McKay Correspondence

To the memory of Peter Slodowy

Iku Nakamura

2005 November

1 Dynkin diagrams and ADE

The following appear in connection with Dynkin

(0) Dynkin diagram ADE

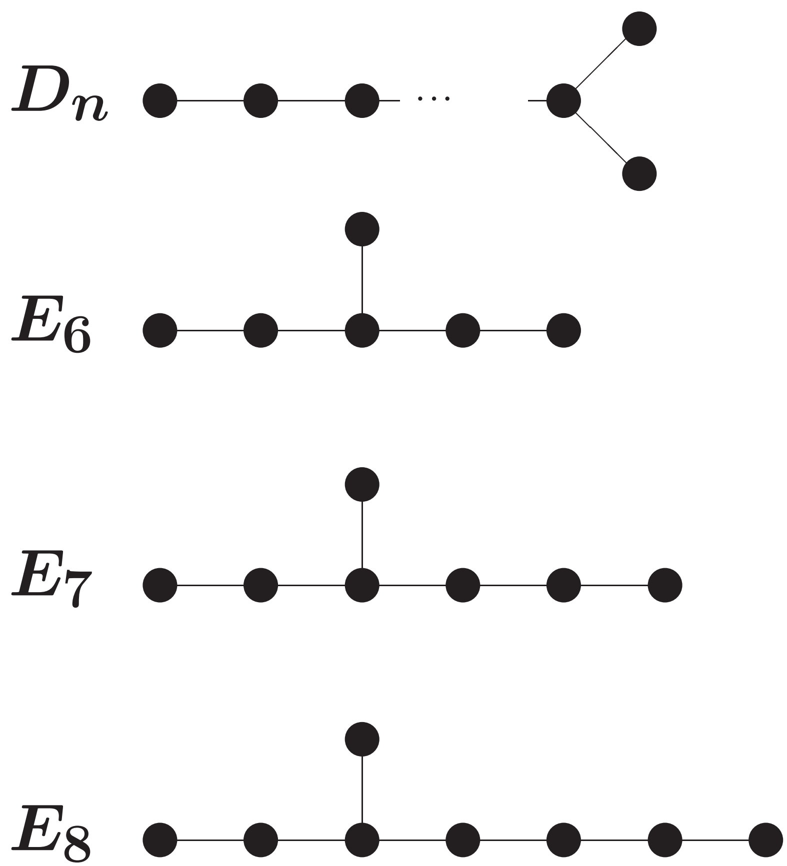
(1) Finite subgroups of $SL(2, C)$, Regular polyhedra

(2) 2-dim simple sing (deform-stable critical points)

(3) simple Lie algebras ADE, II_1-factors etc.

(4) Partition func. of $SL(2, Z)$-inv. conformal field thrys

(5) Finite simple groups Fischer F_{24}, Baby monster B, Monster M, and McKay's 3rd observation on them (E_6, E_7, E_8 correspond to them)


A_n
D_n
E_6
E_7
E_8

Partition funct. of SL(2, Z)-inv. conformal field theorys

Dofinition: $Z = \mathrm{Tr}\, q^{L_0+\bar{L}_0}$ where $q = e^{2\pi\sqrt{-1}\tau}$

Assumptions

1. ∃ Unique vacuum(∃1 state of min. energy)
2. χ_k : a $A_1^{(1)}$-character corresp. to a particle (or an operator in a physical theory)
3. Partition func. Z is a sum of $\chi\overline{\chi'}$, (χ, χ':$A_1^{(1)}$-characters)
4. Z is SL(2, Z)-inv., invariant under $\tau \mapsto -\tau^{-1}$ (the parameter $\tau \in$ H : upper half-plane)

Classification of Z

Cappelli, Itzykson-Zuber, A.Kato

Type	$k+2$	Partition function $Z(q,\theta,\bar{q},\bar{\theta})$
A_n	$n+1$	$\sum_{\lambda=1}^{n} \lvert\chi_\lambda\rvert^2$
D_{2r}	$4r-2$	$\sum_{\lambda=1}^{r-1} \lvert\chi_{2\lambda-1}+\chi_{4r+1-2\lambda}\rvert^2 + 2\lvert\chi_{2r-1}\rvert^2$
D_{2r+1}	$4r$	$\sum_{\lambda=1}^{2r} \lvert\chi_{2\lambda-1}\rvert^2 + \sum_{\lambda=1}^{r-1}(\chi_{2\lambda}\bar{\chi}_{4r-2\lambda}+\bar{\chi}_{2\lambda}\chi_{4r-2\lambda}) + \lvert\chi_{2r}\rvert^2$
E_6	12	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$
E_7	18	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2$
		$+(\chi_3+\chi_{15})\bar{\chi}_9 + \chi_9(\bar{\chi}_3+\bar{\chi}_{15})$
E_8	30	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$

Indices of χ = Coxeter exponents of ADE

Type	Partition function $\boldsymbol{Z}$ and Coxeter exponents
$\boldsymbol{E_6}$	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$ $\mathbf{1, 4, 5, 7, 8, 11}$
$\boldsymbol{E_7}$	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$ $+(\chi_3+\chi_{15})\bar{\chi}_9+\chi_9(\bar{\chi}_3+\bar{\chi}_{15})$ $\mathbf{1, 5, 7, 9, 11, 13, 17}$
$\boldsymbol{E_8}$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$ $\mathbf{1, 7, 11, 13, 17, 19, 23, 29}$

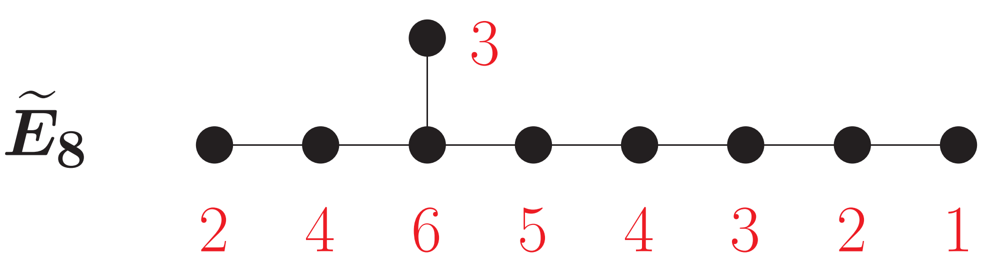
(5) McKay's 3rd observation

$\exists$ only 2 conj. classes of involutions of Monster $\mathbb{M}$.

(Fischer) is one of the classes (Fischer involution).

1. **{the conj. class of $a \cdot b$; $a, b \in$ (Fischer)} = 9 classes**
2. **{order of $a \cdot b$; $a, b \in$ (Fischer)} = {1, 2, 2, 3, 3, 4, 4, 5, 6},**

   the same as mult. of vertices of $\widetilde{E}_8$, extended E_8.



Today

McKay correspondence $(1) \Longrightarrow (0)$

[Ito-N. 1999] explains McKay $(1) \Longrightarrow (0) + (2)$

by Hilbert scheme of G-orbits G-Hilb$(\mathbf{C}^2)$

$G \subset \mathrm{SL}(2)$

Recall:

(0) Dynkin diagrams ADE

(1) Finite subgroups of $\mathrm{SL}(2, \mathbf{C})$

(2) 2-dim. Simple sing. and their resol.

(3) Simple Lie algebras ADE

well known and trivial

1. **From finite groups to singularities : (1) $\Longrightarrow$ (2)**
2. **From singularities to Dynkin diag. : (2) $\Longrightarrow$ (0)**

Recall:

(0) Dynkin diagrams ADE

(1) Finite subgroups of $SL(2, C)$

(2) 2-dim. Simple sing. and their resol.

(3) Simple Lie algebras ADE

well known, but nontrivial

1. From Lie algebra to singulalities : (3) $\Longrightarrow$ (2)

   Grothendieck, Brieskorn, Slodowy

2. From finite groups to Dynkin :(1) $\Longrightarrow$ (0) (McKay)

3. Explanation for McKay : (1) + (2) $\Longrightarrow$ (0) + (2)

   by vector bundles and their Chern classes

   (Gonzalez-Sprinberg, Verdier)

Today [Ito-N. 1999] explains McKay (1) $\Longrightarrow$ (0) + (2)

by Hilbert scheme of G-orbits : G-Hilb(C^2), $G \subset \mathrm{SL}(2)$

More recent progress

Topic(4) : Kawahigashi and other (Ann. Math.)

Topic(5) : Conway, Miyamoto, Lam-Yamada-Yamauchi

Hot news

John McKay received 2003 CRM-Feilds-PIMS prize.

Reason : Discovery of McKay corresp. and McKay's observation about Monster.

History

1978.11	**Moonshine**
—-	**Character table of Monster (Fischer-Thompson)**
1978.12	**McKay correspondence**
1979.02	**Discovery of (5)**
—-	**Griess constructs Monster**

3 dim. McKay corresp. thereafter

$G \subset \mathrm{SL}(3)$, Consider G-Hilb(C^3) only.

1. **Smoothness—G abelian : Nakamura**
2. **Smoothness & 3dim McKay corresp. (G abelian : Ito-Nakajima, G general : Bridgeland-King-Reid)**
3. **Structure —G abelian : Nakamura, Craw-Reid**
4. **Structure of M_θ (variants of G-Hilb(C^3)) —G abelian : Craw-Ishii**
5. **Structure —G non-abelian : Gomi-Nakamura-Shinoda**

Today

McKay correspondence $(1) \Longrightarrow (0)$

[Ito-Nakamura 1999] explain $(1) \Longrightarrow (0) + (2)$

by G-Hilb$(\mathbf{C}^2)$: Hilbert scheme of G-orbits

$G \subset \mathrm{SL}(2)$

Recall:

(0) Dynkin diagrams ADE

(1) Finite subgroups of $\mathrm{SL}(2, \mathbf{C})$

(2) 2-dim. Simple sing. and their resol.

