.6) through a formal calculation, (3.6) hold for all $z \in \mathbf{C}^+$ because of the boundedness of X's and the uniqueness theorem for holomorphic functions. $\square$

This result motivates us to give the following definition.

Definition 3.2 (monotonic convolution). *For a pair of probability measures μ, ν on $\mathbf{R}$, the unique probability measure λ satisfying $H_\lambda(z) = H_\mu(H_\nu(z))$, $z \in \mathbf{C}^+$, is called the monotonic convolution of μ and ν, and denoted by $\lambda = \mu \rhd \nu$.*

The unique existence of such measure λ for given (μ, ν) is shown below. Let us remind of some facts from the theory of Pick-Nevanlinna functions [Bha]. A holomorphic function $f(z)$ on $\mathbf{C}^+$ is said to be a *Pick function* if $\Im f(z) \geq 0$ for all $z \in \mathbf{C}^+$. The following integral representation formula for Pick functions plays a crucial role throughout the paper. Here i is the imaginary unit.

Theorem 3.3 (Nevanlinna's theorem). *[Bha] Let $f(z)$ be a function on $\mathbf{C}^+$. Then the followings two conditions are equivalent.*

(1) $f(z)$ is a Pick function.
(2) There exist a real number α, a non-negative real number $\beta \geq 0$ and a finite positive measure γ on $\mathbf{R}$ such that

$$f(z) \;=\; \alpha + \beta z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x), \qquad z \in \mathbf{C}^+.$$

If the above conditions hold, the triple (α, β, γ) is unique, and satisfies $\alpha = \Re f(i)$, $\gamma(\mathbf{R}) = \Im f(i) - \beta$ and

$$\beta \;=\; \lim_{0<y\to\infty} \frac{\Im f(i\,y)}{y}.$$

For any probability measure μ on $\mathbf{R}$, the reciprocal Cauchy transform $H_\mu(z)$ is a Pick function. Besides it satisfies $\Im H_\mu(z) \geq \Im z$ for all $z \in \mathbf{C}^+$. In this case, the following characterization is known.

Theorem 3.4. *[Maa] Let $f : \mathbf{C}^+ \to \mathbf{C}^+$ be a holomorphic function. Then the following three conditions are equivalent.*

(1) $f(z) = H_\mu(z)$, $z \in \mathbf{C}^+$, for some probability measure μ on $\mathbf{R}$.
(2) There exist a real number $a \in \mathbf{R}$ and a finite positive measure τ on $\mathbf{R}$ such that

$$f(z) \;=\; a + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau(x), \qquad z \in \mathbf{C}^+.$$

(3)

$$\inf_{z\in\mathbf{C}^+} \frac{\Im f(z)}{\Im z} \;=\; 1.$$

Now let us show the well-definedness of the definition 3.2.

Theorem 3.5. *For a pair of probability measures μ, ν on $\mathbf{R}$, there exists a unique probability measure λ on $\mathbf{R}$ such that*

$$H_\lambda(z) \;=\; H_\mu(H_\nu(z)), \quad z \in \mathbf{C}^+.$$

Proof. From Theorem 3.4, there exists a real number a and a finite positive measure τ on $\mathbf{R}$ such that

$$H_\nu(z) \;=\; a + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau(x),$$

from which we have

$$H_\nu(z) - y \;=\; (a-y) + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau(x).$$

Again from Theorem 3.4, there exists a unique probability measure ν_y such that

$$H_{\nu_y}(z) \;=\; (a-y) + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau(x).$$

Note that $H_{\nu_y}(z)$ is continuous in y. So we have

$$\begin{aligned} G_\mu(H_\nu(z)) &= \int_{-\infty}^{+\infty} \frac{1}{H_\nu(z)-y} d\mu(y) \\ &= \int_{-\infty}^{+\infty} G_{\nu_y}(z)\, d\mu(y) \\ &= \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} \frac{1}{z-x} d\nu_y(x) \Big) d\mu(y) \\ &= \int_{-\infty}^{+\infty} \frac{1}{z-x} d\Big(\int_{-\infty}^{+\infty} \nu_y(x) d\mu(y) \Big) \\ &= G_\lambda(z). \end{aligned}$$

Here the measure λ is defined by $\lambda(\cdot) = \int_{-\infty}^{+\infty} \nu_y(\cdot) d\mu(y)$. This implies $H_\lambda(z) = H_\mu(H_\nu(z))$. $\square$

The monotonic convolution $\mu \rhd \nu$ satisfies the following properties. (Denote by δ_0 the point measure at the origin $x = 0$.)

(1) $\delta_0 \rhd \mu \ = \ \mu \rhd \delta_0 \ = \ \mu$;
(2) $(\lambda \rhd \mu) \rhd \nu \ = \ \lambda \rhd (\mu \rhd \nu)$;
(3) the map $\mu \mapsto \mu \rhd \nu$ is affine;
(4) the map $\mu \mapsto \mu \rhd \nu$ (resp. $\nu \mapsto \mu \rhd \nu$) is weak* continuous.

Note that the monotonic convolution is not commutative in general: $\mu \rhd \nu \neq \nu \rhd \mu$.

4. Monotonic Lévy-Hinčin formula in terms of semigroups

In this section, we investigate the "infinite divisibility" with respect to the monotonic convolution, and prove a monotonic analogue of Lévy-Hinčin formula in terms of continuous one-parameter monotonic convolution semigroups of probability measures.

In the setting of "monotonic probability," we formulate, in an appropriate sense, the following three objects:

(A) infinitely divisible distribution ;
(B) continuous one-parameter convolution semigroup ;
(C) (certain) integral representation (= "Lévy measure").

We wish to establish the equivalence beteween among three objects (A), (B) and (C). This should be the content of "monotonic Lévy-Hinčin formula." The equivalence between (B) and (C) will be established in this section in the general setting (Theorem 4.7). On the other hand, the equivalence between (A) and (B) will be established in the next section, but in the restricted class of compactly supported probability measures (Theorem 5.1).

Let us give the definitions of notions concerning the "infinite divisibility."

Definition 4.1. *A probability measure μ on $\mathbf{R}$ is said to be $\rhd$-infinitely divisible if, for each $n \in \mathbf{N}^*$, there exists some probability measure ν on $\mathbf{R}$ such that*

$$\mu \ = \ \overbrace{\nu \rhd \nu \rhd \cdots \rhd \nu}^{n} .$$

Definition 4.2. *A real one-parameter family $\{\mu_t\}_{t \geq 0}$ of probability measures on $\mathbf{R}$ is said to be a weak* continuous one-parameter $\rhd$-semigroup if the following* *conditions are satisfied: (1)* $\mu_0 = \delta_0$; *(2)* $\mu_{s+t} = \mu_s \rhd \mu_t$; *(3) the map* $t \mapsto \mu_t$ *is weak* continuous.*

Definition 4.3. *A real one-parameter family* $\{H_t(z)\}_{t\geq 0}$ *of reciprocal Cauchy transforms of probability measures on* $\mathbf{R}$ *is said to be a continuous one-parameter semigroup of reciprocal Cauchy transforms if the following conditions are satisfied: (1)* $H_0(z) = z$; *(2)* $H_{s+t}(z) = H_s(H_t(z))$; *(3) the map* $t \mapsto H_t(z)$ *is continuous for each fixed* $z \in \mathbf{C}^+$.

There is the natural bijective correspondence beteween the above two kinds of continuous one-parameter semigroups $\{\mu_t\}_{t\geq 0}$ and $\{H_t(z)\}_{t\geq 0}$ because a sequence $\{\mu_n\}_{n=1}^{\infty}$ of probability measures on $\mathbf{R}$ converges in the weak* topology to a probability measure μ if and only if $H_{\mu_n}(z) \to H_\mu(z)$ as $n \to \infty$ for all $z \in \mathbf{C}^+$. Besides there is the natural correspondence from the set of all weak* continuous one-parameter $\rhd$-semigroups $\{\mu_t\}_{t\geq 0}$ to the set of all $\rhd$-infinitely divisible distributions μ given by the specialization (t:=1) : $\{\mu_t\}_{t\geq 0} \mapsto \mu_1$. (In the next section we will construct a partial converse $\mu \mapsto \{\mu_t\}_{t\geq 0}$ for the class of $\rhd$-infinitely divisible distributions with compact supports (Proposition 5.4).)

Let us give some examples of continuous one-parameter semigroups $\{H_t(z)\}_{t\geq 0}$ and its associated $\rhd$-infinitely divisible distributions $\mu = \mu_1$. Denote by χ_I the indicator function of an interval I. Denote by μ_{ac} (resp. μ_s) the absolutely continuous part (resp. the singular part) of μ w.r.t. the Lebesgue measure dx. Denote by E_n^{-1} the nth branch of the product log function E^{-1} (= the inverse analytic function of an entire function $E(z) = z\,e^z$ [Mu3]). E_*^{-1} denotes an appropriate branch composed from E_0^{-1} and E_{-1}^{-1} [Mu3]. Also we denote E_{-1}^{-1} by E^{-1} for short.

Example 4.4. (1) Point measure: $H_t(z) = z - at, \quad \mu = \delta_a.$

(2) Arcsine distribution (= monotonic Gaussian distribution) [Mu1]:

$$H_t(z) = \sqrt{z^2 - 2t}, \quad d\mu(x) = \chi_{(-\sqrt{2},\sqrt{2})}(x) \cdot \frac{1}{\pi\sqrt{2 - x^2}}\,dx.$$

(3) Monotonic Poisson distribution [Mu3]:

$$\begin{aligned}
&H_t(z) = -E_*^{-1}(e^{\lambda t}\,E(-z)),\\
&d\mu_{ac}(x) = \chi_{(a,b)}(x) \cdot \frac{1}{\pi}\Im\frac{1}{E^{-1}(e^\lambda E(-x))}\,dx, \quad \mu_s = c\,\delta_0,\\
&a = -E_0^{-1}(-\frac{1}{e^{1+\lambda}}), \quad b = -E_{-1}^{-1}(-\frac{1}{e^{1+\lambda}}), \quad c = \frac{1}{e^\lambda}, \quad (\lambda > 0).
\end{aligned}$$

(4) A deformation of arcsine distribution:

$$\begin{aligned}
&H_t(z) = c + \sqrt{(z-c)^2 - 2t},\\
&d\mu_{ac}(x) = \chi_{(c-\sqrt{2},c+\sqrt{2})}(x) \cdot \frac{1}{\pi}\frac{\sqrt{2-(x-c)^2}}{c^2 + 2 - (x-c)^2}\,dx, \ \left(\mu_{ac}(\mathbf{R}) = 1 - \frac{|c|}{\sqrt{2+c^2}}\right),\\
&d\mu_s(x) = A\,\delta_{c-\sqrt{2+c^2}} + B\,\delta_{c+\sqrt{2+c^2}}, \quad (A, B : \text{the normalization constants}).
\end{aligned}$$

(5) Cauchy distribution:

$$H_t(z) \ = \ z + ibt, \qquad d\mu(x) \ = \ \frac{1}{\pi}\frac{b}{x^2 + b^2}\,dx \qquad (b > 0).$$

From these examples, the following two features of "monotonic probability" can be read out.

- It is often that important probability distributions may have the reciprocal form: $\frac{1}{\text{some function}}$. (Of course this is an immediate effect of the reciprocal Cauchy transform.) It can be said that, in a sense, "monotonic probability" is a "reciprocal probability."

- It is often that the reciprocal Cauchy transform $H_\mu(z)$ of $\triangleright$-infinitely divisible distribution μ includes a pair consisting of some function f and its inverse function f^{-1}. In fact, this is a general phenomenon as shown in the following Theorem 4.5.

A continuous one-parameter semigroup $\{H_t(z)\}_{t\geq 0}$ of reciprocal Cauchy transforms is said to be *trivial* if $H_t(z) \equiv z$ for all $t \geq 0$ and all $z \in \mathbf{C}^+$.