2 Review : what is McKay correpondence ?

Exam 1

$$G = G(D_5) : \text{ a dihedral group of order } 12$$

$$G : \text{ generated by } \sigma \text{ and } \tau$$

$$\sigma = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$Repres$	$\mathrm{Tr}(\rho)$	e	$-e$	σ	σ^2	τ	τ^3
ρ_0	χ_0	1	1	1	1	1	1
ρ_1	χ_1	1	1	1	1	-1	-1
ρ_2	χ_2	2	-2	1	-1	0	0
ρ_3	χ_3	2	2	-1	-1	0	0
ρ_4	χ_4	1	-1	-1	1	i	$-i$
ρ_5	χ_5	1	-1	-1	1	$-i$	i

Let $\rho_{\rm nat} : G \subset \mathrm{SL}(2,\mathbf{C})$ (the natural inclusion)

Then repres.ρ_i and their tensor products with $\rho_{\rm nat}$

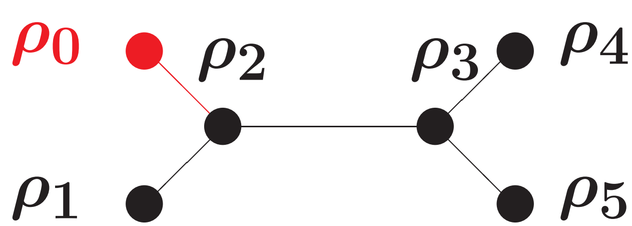
$$\rho_2 \otimes \rho_{\rm nat} = \rho_0 + \rho_1 + \rho_3, \quad \rho_0 \otimes \rho_{\rm nat} = \rho_1 \otimes \rho_{\rm nat} = \rho_2,$$

$$\rho_3 \otimes \rho_{\rm nat} = \rho_2 + \rho_4 + \rho_5, \quad \rho_4 \otimes \rho_{\rm nat} = \rho_3,$$

$$\rho_5 \otimes \rho_{\rm nat} = \rho_3.$$

Draw a graph as follows:

$\widetilde{D}_5$



Rule : Connect ρ_i and $\rho_j \iff \rho_i \otimes \rho_{\rm nat} = \rho_j + \cdots$

Remove ρ_0 (triv. repres.).Then we get a Dynkin.

Exam 2 (Continued)

$$G = G(D_5) \text{ : a dihedral group of order 12}$$

$$G \text{ : generated by } \sigma \text{ and } \tau$$

$$\sigma = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The quotient C^2/G has a unique sing.

Invariant polynomials of $G(D_5)$ and their relation are

$$F = x^6 + y^6, G = x^2y^2, H = xy(x^6 - y^6)$$

$$G^4 - GF^2 + H^2 = 0$$

Invariant polynomials of $G(D_5)$ and their relation are

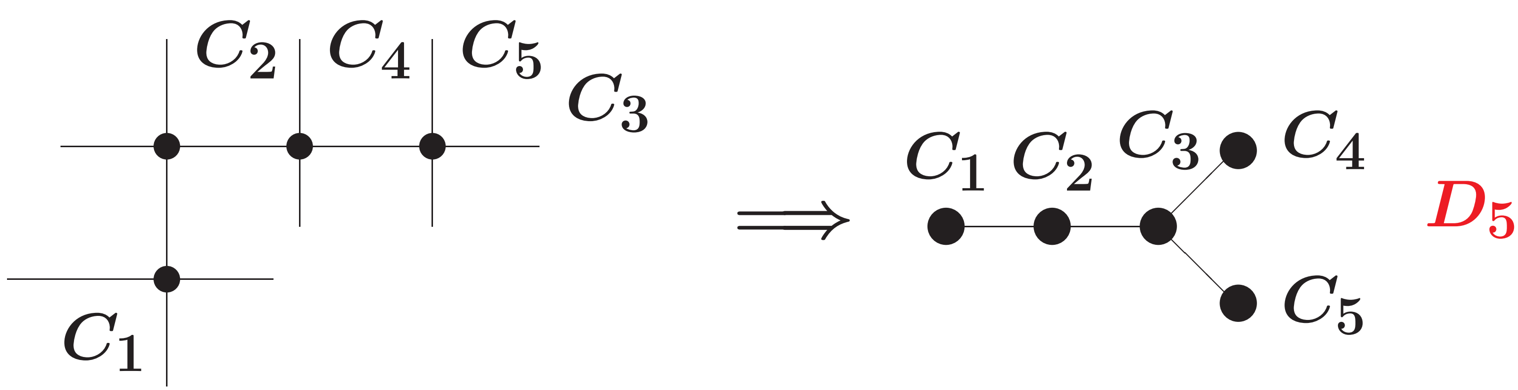
$$G^4 - GF^2 + H^2 = 0$$

The equation of D_5 is usually refered to as

$$X^4 + XY^2 + Z^2 = 0$$

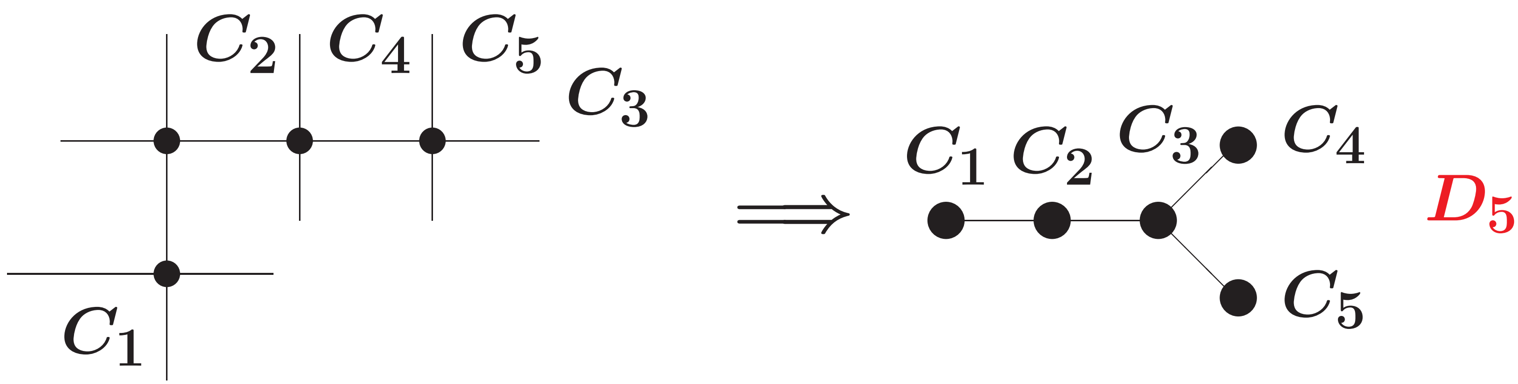
A unique singularity $(F, G, H) = (0, 0, 0)$

The exceptional set for the sing.$(0, 0, 0)$



C_i is P^1 (a line), intersecting at most transversely

The exceptional set for the sing. $(0,0,0)$



C_i is P^1 (a line), intersecting at most transv.

The dual grapf of it is a Dynkin diagram D_5

Rule of dual graph : C_i = a vertex, $C_i \cap C_j$ = an edge

Conclusion : both Dynkin diagrams are the same

(McKay correspondence)

McKay corresp. asserts :

$\exists$ a relationship between (1) and (2)

(1) Resolution of sing. of C^2/G

(2) Representation theory of G

Exam 3 (Continued)

$G = G(D_5)$: a dihedral group of order 12

G : generated by σ and τ

$$\sigma = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The quotient C^2/G has a unique sing.

Invariant polynom. of $G(D_5)$ and relation

$$F = x^6 + y^6, G = x^2y^2, H = xy(x^6 - y^6)$$

$$G^4 - GF^2 + H^2 = 0$$

Invariant polynom. of $G(D_5)$ and relation are

$$G^4 - GF^2 + H^2 = 0$$

The usual eq. of D_5 : $X^4 + XY^2 + Z^2 = 0$

A unique singularity $(F, G, H) = (0, 0, 0)$

When resolve sing, take quotients $H/G = \frac{(x^6 - y^6)}{xy}$ etc.

with Denominators/Numerators not $G(D_5)$-invariant.

But xy, $x^6 - y^6$ belong to the same repres. of $G(D_5)$

Therefore Resolution of sing. and

Repres. theory of G relate each other.

3 New resolution of simple sing.

We dare to regard $\mathbf{C}^2/G$ as a moduli space

$\mathbf{C}^2/G$ ={a G-inv. subset consisting of 12 points}

: moduli of geometric G-orbits

Resolution of $\mathbf{C}^2/G$ = moduli of ring-theor. G-orbits

G - Hilb($\mathbf{C}^2$) := {a $O_{\mathbf{C}^2}$ - G - module of length 12}

For a module $M \in G$ - Hilb($\mathbf{C}^2$)

$$0 \to I \to O_{\mathbf{C}^2} \to M \to 0 \text{ (exact)}$$

G-module generating I is almost G-irreducible

Example of generators of I : $F_t = xy - t(x^6 - y^6)$

4 The Hilbert scheme of n points

The Hilbert scheme of n points in the space X

Z : n points of X

n points Z is a formal sum

$$Z = n_1 P_1 + n_2 P_2 + \cdots + n_r P_r \quad (P_i \neq P_j)$$

$$(\text{where } n = n_1 + \cdots + n_r)$$

Exam 4 Assume $X = \mathrm{C}$.

Let $P_i : x = \alpha_i$, $I_Z =$ the ideal of Z

$I_Z = f_Z \cdot \mathrm{C}[x] \Longleftrightarrow Z$ is the zero locus $f_Z(x)$

Z is n points $\overset{\text{equiv.}}{\Longleftrightarrow}$ $\dim_{\mathrm{C}} \mathrm{C}[x]/I_Z = n$

$$f_Z(x) = (x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_r)^{n_r}$$

$$= x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$$

e.g. if $z = n \cdot [0]$, then $I_Z = (x^n)$

$\mathrm{Hilb}^n(\mathrm{C}) = \{n \text{ points of } X\}$

$$\overset{\text{bij.}}{=} \left\{ x^n + \sum_{j=0}^{n-1} a_{n-j} x^j ; a_j \in \mathrm{C} \right\} \cong \mathrm{C}^n$$

Exam 5 Assume $X = \mathbb{C}^2$. Then

$$Z = n_1 P_1 + \cdots + n_r P_r, \quad \text{(formal sum)},$$

$$P_i \neq P_j, \quad \text{namely},$$

$$Z \in \underbrace{X \times \cdots \times X}_{n} / \text{order forgotton} = X^{(n)}$$

$$X^{(n)} = \underbrace{X \times \cdots \times X}_{n} / S_n$$

$X^{(n)}$ is very singular, has a lot of sing.

Caution $X^{(n)}$ is different from $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Assume $X = \mathbf{C}^2$. Let $X^{[n]}$ =Hilb$^n(\mathbf{C}^2)$.

$$X^{[n]} = \{\text{an ideal } I \subset \mathbf{C}[x,y]; \dim \mathbf{C}[x,y]/I = n\}$$

$$= \left\{ \begin{array}{c} I \subset \mathbf{C}[x,y];\ I : \text{a vector subsp of } \mathbf{C}[x,y] \\ xI \subset I, yI \subset I, \\ \dim \mathbf{C}[x,y]/I = n \end{array} \right\}$$

Thm 1 (Fogarty 1968) $X^{[n]}$ is a resolution of $X^{(n)}$.