Theorem 4.5. *Let $\{H_t(z)\}_{t\geq 0}$ be a continuous one-parameter semigroup of reciprocal Cauchy transforms of probability measures on $\mathbf{R}$. Then the followings hold.*

(1) For each fixed $z \in \mathbf{C}^+$, the map $t \mapsto H_t(z)$ is of class C^∞ (in fact, real analytic). In particular, the limit

$$A(z) \;\equiv\; \lim_{0<\delta\to 0} \frac{H_\delta(z) - z}{\delta}, \quad z \in \mathbf{C}^+$$

exists. The function $A(z)$ is a Pick function, and there exists a unique pair (α, γ) of a real number α and a finite positive measure γ on $\mathbf{R}$ such that

$$A(z) \;=\; \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x), \quad z \in \mathbf{C}^+.$$

(2) Assume further that the semigroup $\{H_t(z)\}_{t\geq 0}$ is not trivial. Then,

i) The reciprocal $\dfrac{1}{A(z)}$ exists for all $z \in \mathbf{C}^+$.

ii) Put $F(z) = \displaystyle\int_i^z \frac{1}{A(z)} dz$. Then, for any $z \in \mathbf{C}^+$ and any $t \geq 0$, there exists a unique $w \in \mathbf{C}^+$ such that $F(w) - F(z) = t$. Besides, $H_t(z)$ has the following representation:

$$H_t(z) \;=\; F^{-1}(F(z) + t), \quad t \geq 0, \;\; z \in \mathbf{C}^+.$$

Proof. Step 1 (right differentiability at $t = 0$). By Theorem 3.4, there exists a unique pair (a_t, τ_t) of a real number $a_t \in \mathbf{R}$ and a finite positive measure τ_t on $\mathbf{R}$ such that

$$H_t(z) \;=\; a_t + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau_t(x), \quad z \in \mathbf{C}^+.$$

Hence $\tau_t(\mathbf{R}) = \Im H_t(i) - 1$ is continuous in t. By the way, the difference $H_{t+\varepsilon}(z) - H_t(z)$ is given by

$$H_t(H_\varepsilon(z)) - H_t(z) \;=\; H_\varepsilon(z) - z + (H_\varepsilon(z) - z) \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_\varepsilon(z))(x-z)} d\tau_t(x). \quad (4.1)$$

Put $\varepsilon := \frac{\delta}{n}$ and $t := \frac{k}{n}\delta$ with $\delta > 0$, and take the sum of (4.1) over all $k = 0, 1, 2, \cdots, n-1$, then we get

$$H_\delta(z) - H_0(z) \;=\; n(H_{\frac{\delta}{n}}(z) - z) + (H_{\frac{\delta}{n}}(z) - z) \sum_{k=0}^{n-1} \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_{\frac{\delta}{n}}(z))(x-z)} d\tau_{\frac{k}{n}\delta}(x).$$

That is

$$H_\delta(z)-z \;=\; \delta\cdot\frac{(H_{\frac{\delta}{n}}(z)-z)}{\frac{\delta}{n}}\cdot+\delta\cdot\frac{(H_{\frac{\delta}{n}}(z)-z)}{\frac{\delta}{n}}\cdot\frac{1}{n}\sum_{k=0}^{n-1}\int_{-\infty}^{+\infty}\frac{1+x^2}{(x-H_{\frac{\delta}{n}}(z))(x-z)}d\tau_{\frac{k}{n}\delta}(x).$$

Let us calculate the limit of this equality for $n\to\infty$. At first we have the convergence to the Riemann integral:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\int_{-\infty}^{+\infty}\frac{1+x^2}{(x-H_{\frac{\delta}{n}}(z))(x-z)}d\tau_{\frac{k}{n}\delta}(x) \;=\; \frac{1}{\delta}\int_0^\delta\Big(\int_{-\infty}^{+\infty}\frac{1+x^2}{(x-z)^2}d\tau_t(x)\Big)dt$$

from Lemma 6.1. This integral can be made arbitrarily small when δ is taken to be sufficiently small. This is because of $\tau_t(\mathbf{R})=\Im(H_t(i))-1\to 0\ (t\to 0)$. Hence there exists $\delta_0>0$ such that, for any δ with $0<\delta<\delta_0$, the limit

$$A_\delta(z)\;\equiv\;\lim_{n\to\infty}\frac{H_{\frac{\delta}{n}}(z)-z}{\frac{\delta}{n}}\;=\;\frac{(H_\delta(z)-z)/\delta}{1+\frac{1}{\delta}\int_0^\delta\Big(\int_{-\infty}^{+\infty}\frac{1+x^2}{(x-z)^2}d\tau_t(x)\Big)dt}$$

exists. (Note that "the denominator $\neq 0$" because $\delta>0$ is sufficiently small.)

Now let us show that the limit $A_\delta(z)$ is not dependent on δ. Let r be a fixed arbitrary rational number with $0<r<1$. Let p and q be natural numbers ≥ 1 such that $r=\frac{p}{q}$. Then we have

$$\lim_{n\to\infty}\frac{H_{\frac{r\delta}{n}}(z)-z}{\frac{r\delta}{n}}\;=\;A_{r\delta}(z)$$

because of $0<r\delta<\delta_0$. Let $\{(H_{\frac{r\delta}{n(k)}}(z)-z)/\frac{r\delta}{n(k)}\}_{k=1}^{\infty}$ be the subsequence of the above convergent sequence defined by $n(k)=kp\ (k=1,2,3,\cdots)$. Then we have

$$\lim_{k\to\infty}\frac{H_{\frac{r\delta}{n(k)}}(z)-z}{\frac{r\delta}{n(k)}}\;=\;A_{r\delta}(z).$$

By the way, since $\frac{r\delta}{n(k)}=\frac{\delta}{kq}$, the sequence $\{(H_{\frac{r\delta}{n(k)}}(z)-z)/\frac{r\delta}{n(k)}\}_{k=1}^{\infty}$ can be viewed as a subsequence of another convergent sequence

$$\lim_{n\to\infty}\frac{H_{\frac{\delta}{n}}(z)-z}{\frac{\delta}{n}}\;=\;A_\delta(z),$$

and hence

$$\lim_{k\to\infty}\frac{H_{\frac{r\delta}{n(k)}}(z)-z}{\frac{r\delta}{n(k)}}\;=\;A_\delta(z).$$

The above two limits of the same sequence $\{(H_{\frac{r\delta}{n(k)}}(z)-z)/\frac{r\delta}{n(k)}\}_{k=1}^{\infty}$ must coincide. So we have $A_{r\delta}(z)=A_\delta(z)$ for all rational numbers r with $0<r<1$. By the way, for each fixed $z\in\mathbf{C}^+$, $A_\delta(z)$ is a continuous function of δ because of the expression

$$A_\delta(z)\;=\;\frac{(H_\delta(z)-z)/\delta}{1+\frac{1}{\delta}\int_0^\delta\Big(\int_{-\infty}^{+\infty}\frac{1+x^2}{(x-z)^2}d\tau_t(x)\Big)dt}. \tag{4.2}$$

This implies that $A_{r\delta}(z) = A_\delta(z)$ for all real numbers r with $0 < r < 1$. Furthermore this means that $A_\delta(z)$ is not dependent on δ when δ is sufficiently small ($0 < \delta < \delta_0$). So we write $A(z) \equiv A_\delta(z)$. From the expression (4.2), we have

$$A(z)\Big(1 + \frac{1}{\delta}\int_0^\delta \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)\Big) dt\Big) \quad = \quad \frac{H_\delta(z) - z}{\delta}.$$

Taking the limit $\delta \to 0$, we get

$$A(z) \quad = \quad \lim_{0<\delta\to 0} \frac{H_\delta(z) - z}{\delta}. \tag{4.3}$$

The expression (4.2) also implies that $A(z)$ ($= A_\delta(z)$) is holomorphic on $\mathbf{C}^+$.

Step 2 (differentiability). The substitution $z := H_t(z)$ for the expression (4.3) yields the right differentiability of the map $t \mapsto H_t(z)$ for each fixed $z \in \mathbf{C}^+$, and hence we get the right derivative

$$D_t^+ H_t(z) \quad = \quad A(H_t(z)).$$

Let $T > 0$ and $\delta > 0$ be arbitrary positive real numbers with $0 < T - \delta < T$. Put $\varepsilon := \frac{\delta}{n}$ and $t := T - \frac{k}{n}\delta$ with $k = 1, 2, 3, \cdots, n$, and take the sum of diferrences $H_t(H_\varepsilon(z)) - H_t(z)$ over all $k = 1, 2, 3, \cdots, n$, then we have

$$H_T(z) - H_{T-\delta}(z) = n(H_{\frac{\delta}{n}}(z) - z) + (H_{\frac{\delta}{n}}(z) - z)\sum_{k=1}^n \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_{\frac{\delta}{n}}(z))(x-z)} d\tau_{T-\frac{k}{n}\delta}(x).$$

Devide this equality with δ and take the limit $n \to \infty$, then we get

$$\frac{H_T(z) - H_{T-\delta}(z)}{\delta} \quad = \quad A(z)\Big(1 + \frac{1}{\delta}\int_{T-\delta}^T \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)\Big) dt\Big). \tag{4.4}$$

Here we used the convergence to Riemann integral :

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_{\frac{\delta}{n}}(z))(x-z)} d\tau_{T-\frac{k}{n}\delta}(x) = \frac{1}{\delta}\int_{T-\delta}^T \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)\Big) dt.$$

Furthermore take the limit of (4.4) for $0 < \delta \to 0$ and replace the letter T with t, then we get the left differentiability and the left derivative

$$D_t^- H_t(z) \quad = \quad A(z)\Big(1 + \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)\Big).$$

Since the second factor of the r.h.s. of this equality equals $\frac{\partial}{\partial z} H_t(z)$, we get

$$\begin{aligned}
D_t^+ H_t(z) \quad &= \quad \lim_{0<\delta\to 0} \frac{H_{t+\delta}(z) - H_t(z)}{\delta} \quad = \quad A(H_t(z)), \quad (t \geq 0), \\
D_t^- H_t(z) \quad &= \quad \lim_{0<\delta\to 0} \frac{H_{t-\delta}(z) - H_t(z)}{-\delta} \quad = \quad \Big(\frac{\partial}{\partial z} H_t(z)\Big) A(z), \quad (t > 0).
\end{aligned}$$

The differentiability of $H_t(z)$ can be concluded from

$$\begin{aligned}
D_t^+ H_t(z) \quad &= \quad \lim_{0<\delta\to 0} \frac{H_{t+\delta}(z) - H_t(z)}{\delta} \\
&= \quad \lim_{0<\delta\to 0} \frac{H_t(H_\delta(z)) - H_t(z)}{H_\delta(z) - z} \cdot \frac{H_\delta(z) - z}{\delta} \\
&= \quad \Big(\frac{\partial}{\partial z} H_t(z)\Big) A(z) \quad = \quad D_t^- H_t(z)
\end{aligned}$$

whenever the semigroup $\{H_t(z)\}_{t\geq 0}$ is not trivial. Here the denominator $H_\delta(z) - z$ is not zero because of Lemma 6.2. On the other hand, the differentiability of $H_t(z)$ is obvious when $\{H_t(z)\}_{t\geq 0}$ is trivial. After all the differential coefficient of $H_t(z)$ is given by $D_t H_t(z) = A(H_t(z))$. This yields that the map $t \mapsto H_t(z)$ is of class C^∞ for each fixed $z \in \mathbf{C}^+$. (The analyticity of the map $t \mapsto H_t(z)$ will become clear from the representation formula for $H_t(z)$ at Step 7 in the later.)