A natural map $\pi : X^{[n]} \to X^{(n)}$ is defined,

$\pi : Z \mapsto n_1 P_1 + \cdots + n_r P_r$ where

$|Z| = \{P_1, \cdots, P_r\}, n_i =$ multiplicity of P_i in Z.

Thm 2 (Fogarty 1968)(revisited)

The natural morphism $X^{[N]} = \mathrm{Hilb}^N(\mathrm{C}^2) \xrightarrow{\pi} X^{(N)}$ is a resol (mimimal) of sing.

The map π sends G-fixed points to G-fixed points.

where $G \subset \mathrm{SL}(2)$, $N = |G|$. Then

$$\pi^{G\text{-inv.}} : (X^{[N]})^{G\text{-inv.}} \longrightarrow (X^{(N)})^{G\text{-inv.}}$$

By Thm of Fogarty $(X^{[N]})^{G\text{-inv.}}$ is nonsing.

G -inv. part of Fogarty = Next theorem

5 The G-orbit Hilbert scheme of $\mathbb{C}^2$

Lemma 3 $(X^{(N)})^{G\text{-inv.}} = \mathbb{C}^2/G.$

Def 4 $G\text{-}\mathrm{Hilb}(\mathbb{C}^2) := (X^{[N]})^{G\text{-inv.}}$

$G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ is called the G-orbit Hilbert scheme of $\mathbb{C}^2$.

For $I \in G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$, $\mathbb{C}[x, y]/I \cong \mathbb{C}[G]$: regular repres.

Thm 5 (Ito-Nakamura 1999)

$G\text{-}\mathrm{Hilb}(\mathbb{C}^2)$ is a minimal resol. of $\mathbb{C}^2/G$ with enough information about repres. of G.

Thm 6 (Ito-Nakamura 1999)

G - Hilb($\mathbb{C}^2$) is a minimal resol. of $\mathbb{C}^2/G$ with enough information about repres. of G.

This gives a new explanation for McKay corresp.

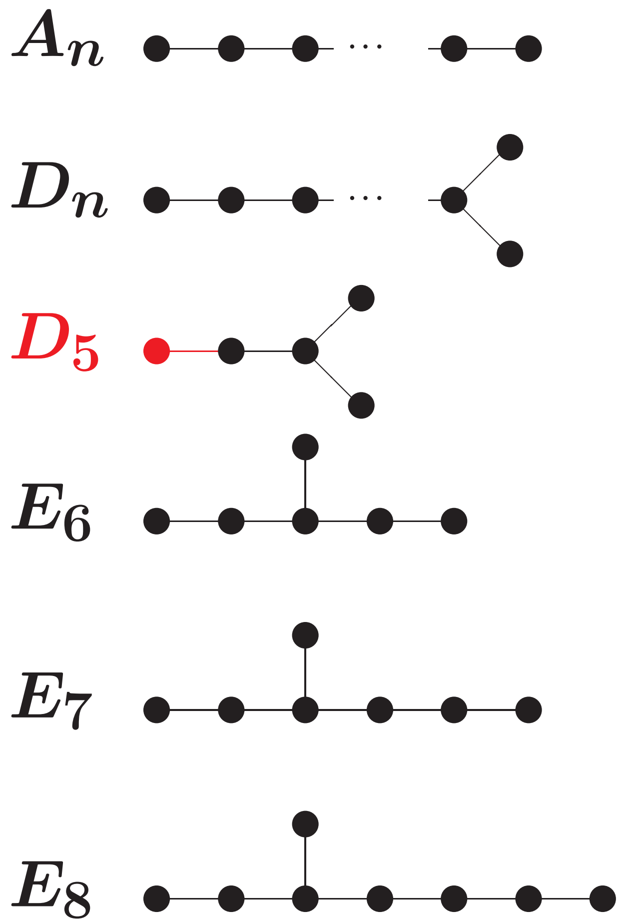
「Vertices of Dynkin diagram」

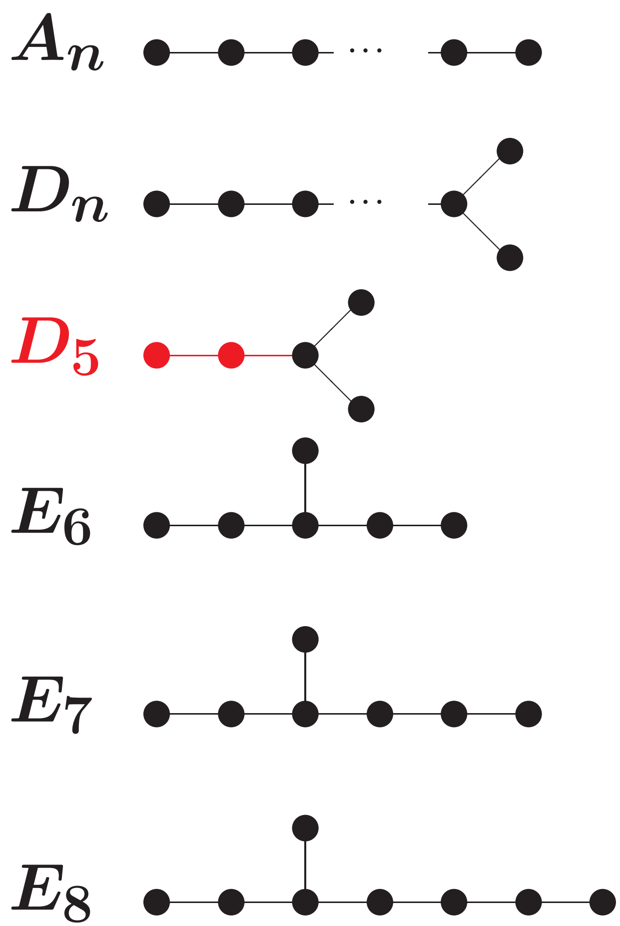
$\Updownarrow$ (bijective)

「Irred. components of except. set」

(bijective) $\Updownarrow$ (McKay corresp.)

「Eq. class of irred. reps($\neq$ trivial) of G」


A_n
D_n
D_5
E_6
E_7
E_8


A_n
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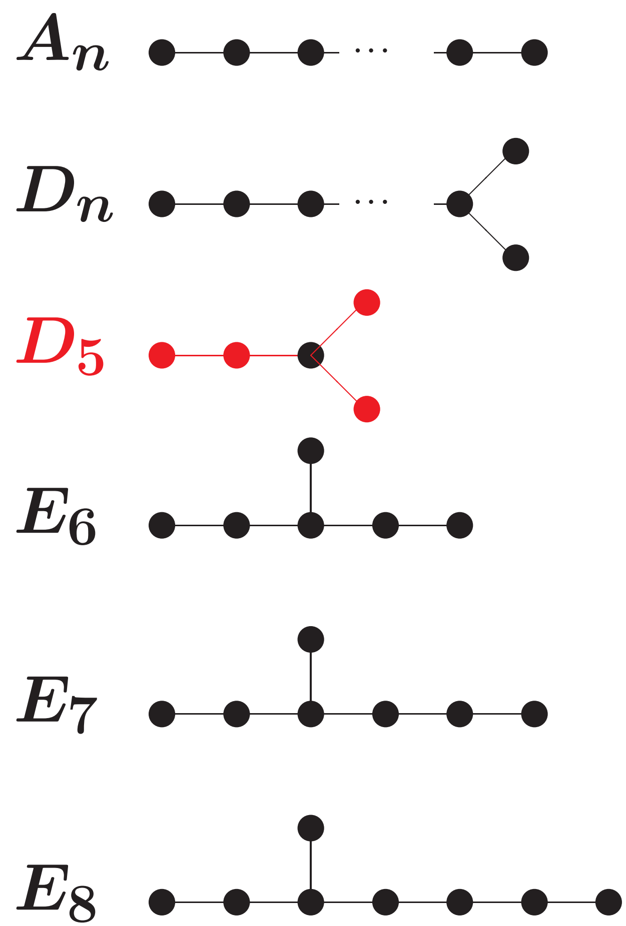

A_n
D_n
D_5
E_6
E_7
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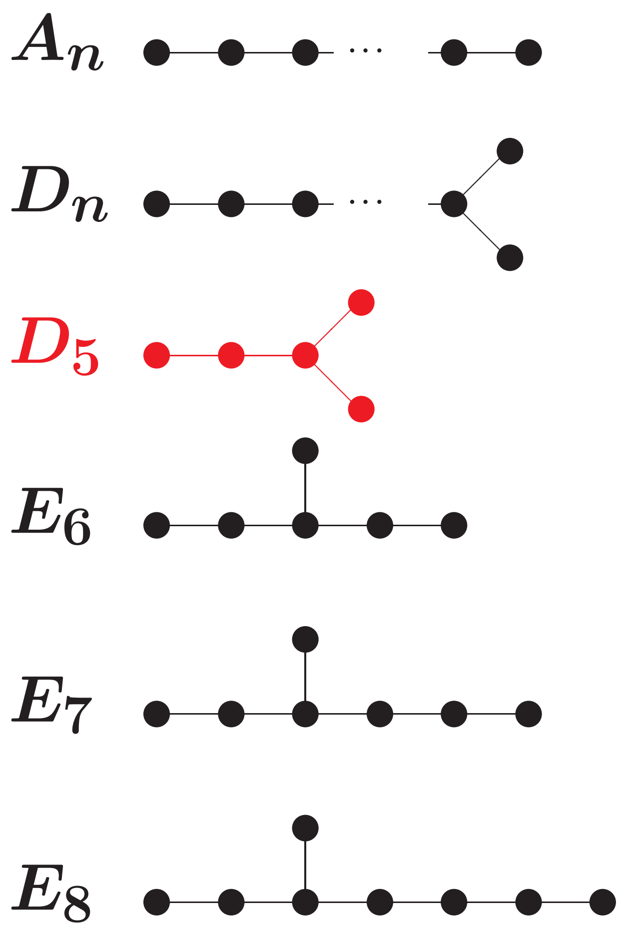
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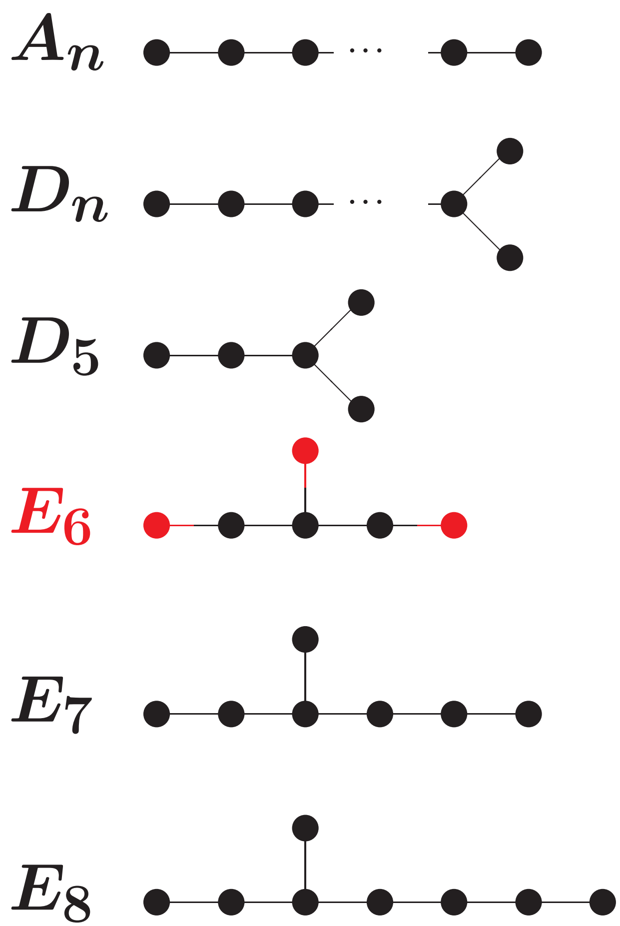

A_n
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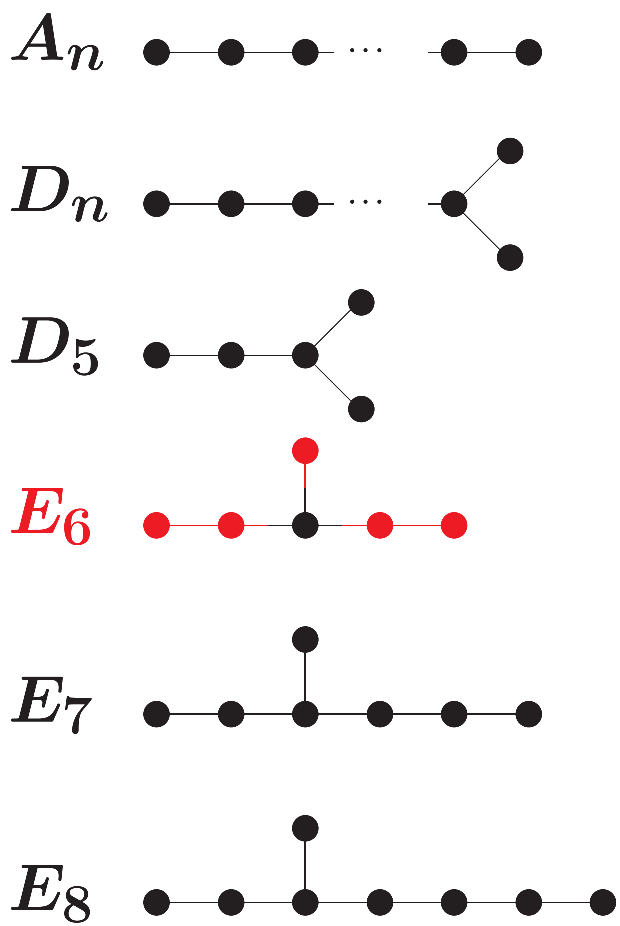
A_n
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E_6
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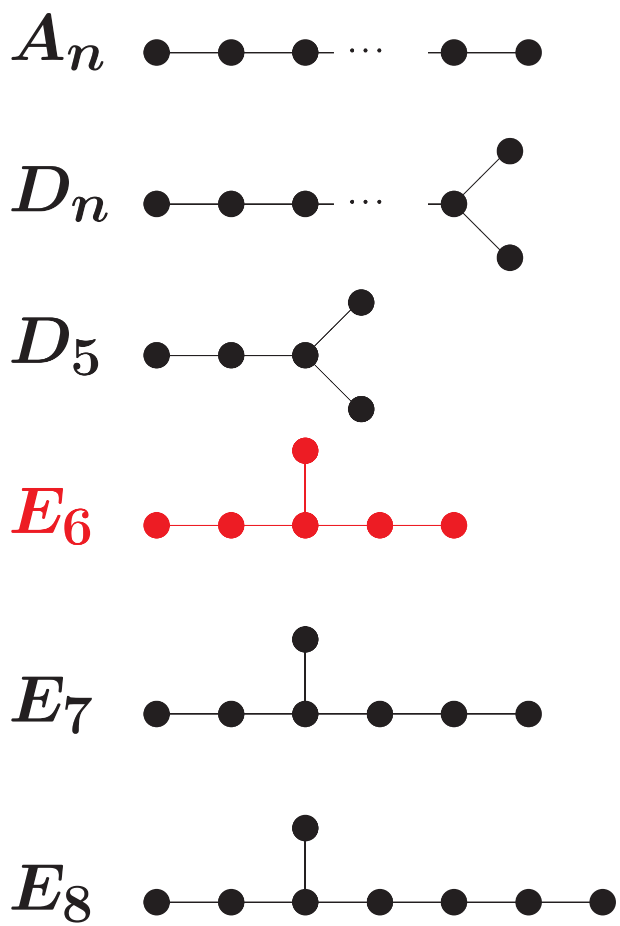
A_n
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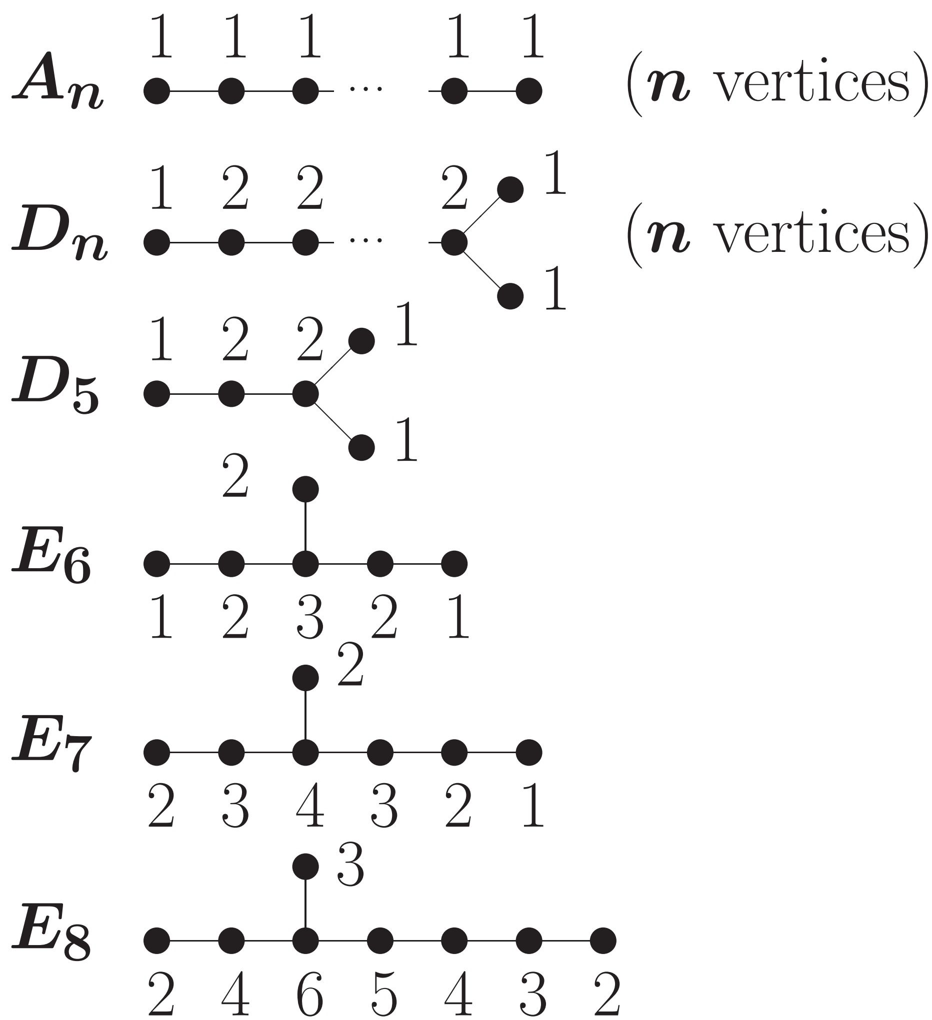

A_n
1 1 1 1 1
(n vertices)
D_n
1 2 2 2 1 1
(n vertices)
D_5
1 2 2 1 1
E_6
2
1 2 3 2 1
E_7
2
2 3 4 3 2 1
E_8
3
2 4 6 5 4 3 2

6 The exceptional set — D_5 case

$\pi : G\text{-Hilb}(\mathrm{C}^2) \to \mathrm{C}^2/G$ the natural ,ap

$G\text{-Hilb}(\mathrm{C}^2) \setminus \pi^{-1}(0) \simeq \mathrm{C}^2/G \setminus \{0\}$ (isom.)

Let $E = \pi^{-1}(0)$ be the exceptional set. Then

$$E = \{I \in G\text{-Hilb}(X); I \subset \mathfrak{m}\}$$

where $\mathfrak{m} = (x, y)\mathrm{C}[x, y]$: the maximal ideal.

For $I \in E$, we set $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$

where $\mathfrak{n} = (F, G, H)\mathrm{C}[x, y]$. $V(I)$ is a G-module of generators of I other than G-inv. For $I \in G\text{-Hilb}(\mathrm{C}^2)$, $\mathrm{C}[x, y]/I = \mathrm{C}[G]$: Regular rep.

Define a subset of E by

$$\begin{aligned} E_1 &:= \{I \in E;\ \rho_1 \subset V(I)\} \\ &= \{I \in G\text{-}\mathrm{Hilb};\ \mathfrak{m} \subset I, \rho_1 \subset V(I)\} \\ E_2 &:= \{I \in E;\quad V(I) \supset \rho_2\}. \end{aligned}$$

We will see

$$E_1 = \{I_1(s); s \in \mathrm{C}\} \cup I_1(\infty) \simeq \mathrm{P}^1,$$

$$E_2 = \{I_2(s); s \in \mathrm{C}^*\} \cup I_2(0) \cup I_2(\infty) \simeq \mathrm{P}^1,$$

$$I_1(\infty) = I_2(0) \text{ (intersection)}$$

<u>The exceptional set E,</u>

<u>E_1 an irred. comp. of E</u>

$$V_2(\rho_1) = \{xy\}, V_6(\rho_1) = \{x^6 - y^6\}$$

$$I_1(s) = \{xy + s(x^6 - y^6)\} + \mathfrak{n}$$

$I_1(s)$ is an ideal generated by $xy + s(x^6 - y^6)$ and $\mathfrak{n} = (F, G, H)$. We see,

$$\dim \mathrm{C}[x, y]/I_1(s) = 12,$$

$$I_1(s) \in G\text{-}\mathrm{Hilb}(\mathrm{C}^2) \quad (\forall s \in \mathrm{C})$$

As $s \longrightarrow \infty$,

If $E \ni I$, we have $I \supset \mathfrak{n} = (F, G, H)$. Hence

$$\mathrm{C}[x, y]/\mathfrak{n} \twoheadrightarrow \mathrm{C}[x, y]/I \text{ (surjective)}$$

$$\therefore E \subset \text{Grass.}(\mathrm{C}[x, y]/\mathfrak{n}, \text{codim } 11)$$

Grass is compact, so that the sequence $I_1(s)$ $(s \in \mathrm{C})$ converges. Now we define:

$$I_1(\infty)/\mathfrak{n} = \lim_{s \longrightarrow \infty} I_1(s)/\mathfrak{n}$$

To compute $I_1(\infty) \Longrightarrow$ McKay corresp. appears !

$$I_1(s) = \{xy + s(x^6 - y^6)\} + \mathfrak{n} \ (s \neq 0)$$
$$= \{\tfrac{1}{s}xy + (x^6 - y^6)\} + \mathfrak{n} \ (s \neq 0)$$

What happens when $s = \infty$? Let $\frac{1}{s} = 0$.