Step 3 (integral representation of a "generator" $A(z)$). The function $A(z)$ constructed above is a Pick function since

$$\Im A(z) \;=\; \lim_{n\to\infty} \Im\left(n(H_{\frac{1}{n}}(z) - z)\right) \;=\; n \lim_{n\to\infty} (\Im(H_{\frac{1}{n}}(z)) - \Im z) \;\geq\; 0, \quad z \in \mathbf{C}^+.$$

So there exist real numbers $\alpha, \beta \in \mathbf{R}$, $\beta \geq 0$ and a finite positive measure γ on $\mathbf{R}$ such that

$$A(z) \;=\; \alpha + \beta z + \int_{-\infty}^{+\infty} \frac{1 + xz}{x - z} d\gamma(x), \quad z \in \mathbf{C}^+$$

from Theorem 3.3. Let us show that $\beta = 0$. Let us remind of the expression

$$A(z) \;=\; \frac{(H_\delta(z) - z)/\delta}{1 + \dfrac{1}{\delta}\displaystyle\int_0^\delta \Big(\int_{-\infty}^{+\infty} \frac{1 + x^2}{(x - z)^2} d\tau_t(x)\Big) dt}. \tag{4.5}$$

From the representation

$$H_\delta(z) \;=\; a_\delta + z + \int_{-\infty}^{+\infty} \frac{1 + xz}{x - z} d\tau_\delta(x),$$

we get

$$\Im(H_\delta(z) - z) \;=\; \Big(\int_{-\infty}^{+\infty} \frac{1 + x^2}{|x - z|^2} d\tau_\delta(x)\Big) \cdot \Im z,$$

and hence

$$\lim_{0<y\to\infty} \frac{\Im(H_\delta(iy) - iy)}{y} \;=\; 0.$$

Put $I_\delta(z) \equiv \dfrac{1}{\delta}\displaystyle\int_0^\delta \Big(\int_{-\infty}^{+\infty} \frac{1 + x^2}{(x - z)^2} d\tau_t(x)\Big) dt$ and let $\delta > 0$ be sufficiently small arbitrary positive real number such that $\sup_{y\geq 1} |I_\delta(iy)| < \varepsilon$. This is possible since we have, for any $y \geq 1$,

$$|I_\delta(iy)| \;\leq\; \frac{1}{\delta}\int_0^\delta \Big(\int_{-\infty}^{+\infty} 1 \cdot d\tau_t(x)\Big) dt \;=\; \frac{1}{\delta}\int_0^\delta \tau_t(\mathbf{R}) dt \;\longrightarrow\; 0 \quad (\delta \to 0)$$

because of the inequality $|\frac{1+x^2}{(x-iy)^2}| \leq 1$ and the continuity of $t \mapsto \tau_t(\mathbf{R})$. Now the imaginary part of (4.5) with $z := iy$ is given by

$$\begin{aligned} \Im A(iy) \;&=\; \Im \frac{(H_\delta(iy) - iy)/\delta}{1 + I_\delta(iy)} \\ &=\; \frac{1}{\delta|1 + I_\delta(iy)|^2} \Big\{ \Im(H_\delta(iy) - iy) + \Im((H_\delta(iy) - iy) I_\delta(iy)) \Big\}. \end{aligned}$$

Since, in the last equality, $I_\delta(iy)$ can be made arbitrarily small ($\leq \varepsilon$), we get

$$\lim_{0<y\to\infty} \frac{\Im A(iy)}{y} \;=\; 0$$

and hence the desired result $\beta = 0$.

Step 4 (existence of the reciprocal $\frac{1}{A(z)}$). The existence of the reciprocal $\frac{1}{A(z)}$, $z \in \mathbf{C}^+$, follows from the expression (4.5) of $A(z)$ and Lemma 6.2.

Step 5 (uniqueness of w). Let us show that, for each $t \geq 0$ and each $z \in \mathbf{C}^+$, the complex number $w \in \mathbf{C}^+$ satisfying $F(w) - F(z) = t$ must be unique (if exists). From the integral representation

$$A(z) \;=\; \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x), \quad z \in \mathbf{C}^+,$$

we get

$$\Im\, A(z) \;=\; \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{|x-z|^2} d\gamma(x)\Big) \cdot \Im\, z.$$

For each fixed $z \in \mathbf{C}^+$, the integrand $\varphi(x) \equiv \frac{1+x^2}{|x-z|^2}$ is a bounded continuous function satisfying $\varphi(x) > 0$ for all $x \in \mathbf{R}$. Hence $A(z)$ is Pick function satisfying $\Im A(z) > 0$ for all $z \in \mathbf{C}^+$, whenever $\gamma(\mathbf{R}) \neq 0$. Note that $(\alpha, \gamma) \neq (0, 0)$ because of the non triviality of $\{H_t(z)\}_{t \geq 0}$.

So let us first consider the case of $\gamma(\mathbf{R}) \neq 0$. Let w_1 and w_2 be arbitrary complex numbers from the upper half plane $\mathbf{C}^+$ such that $w_1 \neq w_2$. Consider the segment $z = z(t) = w_1 + t(w_2 - w_1)$, $0 \leq t \leq 1$, then we have $dz = (w_2 - w_1)dt$, and hence

$$\int_{w_1}^{w_2} \frac{1}{A(z)} dz \;=\; \Big(\int_0^1 \frac{1}{A(z(t))} dt\Big) \cdot (w_2 - w_1).$$

Using the continuity of $A(z(t))$ and the strict positivity $\Im A(z) > 0$, we have

$$\Im\Big(\int_0^1 \frac{1}{A(z(t))} dt\Big) \;=\; -\int_0^1 \frac{\Im\, A(z(t))}{|A(z(t))|^2} dt \;<\; 0.$$

This means $\int_{w_1}^{w_2} \frac{1}{A(z)} dz \neq 0$. Therfore we have $F(w_1) \neq F(w_2)$ from Cauchy's integral theorem. Hence, for a given $z \in \mathbf{C}^+$ and a given $t \geq 0$, the complex number $w \in \mathbf{C}^+$ satisfying $F(w) = F(z) + t$ must be unique (if exists).

Next consider the case of $\gamma(\mathbf{R}) = 0$ and $\alpha \neq 0$. In this case, we have $A(z) = \alpha (\neq 0)$, and hence $\frac{1}{A(z)} = \frac{1}{\alpha}$. So we have

$$F(z) \;=\; \int_i^z \frac{dz}{A(z)} \;=\; \frac{1}{\alpha}(z - i),$$

from which the uniqness of $w \in \mathbf{C}^+$ satisfying $F(w) = F(z) + t$ follows.

Therefore, in any cases, the uniqueness of w holds.

Step 6 (existence of w). Let us show that, for any $t \geq 0$ and any $z_0 \in \mathbf{C}^+$, there exists a complex number $w \in \mathbf{C}^+$ such that $F(w) - F(z_0) = t$. Choose a real number $\beta > 0$ such that $\Im\, z_0 > \beta > 0$ and fix it. Put $\mathbf{C}^+_\beta = \{z \in \mathbf{C}^+ | \Im\, z > \beta\}$. From the expression

$$A(z) \;=\; \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x), \quad z \in \mathbf{C}^+,$$

we have

$$|A(z)| \;\leq\; |\alpha| + \int_{-\infty}^{+\infty} \frac{1}{|x-z|} d\gamma(x) + \Big(\int_{-\infty}^{+\infty} \Big|\frac{x}{x-z}\Big| d\gamma(x)\Big)|z|. \tag{4.6}$$

Put $\varphi(x) \equiv |\frac{x}{x-z}|$. Then we see

$$\sup_{x\in\mathbf{R}} \varphi(x) \;\leq\; 1 \qquad (\text{for all } z \in \mathbf{C}^+) \tag{4.7}$$

from differential calculus. From (4.6) and (4.7), we get an inequality in the domain $\mathbf{C}^+_\beta$:

$$\begin{aligned} |A(z)| &\leq |\alpha| + \frac{\gamma(\mathbf{R})}{\beta} + \gamma(\mathbf{R}) \cdot |z| \\ &= C\,|z| + D \qquad (z \in \mathbf{C}^+_\beta). \end{aligned}$$

Here the constants C and D are given by $C \equiv \gamma(\mathbf{R})$ and $D \equiv |\alpha| + \frac{\gamma(\mathbf{R})}{\beta}$, respectively. So, in the domain $\mathbf{C}^+_\beta$, $|A(z)|$ is domiated from above by a linear function of $|z|$.

From this inequality, we can show that the initial value problem of ordinary differential equation defined by

$$\frac{dz}{dt} \;=\; A(z), \qquad z(0) \;=\; z_0 \tag{4.8}$$

has a global solution $z(t)$, $0 \leq t < +\infty$, in the domain $\mathbf{C}^+_\beta$, as follows.

Let $z = z(t)$ $\;(0 \leq t < t^*)$ be the non-extendable solution of the initial value problem (4.8). Here $t^* \in \mathbf{R}^*_+ \cup \{+\infty\}$. Then we have

$$z(t) \;=\; z_0 + \int_0^t A(z(s))ds, \qquad 0 \leq t < t^*$$

from which we obtain

$$|z(t)| \;\leq\; |z_0| + \int_0^t (C|z(s)| + D)ds \;=\; g(t) + C\int_0^t |z(s)|ds.$$

Here $g(t)$ is defined by $g(t) \equiv |z_0| + D\,t$. From Gronwall's Lemma, we get

$$|z(t)| \;\leq\; g(t) + C\int_0^t e^{C(t-s)}\,g(s)\,ds. \tag{4.9}$$

Put $\varphi(t) \equiv g(t) + C\int_0^t e^{C(t-s)}\,g(s)\,ds$. The non triviality of the semigroup $\{H_t(z)\}_{t\geq 0}$ implies that either $\alpha \neq 0$ or $\gamma(\mathbf{R}) \neq 0$, and hence $D > 0$. Furthermore this implies $\lim_{t\to\infty} \varphi(t) = +\infty$, and hence the function $\varphi(t)$ is a strictly increasing homeomorphism from the interval $[0, +\infty)$ on to the interval $[|z_0|, +\infty)$.

Let $z(t)$ be any solution of (4.8), and let $K \subset \mathbf{C}^+_\beta$ be any compact set containing the point z_0. By the extension theorem for solution of ordinary differential equation [Arn], it is hold that $z(t)$ can be extended forward either indefinitely ($t^* = +\infty$) or up to the boundary ∂K of K. As a compact set K containing z_0, take the following set

$$K_R \;\equiv\; \{z \in \mathbf{C} \mid |z| \leq R \text{ and } \Im z \geq \beta_1\}.$$

Here β_1 is a real number such that $\Im z_0 > \beta_1 > \beta$, and $R > 0$ is a sufficiently large, arbitrary real number so that $R > |z_0|$.

If $z(t)$ can be extended forward up to the boundary ∂K of K, then there exists some $T \in [0, t^*)$ such that $z(T) \in \partial K$. This number T is strictly positive (> 0) because of $z_0 \notin \partial K$. Note that

$$\Im\, z(T) \;=\; \Im\, z_0 + \int_0^T \Im A(z(s))\,ds \;\geq\; \Im\, z_0$$

since $A(z)$ is a Pick function. Because of this inequality, the point $z(T)$ must belong to the semi-circle part of the figure ∂K. So we have $|z(T)| = R$.

Therefore the inequality (4.9) yields

$$R \;=\; |z(T)| \;\le\; \varphi(T).$$

Since $\varphi(t)$ is an isomorphism from the interval $[0, +\infty)$ onto the interval $[|z_0|, +\infty)$ in the sense of ordered sets, we have $T \ge \varphi^{-1}(R)$. Now the number T can be made arbitrarily large when R is prepared to be sufficiently large. This means that the non-extendable solution $z(t)$ is always a global solution ($t^* = +\infty$).

Along with the global solution $z(t)$ of (4.8), let us integrate the equality $\frac{1}{A(z(t))}\frac{dz(t)}{dt} = 1$ from 0 to t with dt, then we get

$$\int_0^t \frac{1}{A(z(t))}\frac{dz(t)}{dt}dt \;=\; \int_0^t dt \;=\; t, \quad 0 \le t < +\infty.$$

That is

$$\int_{z(0)}^{z(t)} \frac{dz}{A(z)} \;=\; t, \quad 0 \le t < +\infty.$$

This means that, for a given $z_0 \in \mathbf{C}^+$ and a given $t \ge 0$, there exists $w \in \mathbf{C}^+$ such that $F(w) = F(z_0) + t$.

Step 7 (representation of $H_t(z)$). For each $t \ge 0$, the function $z \mapsto F^{-1}(F(z) + t) : \mathbf{C}^+ \to \mathbf{C}^+$ is well-defined. Let z be fixed, then the derivative of the function $t \mapsto K_t(z) \equiv F^{-1}(F(z) + t)$ with variable t is given by

$$\frac{d}{dt}K_t(z) \;=\; \frac{1}{F'(F^{-1}(F(z)+t))}\frac{d}{dt}(F(z)+t) \;=\; A(F^{-1}(F(z)+t)).$$

Here $F'(z)$ denotes the derivative of the function $F(z)$. Hence the function $K_t(z)$ satisfies

$$\frac{d}{dt}K_t(z) \;=\; A(K_t(z)), \quad K_0(z) = z.$$

On the other hand, also the function $H_t(z)$ satisfies the same differential equation

$$\frac{d}{dt}H_t(z) \;=\; A(H_t(z)), \quad H_0(z) = z.$$

Therefore we obtain $H_t(z) = K_t(z)$ by the uniqueness of solution for initial value problem of ordinary differential equation. $\square$

Next, let us prove the converse to Theorem 4.5. We say that the function $A(z)$, $z \in \mathbf{C}^+$, is *trivial* if $A(z) \equiv 0$ for all $z \in \mathbf{C}^+$.