If $I_1(\infty) = \{x^6 - y^6\} + \mathfrak{n}$,

Then $I_1(\infty) \notin X^{[12]}$, Absurd.

The answer : $I_1(\infty) = \{x^6 - y^6\} + \{x^2y, xy^2\} + \mathfrak{n}$

Since $I_1(s) \supset \mathfrak{n}$, $x^2y, xy^2 \in I_1(s)$ for $s \neq 0$.

$$V_3(\rho_2) = \{x^2y, xy^2\} = \{x, y\} \cdot \{xy\} = \rho_{\mathrm{nat}} \cdot V_2(\rho_1)$$

This reminds us of the McKay rule : $\rho_2 = \rho_{\mathrm{nat}} \otimes \rho_1$

Recall

$$\boldsymbol{E}_1 := \{\boldsymbol{I} \in \boldsymbol{E};\ \boldsymbol{\rho}_1 \subset \boldsymbol{V}(\boldsymbol{I})\}$$

$$\boldsymbol{E}_2 := \{\boldsymbol{I} \in \boldsymbol{E};\ \boldsymbol{\rho}_2 \subset \boldsymbol{V}(\boldsymbol{I})\}.$$

We saw

$$\boldsymbol{E}_1 = \{\boldsymbol{I}_1(\boldsymbol{s}); \boldsymbol{s} \in \mathrm{C}\} \cup \boldsymbol{I}_1(\infty) \simeq \mathrm{P}^1$$

We will see

$$\boldsymbol{E}_2 = \{\boldsymbol{I}_2(\boldsymbol{s}); \boldsymbol{s} \in \mathrm{C}\} \cup \boldsymbol{I}_2(\infty) \simeq \mathrm{P}^1$$

$$V_3(\rho_2) = \{x^2y, xy^2\},\ \ V_5(\rho_1) = \{-y^5, x^5\}$$

Similarly for $t \neq 0$, we define

$$I_2(t) := (x^2y - ty^5, xy^2 + tx^5) + \mathfrak{n}$$

We see $V(I_2(t)) \simeq \rho_2$. Then,

$$E_2 = \{I \in E;\ \rho_2 \subset V(I)\} \simeq \mathrm{P}^1$$

As $t \to 0$, by the same reason as $I_1(\infty)$,

$$I_2(0) \neq (x^2y, xy^2) + \mathfrak{n}$$

$$I_2(0) = (x^2y, xy^2) + (x^6 - y^6) + \mathfrak{n} = I_1(\infty)$$

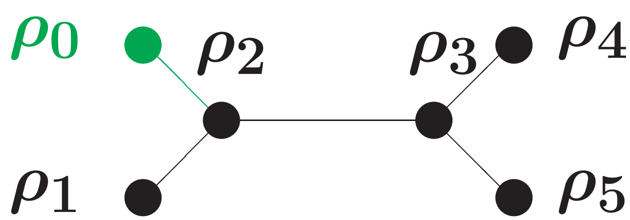
Intersection of E_1 and E_2 !!

This comes from McKay rule because

$$V_6(\rho_1) = \{x^6 - y^6\}$$
$$= \{x, y\} \cdot \{-y^5, x^5\} \quad \text{mod } \mathfrak{n}$$
$$= \rho_{\text{nat}} \cdot V_5(\rho_2) \quad \text{mod } \mathfrak{n} + (x^2y, xy^2)$$

This reminds us of McKay rule : $\rho_1 + \cdots = \rho_{\text{nat}} \otimes \rho_2$

$\widetilde{D}_5$



$$I_1(s) = \{(1/s)(xy) + (x^6 - y^6) + \mathfrak{n}$$

$$I_1(\infty) = (x^2y, xy^2) + (x^6 - y^6) + \mathfrak{n}$$

$$V_3(\rho_2) = \{x^2y, xy^2\} = \{x, y\} \cdot \{xy\} = \rho_{\mathrm{nat}} \cdot V_2(\rho_1)$$

This reminds us of McKay rule : $\rho_2 = \rho_{\mathrm{nat}} \otimes \rho_1$

$$I_2(t) = (x^2y + t(-y^5), xy^2 + tx^5) + \mathfrak{n}$$

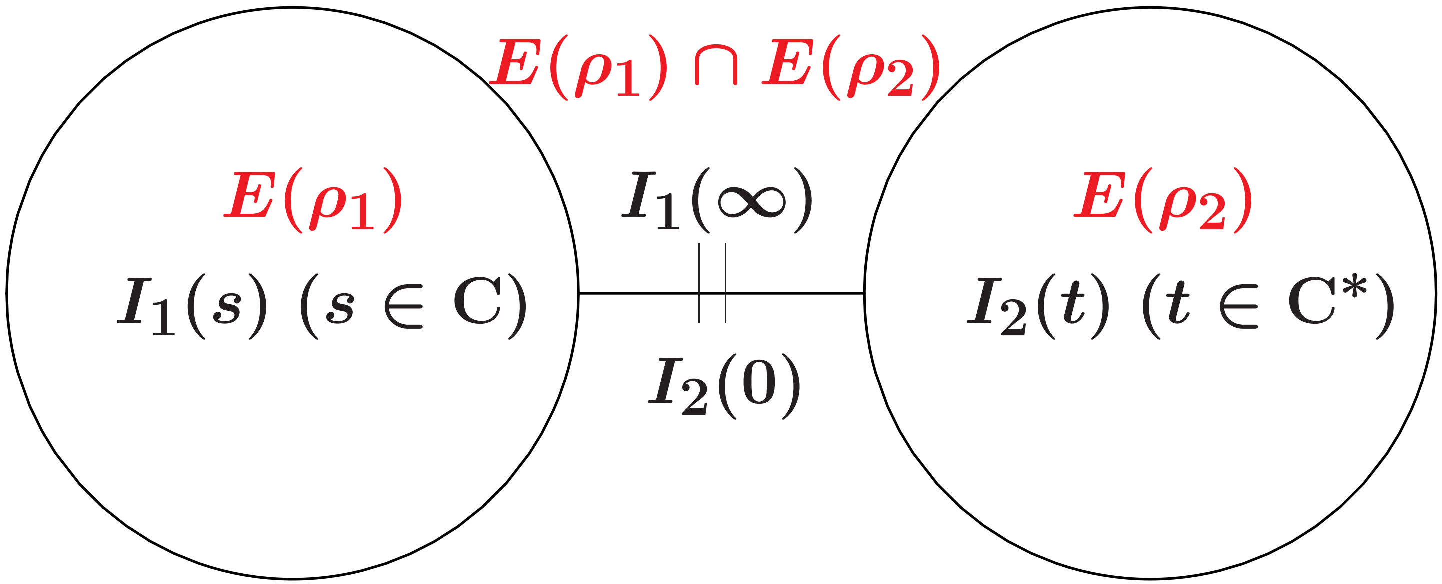
$$I_2(0) = (x^2y, xy^2) + (x^6 - y^6) + \mathfrak{n}$$

$$V_6(\rho_1) \equiv \{x^6 - y^6\} \equiv \{x, y\} \cdot \{-y^5, x^5\}$$

$$\equiv \rho_{\mathrm{nat}} \cdot V_5(\rho_2) \quad \mathrm{mod}\ \mathfrak{n} + (x^2y, xy^2)$$

This reminds us of McKay rule : $\rho_1 + \cdots = \rho_{\mathrm{nat}} \otimes \rho_2$

Let $(1/s) = 0 \quad t = 0$**, Red part changes into Green.**

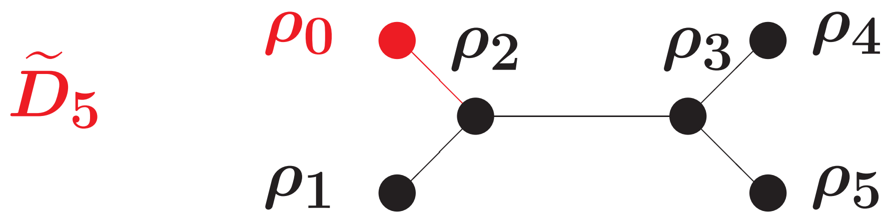


$$V(I_1(s)) \simeq \rho_1,\ V(I_2(t)) \simeq \rho_2,$$

$$V(I_1(\infty)) = V(I_2(0)) \simeq \rho_1 \oplus \rho_2$$

$$s \in \mathrm{C},\ t \in \mathrm{C}\ (s \neq 0)$$

where $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$: generators of I



$$\rho_2 \otimes \rho_{\mathrm{nat}} = \rho_0 + \rho_1 + \rho_3$$

$$\rho_1 \otimes \rho_{\mathrm{nat}} = \rho_2$$

How does the Dynkin diag. come out ?

m	$V_m(\rho)$	Eq. class
1	$\{x, y\}$	ρ_2
2	$\{xy\} \oplus \{x^2, y^2\}$	$\rho_1 + \rho_3$
3	$\{x^2y, -xy^2\} \oplus \{x^3 \pm iy^3\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
4	$\{y^4, x^4\} \oplus \{x^3y, -xy^3\}$	$\rho_3^{\oplus 2}$
5	$\{y^5, -x^5\} \oplus \{xy(x^3 \pm (-iy^3))\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
6	$\{x^6 - y^6\} \oplus \{x^5y, -xy^5\}$	$\rho_1 + \rho_3$
7	$\{xy^6, x^6y\}$	ρ_2

Decomposition of the coinv. alg. into repres. of $G(D_5)$

Coinv. alg. : $\mathrm{C}[x, y]/\mathfrak{n} = \mathrm{C}[x, y]/(F, G, H)$

Degree	**1**	**2**	**3**	**4**
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	$\rho_3^{\oplus 2}$
Degree	**7**	**6**	**5**	
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	

The quiver str. of the red part determ. the Dynkin diag.

$$\text{Let } V_1 = \{x, y\}\ V_1 \cdot V_2(\rho_1) = V_3(\rho_2),$$

$$V_1 \cdot V_3(\rho_2) \subset V_4(\rho_3),\ V_1 \cdot V_3(\rho_4) \subset V_4(\rho_3),$$

Degree	1	2	3	4
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	$\rho_3^{\oplus 2}$
Degree	7	6	5	
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	

Quiver str. of Red part determ. Dynkin.