Theorem 4.6. *Let (α, γ) be a pair of a real number α and a finite positive measure γ on $\mathbf{R}$, and let $A(z)$ be a Pick function given by*

$$A(z) \;=\; \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z}d\gamma(x), \quad z \in \mathbf{C}^+.$$

Assume further that $A(z)$ is not trivial. Then the followings hold.

(1) The reciprocal $\dfrac{1}{A(z)}$ exists for all $z \in \mathbf{C}^+$.

(2) Put $F(z) = \int_i^z \frac{1}{A(z)} dz$*. Then, for each* $z \in \mathbf{C}^+$ *and each* $t \geq 0$*, there exists a unique* $w \in \mathbf{C}^+$ *such that* $F(w) - F(z) = t$*.*

(3) Put

$$K_t(z) \;=\; F^{-1}(F(z)+t), \quad t \geq 0, \;\; z \in \mathbf{C}^+.$$

Then $\{K_t(z)\}_{t \geq 0}$ *is a continuous one-parameter semigroup of reciprocal Cauchy transforms of probability measures on* $\mathbf{R}$*.*

(4) For each fixed $z \in \mathbf{C}^+$*, the map* $t \mapsto K_t(z)$ *is differentiable, and its right derivative at* $t = 0$ *coincides with the Pick function* $A(z)$ *given above.*

Proof. (1) and *(2)* can be shown by the same discussion as in the proof of Theorem 4.5.

(3) At first it is obvious that $\{K_t(z)\}_{t \geq 0}$ satisfies the semigroup property: $K_s(K_t(z)) = K_{s+t}(z)$, $K_0(z) = z$. Let us show that $K_t(z)$ is a Pick function. The range $Ran(F) \equiv \{F(z) \mid z \in \mathbf{C}^+\}$ of F is an open set. From the definition of $w = K_t(z)$:

$$w = K_t(z) \iff \text{unique } w \in \mathbf{C}^+ \; s.t. \; F(w) = F(z) + t,$$

we have the inclusion

$$\{F(z) + t \mid z \in \mathbf{C}^+, \; t \geq 0\} \;\subset\; Ran(F) = \{F(z) \mid z \in \mathbf{C}^+\}.$$

Since the converse inclusion "$\supset$" is obvious, we get the equality

$$\{F(z) + t \mid z \in \mathbf{C}^+, \; t \geq 0\} \;=\; Ran(F).$$

Since $F'(z) = \frac{1}{A(z)} \neq 0$, the inverse function $Ran(F) \ni \zeta \mapsto F^{-1}(\zeta) \in \mathbf{C}^+$ is holomorphic. Hence also the composed funcion $K_t(z) = F^{-1}(F(z) + t)$ is holomorphic. It is obvious that $K_t(\mathbf{C}^+) \subset \mathbf{C}^+$, and hence $K_t(z)$ is a Pick function. For each $z \in \mathbf{C}^+$, the map $t \mapsto K_t(z) = F^{-1}(F(z) + t)$ is continuous. Therefore $\{K_t(z)\}_{t \geq 0}$ is a continuous one-parameter semigroup consisting of of Pick functions.

Now the only thing we must do is to show that $K_t(z)$ is the reciprocal Cauchy transform of some probability measure μ_t on $\mathbf{R}$. For this purpose, we only have to show that

$$\inf_{z \in \mathbf{C}^+} \frac{\Im K_t(z)}{\Im z} \;=\; 1,$$

because of Theorem 3.4. It is easy to see that $K_t(z)$ satisfies the differential equation $\frac{d}{dt} K_t(z) = A(K_t(z))$, and hence

$$K_t(z) - z \;=\; \int_0^t A(K_t(z)) dt. \tag{4.10}$$

Taking the imaginary part of (4.10), we get

$$\Im K_t(z) \geq \Im z \tag{4.11}$$

because $A(z)$ is a Pick function. Take the imaginary part of (4.10) again, but in this time, after the replacement of $A(\zeta)$ in (4.10) with its integral representation. Then we get

$$\Im K_t(z) - \Im z \;=\; \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{1 + x^2}{|x - K_t(z)|^2} d\gamma(x) \Big) dt. \tag{4.12}$$

By the way, the inner integrand in this equality is dominated from above as

$$\frac{1+x^2}{|x-K_t(z)|^2} \;=\; \frac{1+x^2}{(x-\Re K_t(z))^2+(\Im K_t(z))^2} \;\leq\; \frac{1+x^2}{(\Im z)^2}$$

because of $\Im K_t(z) \geq \Im z > 0$. Hence we get the pointwise convergence :

$$\lim_{0<y\to\infty} \frac{1+x^2}{|x-K_t(iy)|^2} \;=\; 0 \qquad (\forall x \in \mathbf{R},\ \forall t \geq 0).$$

with y a real variable. Let us rewrite (4.12) with $z := iy$, as

$$\Im K_t(iy) - y \;=\; \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{d\gamma(x)}{|x-K_t(iy)|^2} \Big) dt + \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{x^2}{|x-K_t(iy)|^2} d\gamma(x) \Big) dt. \tag{4.13}$$

Let $y_n > 0$ be a sequence such that $\lim_{n\to\infty} y_n = +\infty$. Then the second term of the r.h.s. of (4.13) vanishes in the limit $n \to \infty$:

$$\lim_{n\to\infty} \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{x^2}{|x-K_t(iy_n)|^2} d\gamma(x) \Big) dt \;=\; 0$$

because of the dominated convergence theorem and an inequality

$$\frac{x^2}{|x-K_t(z)|^2} \;\leq\; 1 \qquad (\forall x \in \mathbf{R},\ \forall t \geq 0,\ \forall z \in \mathbf{C}^+).$$

Here we applied the dominated convergence theorem to the double integral w.r.t. the product measure $d\gamma(x) \otimes dt$ on $\mathbf{R} \times [0,t]$. After all we get

$$\begin{aligned}
0 \;&\leq\; \Im K_t(iy_n) - y_n \\
&=\; \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{d\gamma(x)}{|x-K_t(iy_n)|^2} \Big) dt + \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{x^2}{|x-K_t(iy_n)|^2} d\gamma(x) \Big) dt \\
&\leq\; \frac{\gamma(\mathbf{R}) \cdot t}{|y_n|^2} + \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{x^2}{|x-K_t(iy_n)|^2} d\gamma(x) \Big) dt \\
&\longrightarrow\; 0 \qquad (n \to +\infty).
\end{aligned}$$

That is

$$\lim_{n\to\infty} \frac{\Im K_t(iy_n)}{y_n} \;=\; 1.$$

With (4.11), we get the desired result

$$\inf_{z\in\mathbf{C}^+} \frac{\Im K_t(z)}{\Im z} \;=\; 1.$$

(4) It is easily checked. □

From Theorem 4.5 and Theorem 4.6, we see that there is the natural bijective correspondence between the set $\mathbf{H}$ of all continuous one-parameter semigroups $\{H_t(z)\}_{t\geq 0}$ of reciprocal Cauchy transforms of probability measures on $\mathbf{R}$ and the set $\mathbf{L}$ of all pairs (α, γ) consisting of real numbers α and finite positive measures γ on $\mathbf{R}$:

$$\mathbf{H} \;\ni\; \{H_t(z)\}_{t\geq 0} \;\longleftrightarrow\; (\alpha,\gamma) \;\in\; \mathbf{L}$$

We summarize this conclusion in the following Theorem. It is a monotonic analogue of Lévy-Hinčin formula in terms of continuous one-parameter semigroups. Note that, for any weak* continuous one-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures, it is hold that either (1) $\mu_t \neq \delta_0$ for all $t>0$, or, (2) $\mu_t = \delta_0$ for all $t \geq 0$. The case (2) occurs if and only if $(\alpha, \gamma) = (0,0)$.

Theorem 4.7 (monotonic Lévy-Hinčin formula in terms of semigroups). *Let $\{\mu_t\}_{t\geq 0}$ be a one-parameter familiy of probability measures on $\mathbf{R}$. Assume that $\mu_t \neq \delta_0$ for all $t>0$. Then the following two conditions are equivalent.*

(1) $\{\mu_t\}_{t\geq 0}$ is a weak continuous one-parameter $\triangleright$-semigroup.*
(2) There exists a pair (α,γ) $(\neq (0,0))$ of a real number α and a finite positive measure γ on $\mathbf{R}$ such that the reciprocal Cauchy transform $H_t(z)$ of μ_t is given by

$$w \;=\; H_t(z) \quad \Longleftrightarrow \quad {}^{\exists 1}w \in \mathbf{C}^+ \;\; s.t. \;\; \int_z^w \frac{dz}{A(z)} \;=\; t,$$

where the function $A(z)$ is defined by

$$A(z) \;=\; \alpha + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\gamma(x). \tag{4.14}$$

If the above conditions hold, (α,γ) and $A(z)$ are unique.

The Pick function $A(z)$ given in Theorem 4.7 is called the *generator* of the semigroup $\{H_t(z)\}_{t\geq 0}$. The pair (α,γ) is called the *Lévy measure* for short although it is not a measure but a pair of a number and a measure. For each semigroup $\{H_t(z)\}_{t\geq 0}$ in Example 4.4, let us give its generator $A(z)$ and the Lévy measure (α,γ) in the standard form (4.14).

Example 4.8. (1) Point measure : $A(z) = -a$, $(\alpha,\gamma) = (-a, 0)$.

(2) Arcsine distribution : $A(z) = -\dfrac{1}{z}$, $(\alpha,\gamma) = (0, \delta_0)$.

(3) Monotonic Poisson distribution: $A(z) = \dfrac{\lambda z}{1-z}$, $(\alpha,\gamma) = \Big(-\dfrac{\lambda}{2}, \dfrac{\lambda}{2}\delta_1\Big)$.

(4) A deformation of arcsine distribution:

$$A(z) = -\frac{1}{z-c}, \quad (\alpha,\gamma) = \Big(\frac{c}{1+c^2}, \frac{\delta_c}{1+c^2}\Big).$$

(5) Cauchy distribution:

$$A(z) = ib = \frac{b}{\pi}\int_{-\infty}^{+\infty} \frac{1+xz}{x-z}\frac{dx}{1+x^2}, \quad \alpha = 0, \quad d\gamma(x) = \frac{b}{\pi}\frac{dx}{1+x^2}.$$

Finally, let us give another standard form (different from (4.14)) for the generator $A(z)$ of the semigroup $\{H_t(z)\}_{t\geq 0}$. This standard form which we give below can be defined for the class of weak* continuous one-parameter $\triangleright$-semigroups $\{\mu_t\}_{t\geq 0}$ such that each μ_t is of finite variance. Denote by $\mathcal{P}$ the set of all probability measures μ on $\mathbf{R}$, by $\mathcal{P}^2$ the set of all μ $(\in \mathcal{P})$ with finite variance, and by $\mathcal{P}_0^2$ the set of all μ $(\in \mathcal{P})$ with finite variance and zero mean. Note that, for a weak* continuous one-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}$, the existence of $t_0 > 0$ such that $\mu_{t_0} \in \mathcal{P}^2$ implies that $\mu_t \in \mathcal{P}^2$ for all $t \geq 0$, because of Lemma 6.3.