Deg.3: $V_1 \cdot \rho_1 = \rho_2$, where, $V_1 = \{x, y\}$

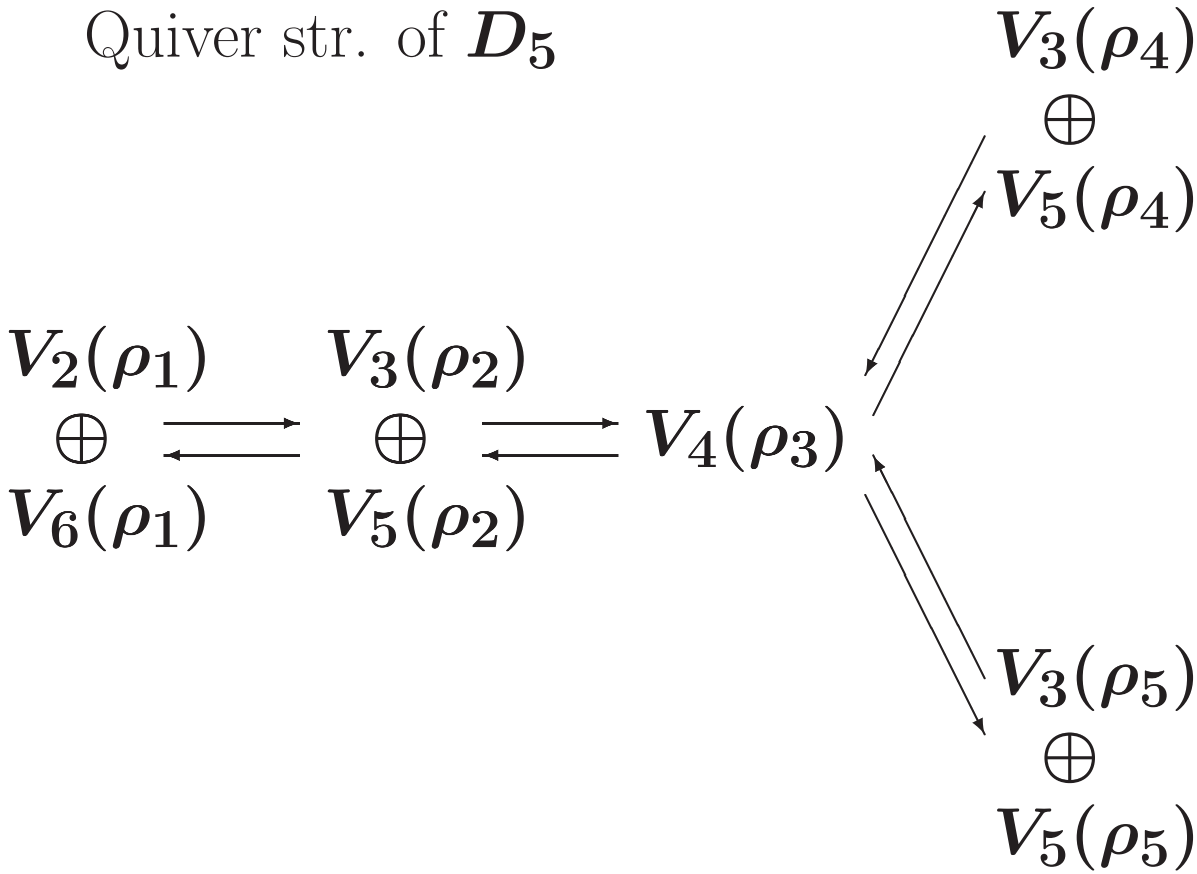
Deg.4: $V_1 \cdot \rho_2 = \rho_3$, $V_1 \cdot \rho_4 = \rho_3$, $V_1 \cdot \rho_5 = \rho_3$

Deg.5: $V_1 \cdot \rho_3^{\oplus 2} \equiv \rho_2 \oplus \rho_4 \oplus \rho_5$

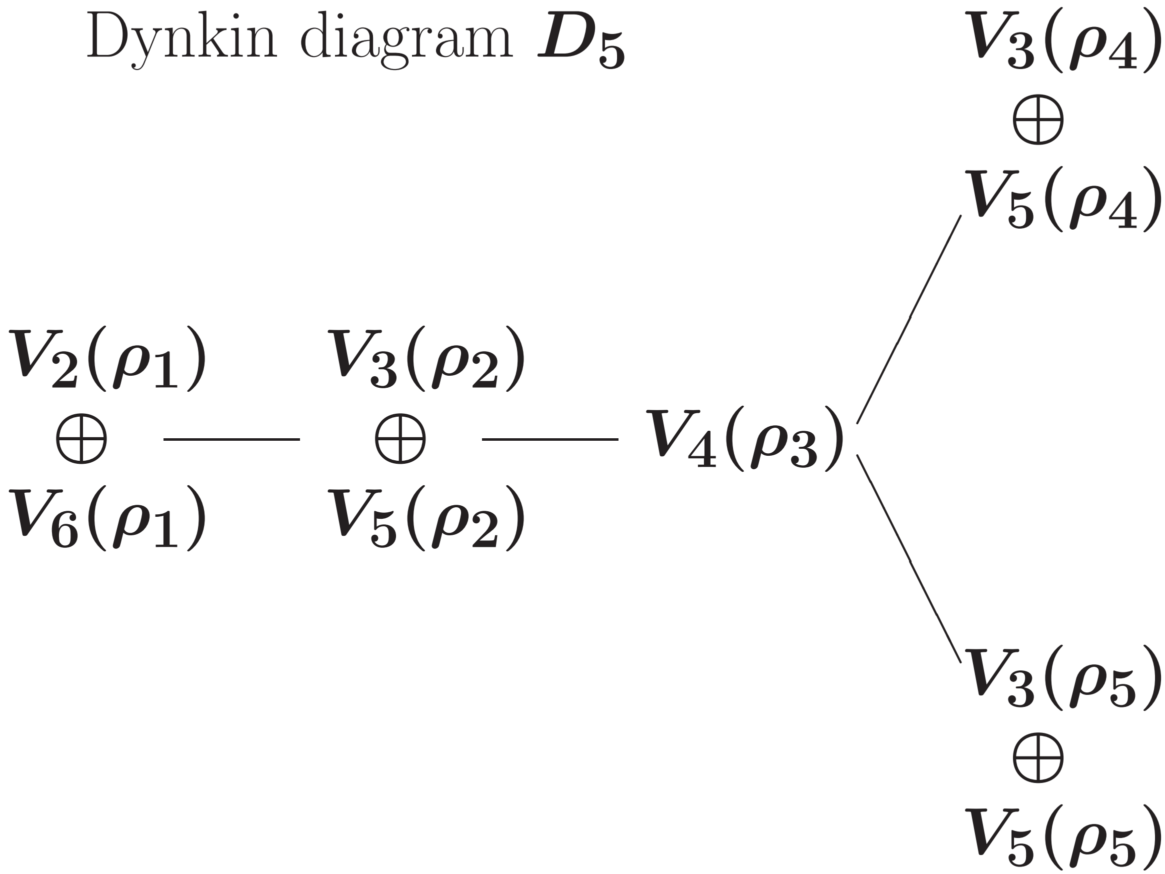
Deg.6: $V_1 \cdot \rho_2 \equiv \rho_1$, $V_1 \cdot \rho_4 \equiv 0$, $V_1 \cdot \rho_5 \equiv 0$

The green part is contained in I

Quiver str. of $\boldsymbol{D_5}$



Dynkin diagram D_5



Compute from lower degrees

$$\boldsymbol{V_2(\rho_1)} \oplus \boldsymbol{V_6(\rho_1)}$$

$$\boldsymbol{V_3(\rho_2)} \oplus \boldsymbol{V_5(\rho_2)}$$

$$\boldsymbol{V_4(\rho_3)}$$

$$\boldsymbol{V_3(\rho_4)} \oplus \boldsymbol{V_5(\rho_4)}$$

$$\boldsymbol{V_3(\rho_5)} \oplus \boldsymbol{V_5(\rho_5)}$$

7 Exceptional set— E_6-case

$G(E_6) \subset \mathrm{SL}(2, \mathrm{C})$ (Binary Tetrahedral Group)

$G(E_6) = \langle \sigma, \tau, \mu \rangle$ order 24

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^7, & \epsilon^7 \\ \epsilon^5, & \epsilon \end{pmatrix},$$

where $\epsilon = e^{2\pi i/8}$.

$G(D_4) := \langle \sigma, \tau \rangle$, normal in $G(E_6)$

$$1 \longrightarrow G(D_4) \longrightarrow G(E_6) \longrightarrow \mathrm{Z}/3\mathrm{Z} \longrightarrow 1.$$