Theorem 4.9. *Let $\{\mu_t\}_{t\geq 0}$ be a weak* continuous one parameter $\triangleright$-semigroup of probability measures on $\mathbf{R}$, and let $\{H_t(z)\}_{t\geq 0}$ its associated reciprocal Cauchy transforms. Then the following two conditions are equivalent.*

(1) $\mu_t \in \mathcal{P}^2$ *for all* $t \geq 0$.
(2) There exists a pair (a, τ) *of a real number* a *and a finite positive measure* τ *on* $\mathbf{R}$ *such that the generator* $A(z)$ *of* $\{H_t(z)\}_{t\geq 0}$ *has a representation*

$$A(z) \;=\; a + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\tau(x), \quad z \in \mathbf{C}^+. \tag{4.15}$$

If the above conditions hold, then the pair (a, τ) *is unique.*

Proof. Step 1 ((1)⇒(2)). Using the characterization theorem for reciprocal Cauchy transforms of the class $\mathcal{P}_0^2$ (Proposition 2.2 in [Maa]), we get

$$H_t(z) \;=\; a_t + z + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\rho_t(x), \quad z \in \mathbf{C}^+,$$

for some real number a_t and some finite positive measure ρ_t on $\mathbf{R}$. Here we have $-a_t = m(\mu_t)$ (= the mean of μ_t) and $\rho_t(\mathbf{R}) = \sigma^2(\mu_t)$ (= the variance of μ_t). Hence the generator $A(z)$ is given by

$$A(z) \;=\; \lim_{n\to\infty} n(H_{\frac{1}{n}}(z) - z) \;=\; \lim_{n\to\infty}\Big(n\, a_{\frac{1}{n}} + \int_{-\infty}^{+\infty} \frac{1}{x-z} dn\rho_{\frac{1}{n}}(x)\Big).$$

Again by the Proposition 2.2 in [Maa], there exists some probability measure $\nu_n \in \mathcal{P}_0^2$ such that

$$H_{\nu_n}(z) \;=\; z + \int_{-\infty}^{+\infty} \frac{1}{x-z} dn\rho_{\frac{1}{n}}(x)$$

and it satisfies

$$|H_{\nu_n}(z) - z| \;\leq\; \frac{\sigma^2(\nu_n)}{\Im z}. \tag{4.16}$$

By the way, from Lemma 6.3, we have

$$\begin{aligned} n\, a_{\frac{1}{n}} \;&=\; -n\, m(\mu_{\frac{1}{n}}) \;=\; -m(\mu_1), \\ \sigma^2(\nu_n) \;&=\; n\, \rho_{\frac{1}{n}}(\mathbf{R}) \;=\; n\, \sigma^2(\mu_{\frac{1}{n}}) \;=\; \sigma^2(\mu_1). \end{aligned}$$

Therefore, taking the limit of (4.16) with $n \to \infty$, we get

$$|A(z) - a_1| \;\leq\; \frac{\sigma^2(\mu_1)}{\Im z}.$$

Applying Proposition 2.2 in [Maa] again to the Pick function $A(z) - a_1 + z$, we obtain the existence of probability measure $\nu \in \mathcal{P}_0^2$ such that

$$A(z) - a_1 + z \;=\; H_\nu(z) \;=\; z + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\tau(x)$$

with some finite positive measure τ on $\mathbf{R}$. This means that

$$A(z) \;=\; a_1 + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\tau(x).$$

Step 2 ((2)⇒(1)). Since $H_t(z)$ satisfies the differential equation $\frac{d}{dt} H_t(z) = A(H_t(z))$ with $H_0(z) = z$, as shown in the proof of Theorem 4.5, we have

$$H_t(z) - z \;=\; \int_0^t A(H_s(z))ds \;=\; \int_0^t \Big(a + \int_{-\infty}^{+\infty} \frac{1}{x - H_s(z)} d\tau(x)\Big) ds$$

from which we get

$$|H_t(z) - a\,t - z| \;\leq\; \int_0^t \Big(\int_{-\infty}^{+\infty} \frac{1}{|x - H_s(z)|} d\tau(x) \Big) ds \;\leq\; \frac{\tau(\mathbf{R})\,t}{\Im z}$$

for all $z \in \mathbf{C}^+$. This implies that $K_t(z) - at$ determines a probability measure in $\mathcal{P}_0^2$ because of Proposition 2.2 in [Maa], and hence $H_t(z)$ determines a probability measure in $\mathcal{P}^2$. □

This standard form (4.15) for $A(z)$ given in Theorem 4.9 will be used in the next section.

5. Infinitely divisible distributions with compact supports

In this section, we give a monotonic Lévy-Hinčin formula in terms of infinitely divisible distributions, but our treatment is restricted to the class of compactly supported probability measures.

Let $\mathcal{P}_c$ be the set of all probability meassures on $\mathbf{R}$ which are compactly supported. We will show

Theorem 5.1 (monotonic Lévy-Hinčin formula for class $\mathcal{P}_c$). *Let μ be a probability measure on $\mathbf{R}$. Assume that $\mu \neq \delta_0$. Then the following three conditions are equivalent.*

(1) μ is $\triangleright$-infinitely divisible and $\mu \in \mathcal{P}_c$.
(2) There exists a weak continuous one-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on $\mathbf{R}$ such that $\mu_1 = \mu$ and $\mu \in \mathcal{P}_c$.*
(3) There exists a pair (a, ρ) $(\neq (0,0))$ of a real number a and a compactly supported finite positive measure ρ on $\mathbf{R}$ such that the Pick function

$$A(z) \;=\; a + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho(x) \tag{5.1}$$

generates $H_\mu(z)$ as

$$w \;=\; H_\mu(z) \quad\Longleftrightarrow\quad {}^{\exists 1}w \in \mathbf{C}^+ \;\; s.t. \; \int_z^w \frac{dz}{A(z)} \;=\; 1.$$

If the above conditions hold, then $\{\mu_t\}_{t\geq 0}$, (a, ρ) and $A(z)$ are unique, and $\mu_t \in \mathcal{P}_c$ for all $t \geq 0$.

For the proof of Theorem 5.1, let us prepare some Lemmas below. A family $\{\mu_r | r \in \mathbf{Q}_+\}$ of probability measures on $\mathbf{R}$ is said to be a *$\mathbf{Q}_+$-parameter $\triangleright$-semigroup* if it satisfies (1) $\mu_0 = \delta_0$; (2) $\mu_{r+s} = \mu_r \triangleright \mu_s$. It naturally corresponds to a $\mathbf{Q}_+$-*parameter semigroup* $\{H_r(z) | r \in \mathbf{Q}_+\}$ of reciprocal Cauchy transforms. Here we made no continuity assumption. They are algebraic objects. Throughout the remainder of this section, we use the term *$\mathbf{R}_+$-parameter $\triangleright$-semigroup* to mean a real one-parameter semigroup $\{\mu_t\}_{t\geq 0}$.

Lemma 5.2 (extension). *Let $\{\mu_r | r \in \mathbf{Q}_+\}$ be a $\mathbf{Q}_+$-parameter $\triangleright$-semigroup of probability measures on $\mathbf{R}$ such that $\lim_{0<r\to 0} \mu_r = \delta_0$ in the weak* topology. Put $H_r(z) = H_{\mu_r}(z)$. Then*

(1) $\lim_{0<r\to 0} H_r(z) = z$, in the compact uniform topology.

(2) The family $\{\mu_r |\; r \in \mathbf{Q}_+\}$ can be uniquely extended to a weak continuous $\mathbf{R}_+$-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures.*

Proof. (1) At first we get the pointwise convergence $H_r(z) \to z$ ($\mathbf{Q}_+^* \ni r \to 0$) from the weak* convergence $\mu_r \to \delta_0$ ($\mathbf{Q}_+^* \ni r \to 0$). The integral representation of $H_r(z)$ is given by

$$H_r(z) \;=\; a_r + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau_r(x), \qquad z \in \mathbf{C}^+$$

with some $a_t \in \mathbf{R}$ and some τ_r. Here $a_r = \Re H_r(i)$ and $\tau_r(\mathbf{R}) = \Im H_r(i) - 1$. We have

$$|H_r(z) - z| \;\leq\; |a_r| + \int_{-\infty}^{+\infty} \left|\frac{1+xz}{x-z}\right| d\tau_r(x).$$

For each compact subset $K \subset \mathbf{C}^+$ ($K \neq \emptyset$), the function $(x,z) \mapsto |\frac{1+xz}{x-z}|$ is bounded on $\mathbf{R} \times K$ by an upper bound $M > 0$, and hence we have

$$\sup_{z \in K} |H_r(z) - z| \;\leq\; |a_r| + M\,\tau_r(\mathbf{R}).$$

By the way, since $a_r = \Re H_r(i)$ and $\tau_r(\mathbf{R}) = \Im H_r(i) - 1$, the convergence $\lim_{0<r\to 0} H_r(z) = z$ yields $\lim_{0<r\to 0} a_r = 0$ and $\lim_{0<r\to 0} \tau_r(\mathbf{R}) = 0$. Hence we get the compact unform convergence: $\sup_{z\in K} |H_r(z) - z| \to 0$ ($0 < r \to 0$) for each compact subset K.

(2) Let $\delta > 0$ be a positive number, and let $r, s \in \mathbf{Q}_+^*$ be positive rational numbers such that $|r - s| < \delta$ and $r > s > 0$. Put $r = s + h$, then we have

$$\begin{aligned} H_r(z) - H_s(z) &= H_s(H_h(z)) - H_s(z) \\ &= (H_h(z) - z)\Big\{1 + \int_{-\infty}^{+\infty} \frac{d\rho_s(x)}{(x - H_h(z))(x-z)}\Big\} \end{aligned}$$

from which we get

$$|H_r(z) - H_s(z)| \;\leq\; |H_h(z) - z|\Big\{1 + \int_{-\infty}^{+\infty} \left|\frac{1}{(x - H_h(z))(x-z)}\right| d\rho_s(x)\Big\}. \tag{5.2}$$

Since (1), for each compact subset $K \subset \mathbf{C}^+$ and any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that, for each δ with $0 < \delta \leq \delta_0$, the r.h.s. of (5.2) is uniformly dominated on K by ε. So we have

$$\sup_{z\in K} |H_r(z) - H_s(z)| < \varepsilon \tag{5.3}$$

for each $r, s \in \mathbf{Q}_+^*$ with $|r - s| < \delta$. Let t_0 be an arbitrary fixed real number. The inequality (5.3) implies that, for any sequence $\{r_n\}$ ($\subset \mathbf{Q}_+$) converging to t_0, the restriction $H_{r_n}^{(K)}(z)$ of $H_{r_n}(z)$ to K is a Cauchy sequence w.r.t. the uniform norm over K, and hence it converges uniformly to a function $H_{t_0}^{(K)}(z)$ which is not depend on the choice of approximate sequence $\{r_n\}$ ($\subset \mathbf{Q}_+^*$). The system $\{H_{t_0}^{(K)}(z) \mid K : \text{compact}\}$ uniquely defines a holomorphic function $H_{t_0}(z)$ on $\mathbf{C}^+$. Besides it is easy to see that the map $\mathbf{R}_+ \ni t \mapsto H_t(z)$ is continuous for each fixed z and that $H_{t_0}(z)$ is a Pick function with $\Im H_{t_0}(z) \geq \Im z$.

Let us show that $H_{t_0}(z)$ is the reciprocal Cauchy transform of some probability measure. Let $\{r_n\} \subset \mathbf{Q}_+$ be a decreasing approximate sequence to t_0 ($r_n \downarrow t_0$), then we have

$$\Im H_{r_n}(z) \;=\; \Im H_{r_n - r_{n+1}}(H_{r_{n+1}}(z)) \;\geq\; \Im H_{r_{n+1}}(z)$$

from which we get $\Im H_{t_0}(z) \;=\; \lim_{n\to\infty} \Im H_{r_n}(z) \;\leq\; \Im H_{r_1}(z)$. So we have

$$1 \;\leq\; \inf_{z\in\mathbf{C}^+} \frac{\Im H_{t_0}(z)}{\Im z} \;\leq\; \inf_{z\in\mathbf{C}^+} \frac{\Im H_{r_1}(z)}{\Im z} \;=\; 1.$$

Here the last equality comes from Theorem 3.4. After all we have $\inf_{z\in\mathbf{C}^+} \frac{\Im H_{t_0}(z)}{\Im z} = 1$. Therefore $H_{t_0}(z)$ must be the reciprocal Cauchy transform of some probability measure because of Theorem 3.4 again.

Let us show the semigroup property: $H_s(H_t(z)) = H_{s+t}(z)$. Let s, t be non negative real numbers, and let $\{s_n\}$, $\{t_n\} \subset \mathbf{Q}_+$ be approximate sequences to s, t, respectively. Let $z \in \mathbf{C}^+$ be fixed. By the continuity of the $w \mapsto H_s(w)$, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|H_s(w) - H_s(H_t(z))| \;<\; \varepsilon \qquad (\, w \in B_\delta(H_t(z))\,).$$

Here $B_\delta(H_t(z))$ denotes the δ-disk of center $H_t(z)$. Since $\{t_n\}$ is an approximate sequence to t, there exists $n_0 = n_0(\delta)$ such that $H_{t_n}(z) \in B_\delta(H_t(z))$ for all $n \geq n_0$, and hence

$$|H_s(H_{t_n}(z)) - H_s(H_t(z))| \;<\; \varepsilon \qquad (n \geq n_0).$$

By the way, the compact uniform convergence $\lim_{n\to\infty} H_{s_n}(z) = H_s(z)$ yields the existence of $n_1 = n_1(\delta)$ such that

$$|H_{s_n}(w) - H_s(w)| \;<\; \varepsilon.$$

for all $w \in \overline{B_\delta(H_t(z))}$ and all $n \geq n_1$. After all we have

$$\begin{aligned} &|H_{s_n}(H_{t_n}(z)) - H_s(H_t(z))| \\ &\qquad \leq\; |H_{s_n}(H_{t_n}(z)) - H_s(H_{t_n}(z))| + |H_s(H_{t_n}(z)) - H_s(H_t(z))| \;\leq\; 2\,\varepsilon. \end{aligned}$$

for all $n \geq \max\{n_0, n_1\}$. This implies $\lim_{n\to\infty} H_{s_n}(H_{t_n}(z)) = H_s(H_t(z))$. Therefore we get the semigroup property as

$$H_{s+t}(z) \;=\; \lim_{n\to\infty} H_{s_n+t_n}(z) \;=\; \lim_{n\to\infty} H_{s_n}(H_{t_n}(z)) \;=\; H_s(H_t(z)). \qquad \square$$

Corollary 5.3. *Let $\{\mu_r | r \in \mathbf{Q}_+\}$ be a $\mathbf{Q}_+$-parameter $\triangleright$-semigroup of probability measures such that $\mu_t \in \mathcal{P}^2$ for all $t \geq 0$. Then $\{\mu_r |\; r \in \mathbf{Q}_+\}$ can be uniquely extended to a weak* continuous $\mathbf{R}_+$-parameter $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures.*

Proof. Let $H_r(z)$ be the reciprocal Cauchy transform of μ_r. Since $\mu_r \in \mathcal{P}^2$, there exists a real number $a_r \in \mathbf{R}$ and a finite positive mesure ρ_r on $\mathbf{R}$ such that

$$H_r(z) \;=\; a_r + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho_r(x).$$

Here $a_r = -m(\mu_r)$ and $\rho_r(\mathbf{R}) = \sigma^2(\mu_r)$ ([Maa] Prop. 2.2). By the semigroup property of $\{H_r(z)|\; r \in \mathbf{Q}_+\}$ and Lemma 6.3, we have $m(\mu_r) = r\, m(\mu_1)$ and $\sigma^2(\mu_r) = r\,\sigma^2(\mu_1)$ for all $r \in \mathbf{Q}_+$. Hence we get

$$\begin{aligned} |H_r(z) - z| \;&\leq\; |a_r| + \int_{-\infty}^{+\infty} \left|\frac{1}{x - z}\right| d\rho_r(x) \\ &\leq\; |m(\mu_r)| + \frac{\rho_r(\mathbf{R})}{\Im z} \\ &=\; \left| m(\mu_1) + \frac{\sigma^2(\mu_1)}{\Im z} \right| \cdot r \;\longrightarrow\; 0 \quad (r \to 0). \end{aligned}$$

for all $r \in \mathbf{Q}_+$ and all $z \in \mathbf{C}^+$. This means the weak* convergence $\mu_r \to \delta_0$ $(\mathbf{Q}_+^* \ni r \to 0)$. So we get the desired result because of Lemma 5.2. $\square$

Let $\mu, \nu \in \mathcal{P}$. We say that ν is an *nth root* of μ if $\mu = \overbrace{\nu \rhd \nu \rhd \cdots \rhd \nu}^{n}$.

Proposition 5.4. *Let μ be a $\rhd$-infinitely divisible probability measure on $\mathbf{R}$ which is compactly supported. Then*

(1) The nth root of μ is unique. (We denote it by $\mu_{\frac{1}{n}}$.)

(2) The family $\{\mu_r \mid r = \frac{1}{n},\ n \in \mathbf{N}^*\}$ can be uniquely extended to a weak continuous $\mathbf{R}_+$-parameter $\rhd$-semigroup $\{\mu_t\}_{t \geq 0}$ of probability measures.*

(3) $\mu_t \in \mathcal{P}_c$ for all $t \in \mathbf{R}_+$.

Proof. *(1)* Let n be fixed, and let $\nu \in \mathcal{P}$ be an n th root of μ. From Lemma 6.4, ν must be compactly supported. From Corollary 2.2, there exist monotonically independent self-adjoint random variables $X_1, X_2, \cdots, X_n \in B(H)$ on some Hilbert space with unit vector (H, ξ) such that the probability distribution μ_{X_i} of X_i coincides with ν for each $i = 1, 2, \cdots, n$. Put $X = X_1 + X_2 + \cdots + X_n$. Then we get

$$H_X(z) \;=\; H_{X_1}(H_{X_2}(\cdots(H_{X_n}(z))\cdots))$$

from Theorem 3.1. That is, the probability distribution μ_X of X under $\langle \xi | \cdot \xi \rangle$ is given by the monotonic convolution $\mu_X = \nu \rhd \nu \rhd \cdots \rhd \nu$. Hence $\mu_X = \mu$. Therefore, for the proof of the uniqueness of n th root of μ, we have only to show that the measure ν can be uniquely dertermined from the measure μ_X. Let m_p (resp. λ_p) be the pth moment of X (resp. X_i). Then, from the formula (3.1) and the recurrence relations for $V(g)$ given in the proof of Theorem 3.1, we see that λ_p is a polynomial in p variables $m_1, m_2, \cdots, m_p$. Hence ν is uniquely determined by μ_X.

(2) By the uniqueness of nth root $\mu_{\frac{1}{n}}$ of μ, the family $\{\mu_r | \ r = \frac{1}{n},\ n \in \mathbf{N}^*\}$ can be uniquely extended to a $\mathbf{Q}_+$-parameter $\rhd$-semigroup $\{\mu_r | \ r \in \mathbf{Q}_+\}$ by $\mu_r \equiv \mu_{\frac{1}{q}} \rhd \mu_{\frac{1}{q}} \rhd \cdots \rhd \mu_{\frac{1}{q}}$ (convolution of p $\mu_{\frac{1}{q}}$'s) for $r = \frac{p}{q}$, $p, q \in \mathbf{N}^*$. Since $\mu \in \mathcal{P}^2$, the $\mathbf{Q}_+$-parameter $\rhd$-semigroup $\{\mu_r | \ r \in \mathbf{Q}_+\}$ can be uniquely extended to a weak* continuous $\mathbf{R}_+$-parameter $\rhd$-semigroup $\{\mu_t\}_{t \geq 0}$ in virtue of Corollary 5.3.

(3) By Lemma 6.4, for each $n \in \mathbf{N}^*$, $\mu_n = \mu_1 \rhd \mu_1 \rhd \cdots \rhd \mu_1$ (n μ_1's) is compactly supported. Again by Lemma 6.4, for any $t \in \mathbf{R}_+$ with $0 < t < n$, the probability measure μ_t is compactly supported because of $\mu_n = \mu_{1-t} \rhd \mu_t$. $\square$

Now we can give the proof of the main theorem in this section.

Proof of Theorem 5.1. Step 1 ((1)⇔(2)). It is clear from Proposition 5.4.

Step 2 ((2)⇒(3)). Note that $\mu_t \in \mathcal{P}_c$ for all $t \in \mathbf{R}_+$ by the same reason as in the proof of Proposition 5.4 (3). Let $H_t(z)$ be the reciprocal Cauchy transform of μ_t. In virtue of Theorem 4.7 and Theorem 4.9, we have only to show that, in the representation with the standard form (4.15)

$$A(z) \;=\; a + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\tau(x)$$

of the generator $A(z)$ of $\{H_t(z)\}_{t \geq 0}$, the measure τ is compactly supported.

Since $\mu_t \in \mathcal{P}_c \subset \mathcal{P}^2$, $H_t(z)$ has the representation

$$H_t(z) \;=\; a_t + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho_t(x)$$

with $a_t \in \mathbf{R}$ and ρ_t a finite positive measure on $\mathbf{R}$ from Prop. 2.2 in [Maa]. Since μ_t is compactly supported, also ρ_t is compactly supported from Lemma 6.5. Let $\mu_{\frac{1}{n},y}$ be the probability measure defined by $H_{\mu_{\frac{1}{n},y}} = H_{\frac{1}{n}}(z) - y$, then we have

$$\begin{aligned} H_1(z) &= H_{\frac{n-1}{n}}(H_{\frac{1}{n}}(z)) \\ &= a_{\frac{n-1}{n}} + H_{\frac{1}{n}}(z) + \int_{-\infty}^{+\infty} \frac{1}{y - H_{\frac{1}{n}}(z)} d\rho_{\frac{n-1}{n}}(y) \\ &= a_{\frac{n-1}{n}} + \Big(a_{\frac{1}{n}} + z + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\rho_{\frac{1}{n}}(x)\Big) - \int_{-\infty}^{+\infty} \frac{1}{H_{\mu_{\frac{1}{n},y}}(z)} d\rho_{\frac{n-1}{n}}(y) \\ &= (a_{\frac{n-1}{n}} + a_{\frac{1}{n}}) + z + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\Big(\rho_{\frac{1}{n}}(x) + \int_{-\infty}^{+\infty} \mu_{\frac{1}{n},y}(x) d\rho_{\frac{n-1}{n}}(y)\Big) \end{aligned}$$

from which we obtain

$$\rho_1(\cdot) = \rho_{\frac{1}{n}}(\cdot) + \int_{-\infty}^{+\infty} \mu_{\frac{1}{n},y}(\cdot) d\rho_{\frac{n-1}{n}}(y).$$

This implies that the compact support of $\rho_{\frac{1}{n}}$ is contained in the compact support K of ρ_1 for all $n \in \mathbf{N}^*$. Since $\int_{-\infty}^{+\infty} \frac{1}{x-z} dn\rho_{\frac{1}{n}}(x) \to \int_{-\infty}^{+\infty} \frac{1}{x-z} d\tau(x)$, τ is also supported in K.

Step 3 ((3)⇒(2)). By Theorem 4.6, the Pick function $A(z)$ generates a semigroup $\{H_t(z)\}_{t\geq 0}$ of reciprocal Cauchy transforms of some probability measures μ_t. We must show that $\mu_1(=\mu) \in \mathcal{P}_c$ and furthermore that $\mu_t \in \mathcal{P}_c$ for all $t \geq 0$. We can assume that $\rho(\mathbf{R}) \neq 0$ because $\rho(\mathbf{R}) = 0$ imples $\mu_t = \delta_{a\,t}$ for some $a \in \mathbf{R}$. Since ρ is supported in some compact interval $[a, b]$, the reciprocal $\frac{1}{A(z)}$ exists in the domain $\mathbf{C} \setminus [-M, M]$ for sufficiently large any $M > 0$ because of

$$\Im A(z) = \Big(\int_{-\infty}^{+\infty} \frac{d\rho(\xi)}{|\xi - z|^2}\Big) \cdot \Im z \neq 0, \qquad z \in \mathbf{C}^+ \cup \mathbf{C}^-$$

and $A(x) \neq 0$ for $x \in \mathbf{R} \setminus [-M, M]$. Consider the initial value problem of ordinary differential equation:

$$\frac{dz(t)}{dt} = A(z(t)), \qquad z(0) = z_0 \tag{5.4}$$

If $z_0 \in \mathbf{C}^+$ then (5.4) has the global solution by the same reason as that in the step 6 of Proof of Theoreom 4.5. In the same way, if $z_0 \in \mathbf{C}^-$, then (5.4) also has the global solution. So let us consider the case that z_0 is real ($z_0 = x_0 \in \mathbf{R} \setminus [-M, M]$). Consider the ordinary differential equation in the real domain:

$$\frac{dx(t)}{dt} = A(x(t)) \qquad x(0) = x_0 \tag{5.5}$$

Fix a constatn $c > 0$, and let M_1 ($\geq M$) be such that $-M_1 < a - c$ and $a + c < M_1$. Then, for any $x, y \in \Omega \equiv \mathbf{R} \setminus [-M_1, M_1]$, we have

$$|A(x) - A(y)| \leq \Big(\int_{-\infty}^{+\infty} \frac{d\rho(\xi)}{|\xi - x||\xi - y|}\Big) \cdot |x - y| \leq \frac{\rho(\mathbf{R})}{c^2} \cdot |x - y|,$$

that is, $A(z)$ satisfies a Lipschitz condition in Ω with Lipschitz constant $L \equiv \rho(\mathbf{R})/c^2 \neq 0$. Hence, given $r > 0$, the solution $x(t)$ of (5.5) uniquely exists in the interval $\{t \in \mathbf{R} |\, |t| \leq R\}$ whenever the interval $\{x \in \mathbf{R} |\, |x - x_0| \leq r\}$ is included in Ω. Here R is given by

$$R \;\equiv\; \frac{1}{L}\log\Big(1+\frac{L\,r}{N}\Big), \qquad N \;\equiv\; \sup_{x\in\Omega}|A(x)|.$$