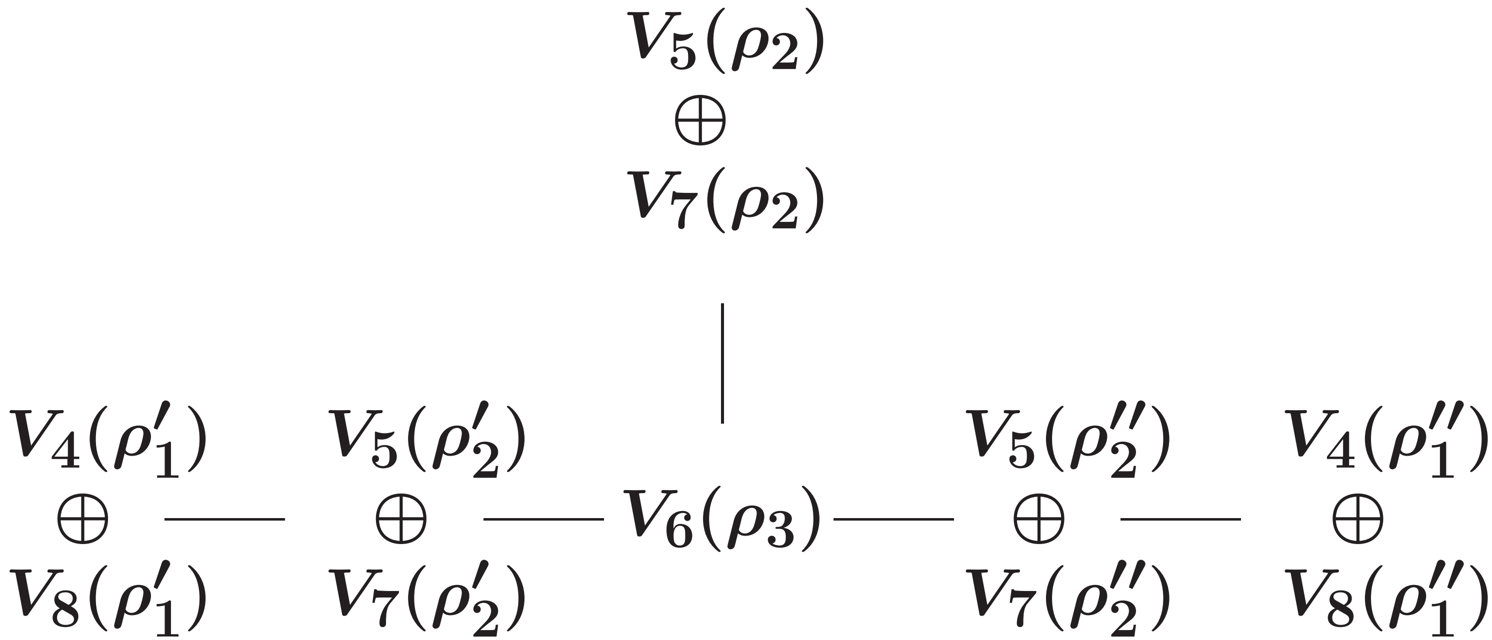
$$\mathrm{C}^2/G(D_4) \xrightarrow{3:1} \mathrm{C}^2/G(E_6)$$

Irred. rep. of E_6 (irred. character)

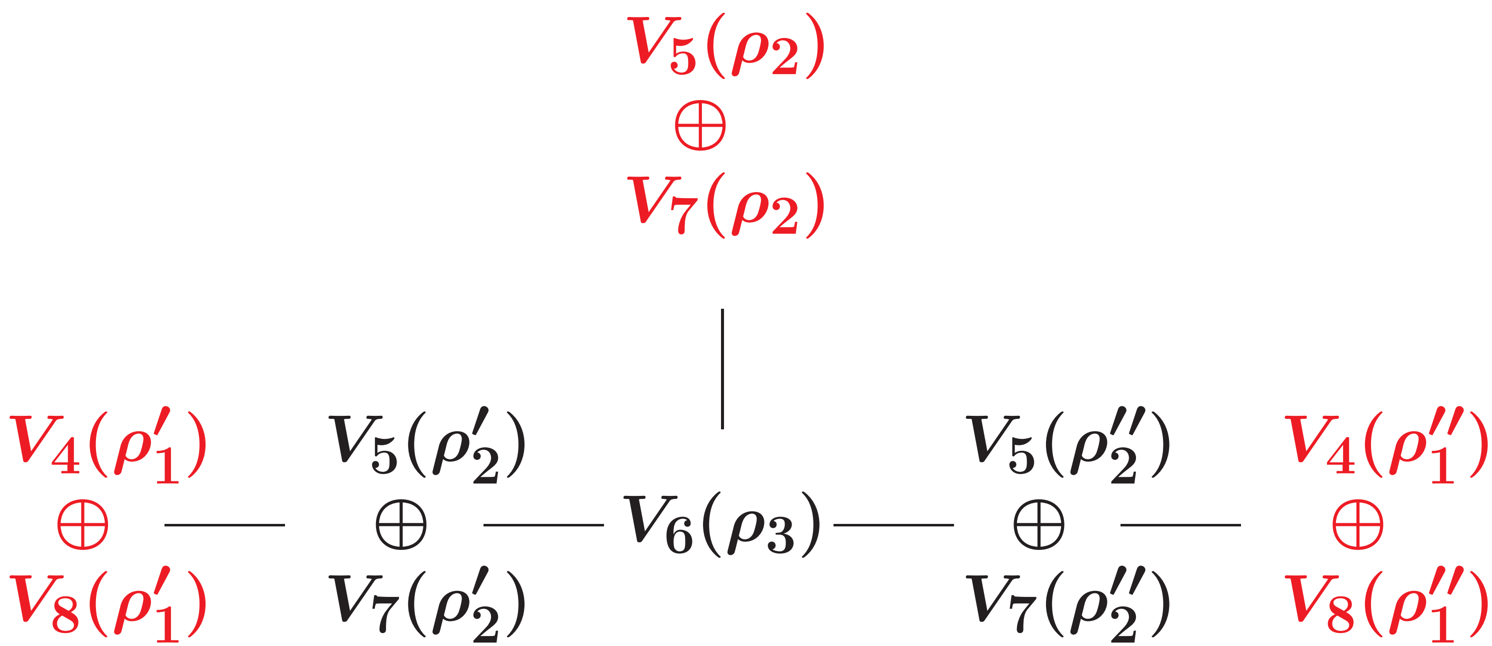
	1	-1	τ	μ	μ^2	μ^4	μ^5
$(\sharp)$	1	1	6	4	4	4	4
ρ_0	1	1	1	1	1	1	1
ρ_2	2	-2	0	1	-1	-1	1
ρ_3	3	3	-1	0	0	0	0
ρ_2'	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω
ρ_1'	1	1	1	ω^2	ω	ω^2	ω
ρ_2''	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2
ρ_1''	1	1	1	ω	ω^2	ω	ω^2

Coinv. alg. of E_6

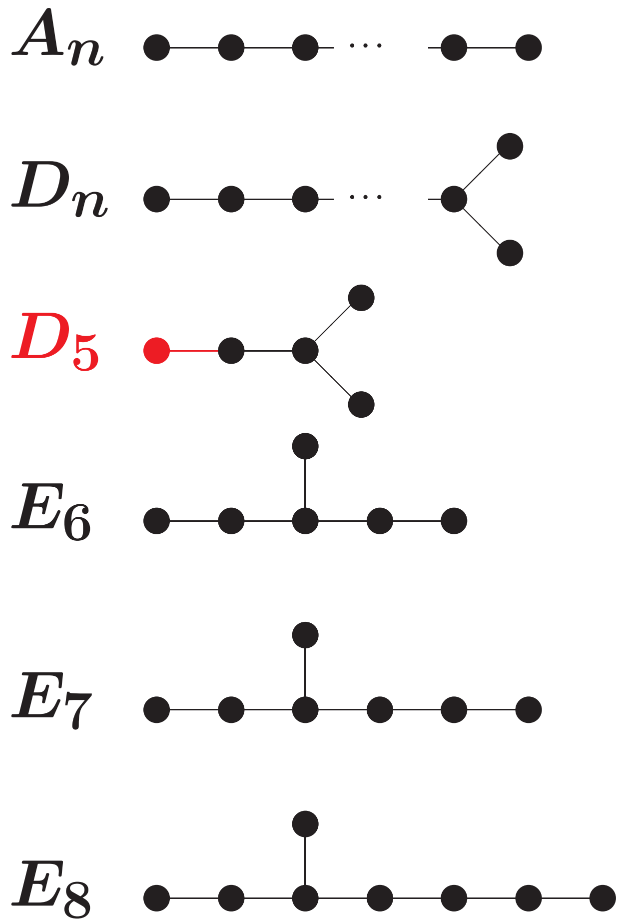
m	$\bar{V}_m$
1	ρ_2
2	ρ_3
3	$\rho_2' + \rho_2''$
4	$\rho_1' + \rho_1'' + \rho_3$
5	$\rho_2 + \rho_2' + \rho_2''$
6	$2\rho_3$
7	$\rho_2 + \rho_2' + \rho_2''$
8	$\rho_1' + \rho_1'' + \rho_3$
9	$\rho_2' + \rho_2''$
10	ρ_3
11	ρ_2