By the way, r can be made arbitrarily small when $|x_0|$ is prepared to be sufficiently large. This means that, for each $T > 0$, there exists M_T $(\geq M_1)$ such that, for any initial value $x_0 \in \mathbf{R} \setminus [-M_T, M_T]$, the solution $x(t)$ of (5.5) uniquely exists in $0 \leq t \leq T$. After all, the solution $z(t)$ of (5.4) uniquely exists in $0 \leq t \leq T$ for every initial value $z_0 \in \mathbf{C} \setminus [-M_T, M_T]$. We denote it by $z(t) = \varphi_t(z_0)$, $0 \leq t \leq T$. Note that $\varphi_t(z) = H_t(z)$ for all $z \in \mathbf{C}^+$. By the way, the theorem on continuous dependence [Arn] yields

$$\lim_{0<y\to 0}\Im H_t(x+iy) \;=\; \lim_{0<y\to 0}\Im\varphi_t(x+iy) \;=\; \Im\varphi_t(x) \;=\; 0$$

for $x \in \mathbf{R} \setminus [-M_T, M_T]$ and $0 < t < T$. Hence we get

$$\lim_{0<y\to 0}\Im G_t(x+iy) \;=\; -\lim_{0<y\to 0}\frac{\Im H_t(x+iy)}{|H_t(x+iy)|^2} \;=\; 0.$$

Let $[c, d] \subset \mathbf{R}\setminus[-M_T, M_T]$ be an arbitrary compact interval, then $\Im G_t(x+iy)$ is uniformly continuous on the compact rectangle $K = \{z \in \mathbf{C} |\; c \leq x \leq d;\ -1 \leq y \leq 1\}$, and hence,

$$\lim_{0<y\to 0}\int_c^d \Im G_t(x+iy)\,dx \;=\; 0.$$

This means $\mu_t(\mathbf{R} \setminus [-M_T, M_T]) = 0$ because of the Stieltjes inversion formula for Cauchy transforms. Therefore $\mu_t \in \mathcal{P}_c$. $\square$

Remark. In the statement of Theorem 5.1, we made the assumption of compact support. This assumption was made to assure the unique existence of nth root $\mu_{\frac{1}{n}}$ for a $\rhd$-infinitely divisible distribution μ. Up to now, we do not know if this assumption can be removed or not when, in Theorem 5.1, we replace the standard form (5.1) for $A(z)$ with another standard form (4.14) given in Theorem 4.7.

Finally, let us give a limit theorem. The following theorem is a generalization of monotonic law of small numbers given in [Mu3]. The limit distribution μ_τ is a *monotonic analogue of compound Poisson distribution* (of restricted class so that the mixing measure τ is compactly supported).

Theorem 5.5 (limit theorem). *Let τ be a compactly supported finite positive measure on $\mathbf{R}$ and let $\{\lambda_p\}_{p=1}^{\infty}$ be the sequence defined by $\lambda_p = \int_{-\infty}^{+\infty} x^p d\tau(x)$. Furthermore let $\{X_i^{(n)} |\; 1 \leq i \leq n;\ n \in \mathbf{N}^*\} \subset \mathcal{A}$ be self-adjoint random variables on a C^*-probability space $(\mathcal{A}, \phi)$ satisfying*

i) $X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)}$ are monotonically independent and identically distributed w.r.t. ϕ for each $n \in \mathbf{N}^*$;
ii) $\lim_{n\to\infty} n\,\phi((X_i^{(n)})^p) = \lambda_p$ for each $p \in \mathbf{N}^*$.

Then the probability distribution of $X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)}$ under ϕ converges in the weak topology as $n \to \infty$ to a unique probability measure μ_τ. This μ_τ is a compactly supported $\triangleright$-infinitely divisible distribution with the generator*

$$A(z) \;=\; \int_{-\infty}^{+\infty} \frac{xz}{x-z} d\tau(x). \tag{5.6}$$

Proof. The unique existence of the limit distribution μ_τ can be easily verified. The limit of pth moments $m_p^{(\infty)} \equiv \lim_{n\to\infty} \langle (X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)})^p \rangle$ exists and it is given by

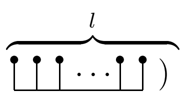
$$m_p^{(\infty)} \;=\; \sum_{g \in \mathrm{NCD}(p)} V(g),$$

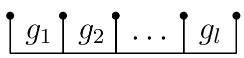
where $V(g)$ is defined by

$$V(g) \;=\; \lim_{n\to\infty} \sum_{\substack{(i_p \cdots i_2 i_1) \\ \in \{1,2,\cdots,n\}^{\mathrm{M}_g}}} \prod_{v \in \mathcal{P}(g)} \langle\, (X_1^{(n)})^{|v|} \,\rangle.$$

From the scaling condition ii) and the monotonic property (ii) in Lemma 1.2, we see that $V(g)$ satisfies the following relations.

Recurrence relations:

i) $V($  $) \;=\; \lambda_l,$

ii) $V($  $)$

$$= \frac{\lambda_{l+1}}{\#\{\text{blocks in } g_1\} + \cdots + \#\{\text{blocks in } g_l\} + 1} V(g_1)V(g_2)\cdots V(g_l),$$

iii) $V(g_1 g_2 \cdots g_l) \;=\; V(g_1)V(g_2)\cdots V(g_l).$

Here we made a convention that $V(\Lambda) \equiv 1$ for the empty diagram Λ.

Denote by $f(s) = \sum_{p=0}^{\infty} m_p^{(\infty)} s^p$ be the moment generating function of μ_τ, by $G(z)$ the Cauchy transform of μ_τ and by $H(z)$ the reciprocal Cauchy transform of μ_τ. Let us obtain the integral representation for $w = H(z)$ in a formal way.

For each noncrossing diagram $g \in \mathrm{NCD} = \cup_{r=0}^{\infty} \mathrm{NCD}(r)$, put $p(g) = \#\{\text{points in } g\}$ and $n(g) = \#\{\text{blocks in } g\}$. Of course $p(\Lambda) \equiv 0$ and $n(\Lambda) \equiv 0$. Let $f(s,t)$ be the (formal) generating function for $V(g)$ defined by

$$f(s,t) \;=\; \sum_{g \in \mathrm{NCD}} V(g)\, s^{p(g)}\, t^{n(g)}.$$

Then, in the same way as in [Mu3], we get

$$f(s,t) \;=\; \frac{1}{1 - g(s,t)}.$$

where

$$g(s,t) \;=\; \sum_{\substack{h \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(h) = 1}} V(h)\, s^{p(h)}\, t^{n(h)}.$$

By the way, this $g(s,t)$ can be rewritten as

$$
\begin{aligned}
g(s,t) &= \sum_{\substack{h \,\in\, \mathrm{NCD}^* \\ \mathrm{out}(h)\,=\,1}} V(h)\; s^{p(h)}\; t^{n(h)} \\
&= \sum_{l=0}^{\infty} \sum_{g_1, g_2, \cdots, g_l \in \mathrm{NCD}} V(\, \underline{\; g_1 \; g_2 \; \cdots \; g_l \;}\,) \\
&\qquad \times s^{p(g_1)+\cdots+p(g_l)+l+1}\; t^{n(g_1)+\cdots+n(g_l)+1} \\
&= \lambda_1\, s\, t \;+\; \sum_{l=1}^{\infty} \sum_{g_1, g_2, \cdots, g_l \in \mathrm{NCD}} \frac{\lambda_{l+1}}{n(g_1)+n(g_2)+\cdots+n(g_l)+1} \\
&\qquad \times V(g_1)\, s^{p(g_1)}\, t^{n(g_1)}\;\; V(g_2)\, s^{p(g_2)}\, t^{n(g_2)} \;\cdots\; V(g_l)\, s^{p(g_l)}\, t^{n(g_l)}\; s^{l+1}\, t^{1} \\
&= \lambda_1\, s\, t \;+\; \sum_{l=1}^{\infty} \lambda_{l+1} \int_0^t dt\, \left\{ \sum_{g \in \mathrm{NCD}} V(g)\; s^{p(g)}\; t^{n(g)} \right\}^{l} s^{l+1} \\
&= \int_0^t dt\, \sum_{l=0}^{\infty} \lambda_{l+1}\; f(s,t)^l\; s^{l+1} \\
&= \int_0^t dt\, \left\{ \sum_{l=0}^{\infty} \lambda_{l+1}\; (s\, f(s,t))^l\; s \right\}.
\end{aligned}
$$

Put $h(t) = \sum\limits_{l=1}^{\infty} \lambda_l\, t^l$, then we get

$$
g(s,t) \;=\; \int_0^t dt\; \frac{h(s\, f(s,t))}{f(s,t)}. \tag{5.7}
$$

Since $f = \frac{1}{1-g}$, (5.7) yields the differential equation

$$
\frac{f'}{f} \;=\; h(s\, f(s,t)), \tag{5.8}
$$

where $' = \frac{d}{dt}$. Put $y = f(s,t)$, then (5.8) can be rewritten as

$$
\frac{y'}{y} \;=\; h(s\, y),
$$

and hence we get

$$
\int_1^y \frac{dy}{y\, h(s\, y)} \;=\; \int_0^t dt \;=\; t.
$$

By the specialization $t := 1$ and the change of variables $u = s\, y$, we get the functional equation for the generating function $y = f(s)$:

$$
\int_s^{s\, y} \frac{du}{u\, h(u)} \;=\; 1.
$$

By the change of variables $s = \frac{1}{z}$, we get the functional equation for the Cauchy transform $\omega = G(z) = \frac{1}{z} f(\frac{1}{z})$:

$$
\int_{1/z}^{\omega} \frac{du}{u\, h(u)} \;=\; 1.
$$

Furthermore, by the change of variables $u = \frac{1}{\zeta}$, we get the functional equaion for the reciprocal Cauchy transform $w = H(z) = \frac{1}{G(z)}$:

$$\int_z^w -\frac{d\zeta}{\zeta\, h(\frac{1}{\zeta})} \;=\; 1. \tag{5.9}$$

Since the denominator $-\zeta\, h(\frac{1}{\zeta})$ in (5.9) can be rewritten as $-\zeta\, h(\frac{1}{\zeta}) = \int_{-\infty}^{+\infty} \frac{x\zeta}{x-\zeta} d\tau(x)$, we define

$$A(z) \;\equiv\; \int_{-\infty}^{+\infty} \frac{xz}{x-z} d\tau(x), \qquad z \in \mathbf{C}^+. \tag{5.10}$$

Then $A(z)$ is a Pick function, and (5.9) can be rewritten as

$$\int_z^w \frac{dz}{A(z)} \;=\; 1. \tag{5.11}$$

Although we get this expression (5.11) through a formal calculation, we see that (5.11) hold for all $z \in \mathbf{C}^+$ because of the assumption of compact support for τ and the uniqueness theorem for holomorphic functions. From Theorem 4.7, the Pick function $A(z)$ determines a weak* continuous $\triangleright$-semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures and hence a $\triangleright$-infinitely divisible distribution μ_1. Therefore μ_τ $(= \mu_1)$ is $\triangleright$-infinitely divisible. Since (5.10) can be rewrittren as

$$A(z) \;=\; a + \int_{-\infty}^{+\infty} \frac{1}{x-z} d\sigma(x).$$

with $a = -\int_{-\infty}^{+\infty} x d\tau(x)$ and $d\sigma(x) = x^2 d\tau(x)$ (σ is compactly supported), we conclude, from Theorem 5.1, that μ_τ is compactly supported. □

Note that the case $\tau = \lambda\,\delta_1$ (hence $\lambda_p = \lambda$ for all $p \in \mathbf{N}^*$) corresponds to the monotonic law of small numbers [Mu3].

6. Appendix

In this Appendix, we collected some Lemmas needed in the preceding sections.

Let $\{H_t(z)\}_{t\geq 0}$ be a continuous one-parameter semigroup of reciprocal Cauchy transforms of probability measures on $\mathbf{R}$, and let $H_t(z) = a_t + z + \int_{-\infty}^{+\infty} \frac{1+xz}{x-z} d\tau_t(x)$ be its integral representation due to Theorem 3.4. Then the followings hold.