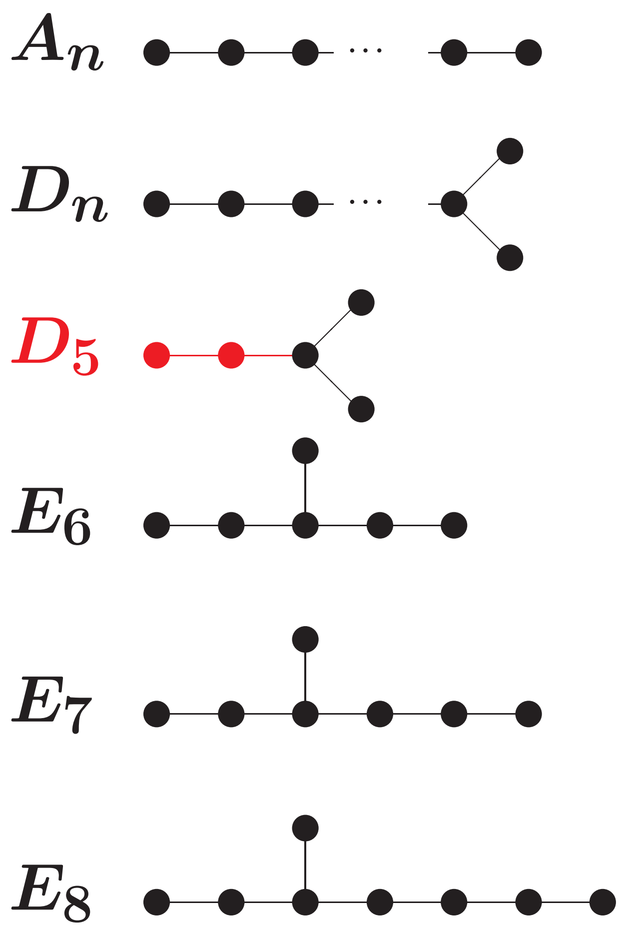
Dynkin diagram $\boldsymbol{E_6}$



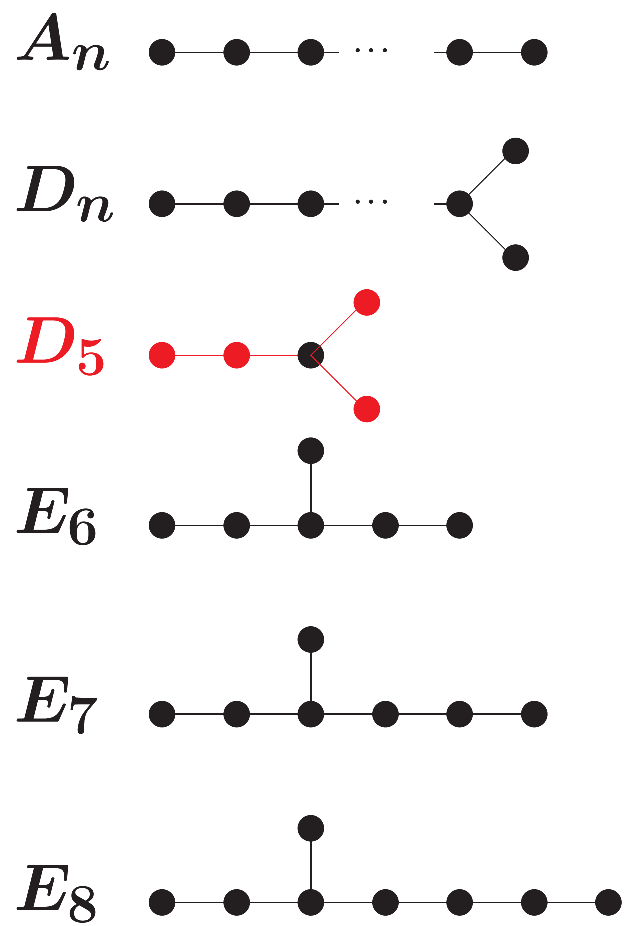
Dynkin diagram $\boldsymbol{E_6}$



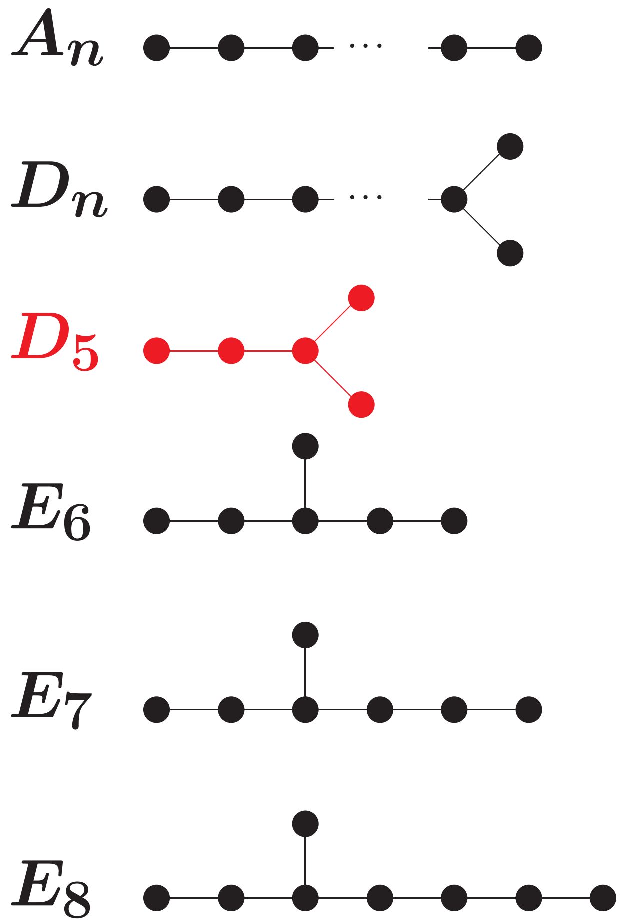
A_n
D_n
D_5
E_6
E_7
E_8



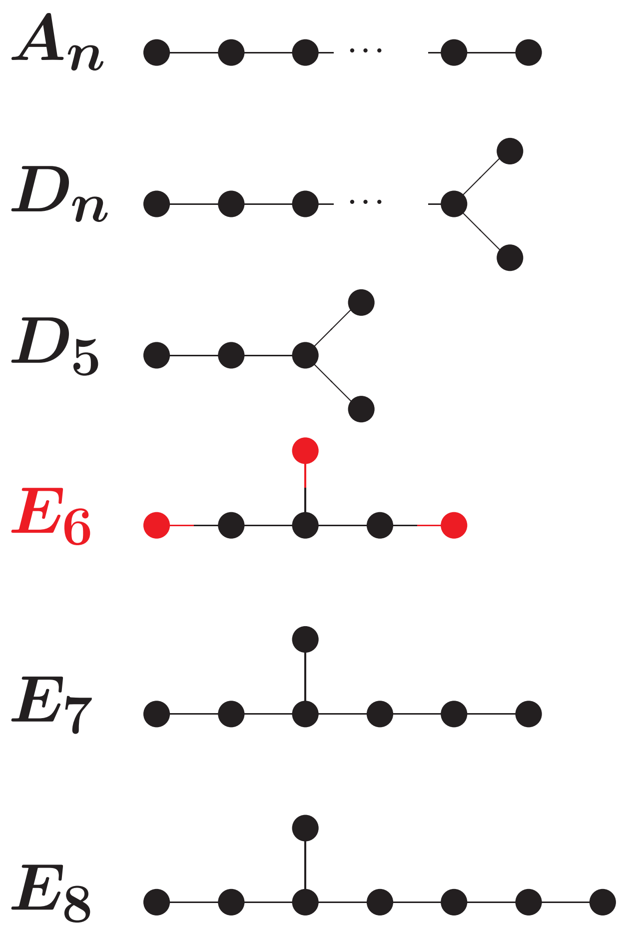
A_n
D_n
D_5
E_6
E_7
E_8

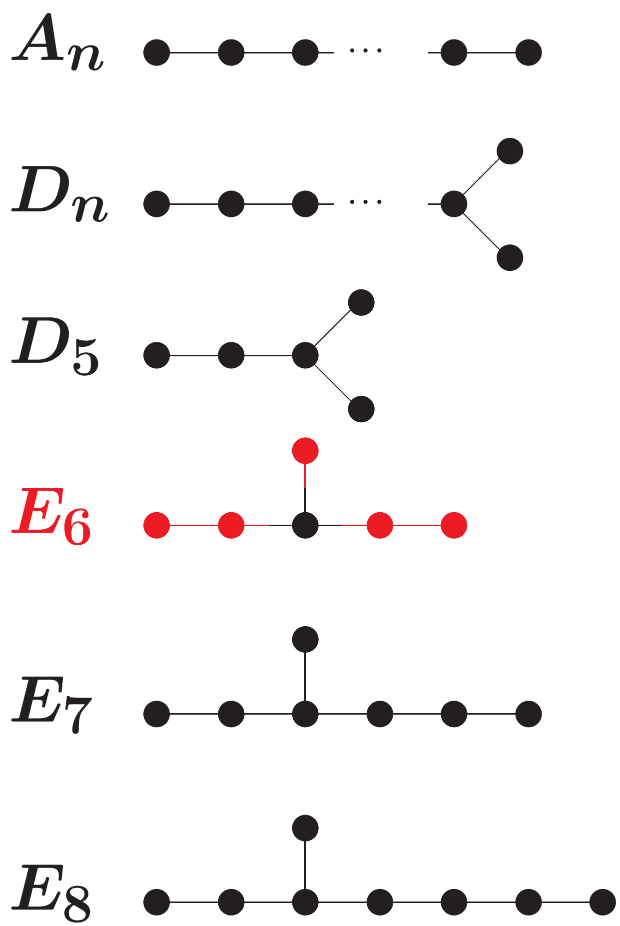


A_n
D_n
D_5
E_6
E_7
E_8

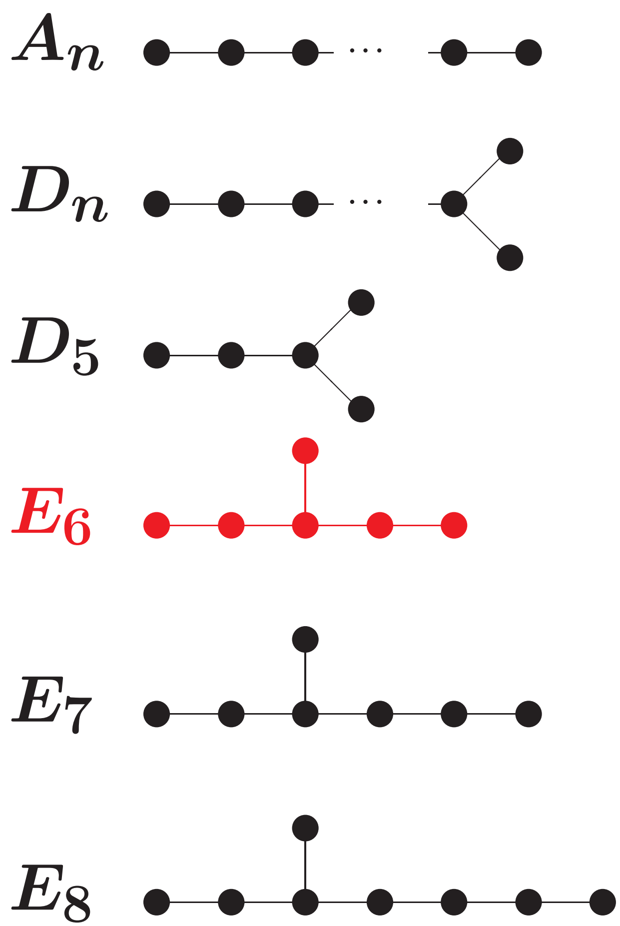


A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8



A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8

8 Summary

- Need only to compute E the exceptional set.
- Need only to see the coinv. alg. $\mathbb{C}[x,y]/(F,G,H)$
- G-inv. polyn. are zero in the coinv. alg.
- Look at coinv. alg. $\Leftrightarrow$ Forget trivial rep. in $\widetilde{D}$, $\widetilde{E}$.
- The red part of the coinv. alg. determines Dynkin.

Degree	1	2	3	4
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	$\rho_3^{\oplus 2}$
Degree	7	6	5	
Repr.	ρ_2	$\rho_1 + \rho_3$	$\rho_2 + \rho_4 + \rho_5$	

For $I \in E$: (except. set), we define

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n})$$

Then $V(I)$ is either irred. or $\rho \oplus \rho'(\rho, \rho'$ irred.)

For ρ, ρ' irred rep of $G, \rho \neq \rho'$, define subsets of E by

$$E(\rho) := \{I \in E; V(I) \supset \rho\}$$

$$P(\rho, \rho') := \{I \in E; V(I) \supset \rho \oplus \rho'\}$$

Thm 7 (Ito-Nakamura 1999)

Let G be a finite subgroup of $\mathrm{SL}(2, \mathrm{C})$. Then

(1) G-Hilb(C^2) is a min. resol. of C^2/G.

(2) For any irred rep. ρ of G, $\rho \neq$ trivial, $E(\rho) = \mathrm{P}^1$,

The map $\rho \mapsto E(\rho)$ is McKay corresp.

(3) Decomp. rule of $\rho_{\mathrm{nat}} \otimes$ into irred. rep. determine the quiver str. of the coinv. algebra, from which the intersection $P(\rho, \rho')$ between $E(\rho)$ come out.

End

Thank you for your attention.