Lemma 6.1.
(1) For each $z \in \mathbf{C}^+$, the function $t \mapsto \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)$ is continuous.
(2) For any $\delta > 0$ and any $z \in \mathbf{C}^+$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_{\frac{\delta}{n}}(z))(x-z)} d\tau_{\frac{k}{n}\delta}(x) \;=\; \frac{1}{\delta} \int_0^\delta \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x) \Big) dt.$$

Proof. *(1)* Let $z \in \mathbf{C}^+$ be fixed, and let $\{z_n\}_{n=1}^\infty \subset \mathbf{C}^+$ be a sequence such that $z_n \neq z$ and $\lim_{n\to\infty} z_n = z$. Put

$$f_n(t) \;=\; \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z_n)(x-z)} d\tau_t(x), \qquad f(t) \;=\; \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x),$$

then we have

$$|f_n(t) - f(t)| \ \leq \ \int_{-\infty}^{+\infty} \frac{1+x^2}{|x-z|^2} \frac{|z_n - z|}{\Im z - |z_n - z|} d\tau_t(x). \tag{6.1}$$

Since $\frac{1+x^2}{|x-z|^2}$ is a bounded function of x, we put $M \equiv \sup\limits_{x \in \mathbf{R}} \frac{1+x^2}{|x-z|^2}$ $(< \infty)$. From (6.1) we have

$$\sup_{a \leq t \leq b} |f_n(t) - f(t)| \ \leq \ M \cdot \frac{|z_n - z|}{\Im z - |z_n - z|} \cdot \sup_{a \leq t \leq b} \tau_t(\mathbf{R})$$

for each compact interval $[a, b]$ with $0 \leq a \leq b < \infty$. By the way, we have $\sup\limits_{a \leq t \leq b} \tau_t(\mathbf{R}) < \infty$ because of the continuity of $t \mapsto \tau_t(\mathbf{R}) = \Im(H_t(i)) - 1$. So we get the uniform convergence $\sup\limits_{a \leq t \leq b} |f_n(t) - f(t)| \to 0$ $(n \to \infty)$ for each $[a, b]$. Since

$$f_n(t) \ = \ \frac{1}{z_n - z} \Big\{ H_t(z_n) - H_t(z) + (z - z_n) \Big\}$$

is a continuous function of t for each n, we conclude that $f(t)$ is a continuous function.

(2) Let $z \in \mathbf{C}^+$ be fixed. Since (1), the function $t \mapsto \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x)$ is integrable on $0 \leq t \leq \delta$ in the sense of Riemann integral. Put

$$I \ = \ \int_0^{\delta} \Big(\int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_t(x) \Big) dt, \qquad I_n \ = \ \frac{\delta}{n} \sum_{k=0}^{n-1} \int_{-\infty}^{+\infty} \frac{1+x^2}{(x-z)^2} d\tau_{\frac{k}{n}\delta}(x).$$

Here I_n is the nth approximation to the Riemann integral I. Put

$$J_n \ = \ \frac{1}{n} \sum_{k=0}^{n-1} \int_{-\infty}^{+\infty} \frac{1+x^2}{(x - H_{\frac{\delta}{n}}(z))(x-z)} d\tau_{\frac{k}{n}}(x),$$

then we have

$$|I_n - J_n| \ \leq \ \frac{1}{n} \sum_{k=0}^{n-1} \int_{-\infty}^{+\infty} \frac{1+x^2}{|x-z|^2} \frac{|H_{\frac{\delta}{n}}(z) - z|}{\Im z - |H_{\frac{\delta}{n}}(z) - z|} d\tau_{\frac{k}{n}\delta}(x).$$

for sufficiently large arbitrary n. Put $M \equiv \sup\limits_{x \in \mathbf{R}} \frac{1+x^2}{|x-z|^2}$ $(< \infty)$, then we get

$$|J_n - I_n| \ \leq \ M \cdot \frac{|H_{\frac{\delta}{n}}(z) - z|}{\Im z - |H_{\frac{\delta}{n}}(z) - z|} \cdot \frac{\delta}{n} \sum_{k=0}^{n-1} \tau_{\frac{k}{n}\delta}(\mathbf{R}) \ \to \ 0 \cdot \int_0^{\delta} \tau_t(\mathbf{R}) dt \ = \ 0 \quad (n \to \infty).$$

Here we used the continuity of the function $t \mapsto \tau_t(\mathbf{R}) = \Im H_t(i) - 1$. So we get the desired result $J_n \to I$ $(n \to \infty)$. $\square$

Lemma 6.2. *If the semigroup $\{H_t(z)\}_{t \geq 0}$ is not trivial, then $H_t(z) \neq z$ for all $t > 0$ and all $z \in \mathbf{C}^+$.*

Proof. Let us suppose that there exist some $t_0 > 0$ and some $z_0 \in \mathbf{C}^+$ such that $H_{t_0}(z_0) = z_0$. Then we have $\Im(H_{t_0}(z_0)) = \Im z_0$. It is known that any reciprocal

Cauchy transform with such a properpty must correspond to some point measure ([Maa], Prop.2.1.). Hence the Nevanlinna representation of $H_{t_0}(z)$ must be

$$H_{t_0}(z) = a_{t_0} + z, \quad z \in \mathbf{C}^+,$$

that is $\tau_{t_0}(\mathbf{R}) = 0$. By the way, from $\tau_t(\mathbf{R}) = \Im H_t(i) - 1$, we have

$$\tau_{t+\varepsilon}(\mathbf{R}) \;=\; \Im H_\varepsilon(H_t(i)) - 1 \;\geq\; \Im H_t(i) - 1 \;=\; \tau_t(\mathbf{R}),$$

so $\tau_t(\mathbf{R})$ is a nondecreasing function of t. Therefore we have $0 \leq \tau_t(\mathbf{R}) \leq \tau_{t_0}(\mathbf{R}) = 0$ for each $t \in [0, t_0]$. Let $T > 0$ be an arbitrary positive real number, then there exists some natural number $n \geq 1$ such that $\frac{T}{n} \in [0, t_0]$. The probability measure $\mu_{\frac{T}{n}}$ corresponding to $H_{\frac{T}{n}}(z)$ must be a point measure since $\tau_{\frac{T}{n}}(\mathbf{R}) = 0$. So we have $H_{\frac{T}{n}}(z) = a_{\frac{T}{n}} + z$, $z \in \mathbf{C}^+$, and hence

$$H_T(z) \;=\; H_{\frac{T}{n}}(H_{\frac{T}{n}}(\cdots(H_{\frac{T}{n}}(z))\cdots)) \;=\; na_{\frac{T}{n}} + z$$

by n times iteration. So the probability measure μ_T must be a point measure. That is

$$H_T(z) \;=\; a_T + z, \quad z \in \mathbf{C}^+$$

for all $T \geq 0$. Note that $a_T = T \cdot a_1$ because of the semigroup property of $\{H_t(z)\}_{t\geq 0}$. By the way, since we have $z_0 = H_{t_0}(z_0) = t_0 \cdot a_1 + z_0$, we obtain $a_1 = 0$. After all we have $H_T(z) = z$ for all $T > 0$ and all $z \in \mathbf{C}^+$. But this contradicts to the assumption that the semigroup $\{H_t(z)\}_{t\geq 0}$ is not trivial. Therfore we conclude that $H_t(z) \neq z$ for all $t > 0$ and all $z \in \mathbf{C}^+$. □

Lemma 6.3. *Let μ, ν be probability measures on $\mathbf{R}$ with finite variances. Then $\lambda = \mu \triangleright \nu$ also has finite variance. Besides the mean m and variance σ^2 are additive:*

$$m(\lambda) \;=\; m(\mu) + m(\nu), \qquad \sigma^2(\lambda) \;=\; \sigma^2(\mu) + \sigma^2(\nu).$$

Proof. Since $\nu \in \mathcal{P}^2$, there exists a real number a and a finite positive measure ρ such that

$$H_\nu(z) \;=\; a + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho(x)$$

by the characterization theorem ([Maa], Prop. 2.2). Here a and ρ satisfy $a = -m(\nu)$ and $\rho(\mathbf{R}) = \sigma^2(\nu)$. For each $y \in \mathbf{R}$, we have

$$H_\nu(z) - y \;=\; -(m(\nu) + y) + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho(x).$$

This integral representation uniquely determines a probability measure $\nu_y \in \mathcal{P}^2$ by $H_\nu(z) - y = H_{\nu_y}(z)$. Note that $m(\nu_y) = m(\nu) + y$ and $\sigma^2(\nu_y) = \rho(\mathbf{R}) = \sigma^2(\nu)$. As seen in the proof of Theorem 3.5, the monotonic convolution measure $\lambda = \mu \triangleright \nu$ is given by

$$\lambda(\cdot) \;=\; \int_{-\infty}^{+\infty} \nu_y(\cdot) d\mu(y). \tag{6.2}$$

Using the relations $m(\nu_y) = m(\nu) + y$ and (6.2), we get

$$m(\lambda) \;=\; m(\mu) + m(\nu). \tag{6.3}$$

Also, using the relations $\sigma^2(\nu_y) = \sigma^2(\nu)$, (6.2) and (6.3), we get

$$\sigma^2(\lambda) = \sigma^2(\mu) + \sigma^2(\nu).$$

□

Lemma 6.4. *Let $\lambda, \mu, \nu \in \mathcal{P}$ such that $\lambda = \mu \rhd \nu$. Then*

(1) If μ and ν are compactly supported, then λ is cpompactly supported.

(2) If λ is compactly supported, then ν is compactly supported.

Proof. (1) From Corollary 2.2, there exist monotonically independent bounded self-adjoint random variables X, Y in some Hilbert space with unit vector (H, ξ) such that $\mu_X = \mu$ and $\mu_Y = \nu$. By theorem 3.1, we have

$$\mu_{X+Y} \;=\; \mu_X \rhd \mu_Y \;=\; \mu \rhd \nu \;=\; \lambda.$$

Obviously $X + Y$ is a bounded self-adjoint operator, and hence λ is compactly supported.

(2) Remind of the composition formula

$$\lambda(\cdot) \;=\; \int_{-\infty}^{+\infty} \nu_x(\cdot) d\mu(x).$$

where $\nu_x \in \mathcal{P}$ is defined by $H_{\nu_x}(z) = H_\nu(z) - x$. Since λ is compactly supported, the above expression yields the existence of $x_0 \in \mathbf{R}$ such that ν_{x_0} is compactly supported. Assume that ν_{x_0} is supported in a compact interval $[-M, M]$. Then, for any $\varepsilon > 0$, there exists $M_1 (\geq M)$ such that $G_{\nu_{x_0}}(z)$ is holomorphic and satisfies $0 < |G_{\nu_{x_0}}(z)| < \varepsilon$ on $|z| > M_1$. Take $\varepsilon > 0$ so that $\frac{1}{\varepsilon} > |x_0|$, then we have

$$\frac{1}{\varepsilon} \;<\; |H_{\nu_{x_0}}(z)| \;=\; |H_\nu(z) - x_0| \;\leq\; |H_\nu(z)| + |x_0|$$

and hence

$$|H_\nu(z)| \;\geq\; \frac{1}{\varepsilon} - |x_0| \;>\; 0$$

for $|z| > M_1$. This means that $H_\nu(z)$ has the reciprocal $G_\nu(z) = \frac{1}{H_\nu(z)}$ on $|z| > M_1$. Therfore ν is compactly supported. $\square$

Lemma 6.5. *Let μ be a probability measure on $\mathbf{R}$. Then the followings are equivalent.*

(1) μ is compactly supported.

(2) There exists a real number a and a compactly supported finite positive measure ρ on $\mathbf{R}$ such that

$$H_\mu(z) \;=\; a + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho(x).$$

Proof. We can assume that $\mu \in \mathcal{P}^2$ because of Proposition 2.2 in [Maa]. Hence $H_\mu(z)$ has a representation

$$H_\mu(z) \;=\; a + z + \int_{-\infty}^{+\infty} \frac{1}{x - z} d\rho(x)$$

with $a \in \mathbf{R}$ and ρ a finite positive measure on $\mathbf{R}$. μ is compactly supported if and only if $G_\mu(z)$ (and hence $H_\mu(z)$) is holomorphic on $\mathbf{C} \setminus [-M, M]$ for some $M > 0$. Furthermore this case is equivalent to that $\int_{-\infty}^{+\infty} \frac{d\rho(x)}{x-z}$ $(= H_\mu(z) - a - z)$ is holomorphic on $\mathbf{C} \setminus [-M, M]$, that is, ρ is compactly supported. $\square$

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