MCKAY CORRESPONDENCE

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Abstract. We discuss the two-dimensional McKay correspondence from the view point of Hilbert schemes.

0. Introduction

There is a whole series of apparently unrelated phenomena that are governed by the so-called ADE Dynkin diagram scheme. It is widely believed that, despite the diverse nature of the objects concerned, there must be some hidden reasons for these coincidences. The ADE Dynkin diagrams provide a classification of the following types of objects:

(1) simple singularities (rational double points) of complex surfaces,
(2) finite subgroups of $\mathrm{SL}(2,\mathbb{C})$,
(3) simple Lie groups and simple Lie algebras,
(4) the following three finite simple groups, the derived group $\mathbb{F}_{24}'$ of the Fischer $\mathbb{F}_{24}$, the Baby monster $\mathbb{B}$ and the Monster $\mathbb{M}$ are related with E_6, E_7 and E_8 respectively.

Meanwhile there are three outstanding McKay observations. The first McKay observation made in November, 1978 was concerned with the so-called moonshine, and the second in December, 1978 with the connection between the above items (1) and (2), while the third in February, 1979 with the above item (4). It is the second McKay observation that we discuss in this article, which we refer to as the McKay correspondence. The purpose of this article is to discuss the McKay correspondence in detail, partially based on [IN99].

For a given finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, the McKay correspondence is incorporated into a quiver, called the McKay quiver, in the quotient of the coinvariant algebra. The McKay quiver gives the Dynkin diagram of the exceptional set of the corresponding simple singularity ADE. Moreover, the natural strata of the exceptional set are understood via the subquivers of the McKay quiver, which are easily described by directed Dynkin diagrams (See Figures 7-16). The McKay correspondence is also understood as a natural

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bijective correspondence in the irreducible decomposition of a certain coherent sheaf over the G-orbit Hilbert scheme. Any extended Dynkin diagram of ADE is realized by a quiver in the symmetric algebra extending the McKay quiver.

The present article is organized as follows. In section one we recall the simple singularities, the McKay correspondence and related notions. In section two we recall the notion of G-orbit Hilbert schemes. For a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, the G-orbit Hilbert scheme is a minimal resolution of the quotient $\mathbb{A}^2/G$ by [IN99]. In section three we recall the main theorem of [IN99] and state a new theorem which sharpens the main theorem. In section 4 we prove the McKay correspondence for D_5 in full detail, partially based on [IN99]. We introduce the (extended) McKay quiver, and the subquivers of it so that we will have a transparent overview of the strata of the exceptional set of the G-orbit Hilbert scheme. In section 5 we briefly discuss E_6 along the same line. In section 6 we prove our new theorem mentioned above.

1. Simple Singularities and McKay Correspondence

1.1. **Simple singularities.** We first recall the definition of simple singularities. A germ of a two-dimensional isolated hypersurface singularity is called a *simple singularity* if it is isomorphic to one of the following germs at the origin

$$\begin{array}{lcll} A_n & : & x^{n+1}+y^2+z^2=0 & \text{for } n\geq 1,\\ D_n & : & x^{n-1}+xy^2+z^2=0 & \text{for } n\geq 4,\\ E_6 & : & x^4+y^3+z^2=0, & \\ E_7 & : & x^3y+y^3+z^2=0, & \\ E_8 & : & x^5+y^3+z^2=0. & \end{array}$$

It is also a quotient of the germ $(\mathbb{C}^2,0)$ by a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Moreover it has a minimal resolution of singularities with exceptional set consisting of smooth rational curves of self-intersection -2 intersecting transversally. See Figure 1 for the Dynkin diagram involved.

1.2. **Finite subgroups of** $\mathrm{SL}(2,\mathbb{C})$**.** Up to conjugacy, any finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ is one of the subgroups listed in Table 1; see [Klein]. The triple (d_1,d_2,d_3) specifies the degrees of the generators of the G-invariant polynomial ring. The integer h is the Coxeter number of the Lie algebra of the type involved (see Table 1), which we also call the Coxeter number of the simple singularity $(\mathbb{A}^2/G,0)$.

1.3. **Dynkin diagrams.** Let $(S,0)$ be a germ of a simple singularity, $\pi\colon X\to S$ its minimal resolution, $E:=\pi^{-1}(0)_{\mathrm{red}}$ and E_i for $1\leq i\leq r$ the irreducible component of E. It is known that $E_i\simeq\mathbb{P}^1$ with self-intersection (-2). Let $\mathrm{Irr}\,E$ be the set $\{E_i; 1\leq i\leq r\}$ and $H^2:=H^2(X,\mathbb{Z})$. We see that $H_2=\bigoplus_{1\leq i\leq r}\mathbb{Z}[E_i]$. Then H_2 has a negative definite intersection pairing $(\ ,\ )_{\mathrm{SING}}\colon H_2\times H_2\to\mathbb{Z}$. Since $(E_iE_j)_{\mathrm{SING}}=0$ or 1 for $i\neq j$, the pairing

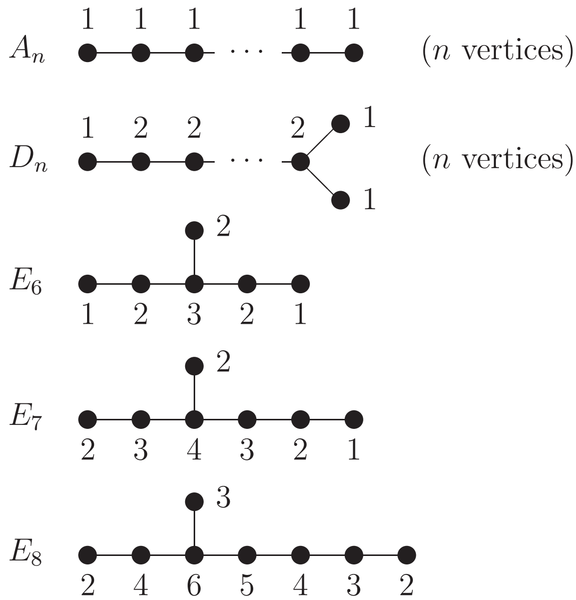

FIGURE 1. The Dynkin diagrams ADE

Type	G	name	$\|G\|$	h	(d_1, d_2, d_3)
A_n	$\mathbb{Z}_{n+1}$	cyclic	$n+1$	$n+1$	$(2, n+1, n+1)$
D_n	$\mathbb{D}_{n-2}$	binary dihedral	$4(n-2)$	$2n-2$	$(4, 2n-4, 2n-2)$
E_6	$\mathbb{T}$	binary tetrahedral	24	12	$(6, 8, 12)$
E_7	$\mathbb{O}$	binary octahedral	48	18	$(8, 12, 18)$
E_8	$\mathbb{I}$	binary icosahedral	120	30	$(12, 20, 30)$

TABLE 1. Finite subgroups of $\mathrm{SL}(2, \mathbb{C})$

$(\ ,\ )_{\mathrm{SING}}$ can be expressed by a finite graph with simple edges. We rephrase this as follows: we associate a vertex $v(E')$ to any irreducible component E' of E, and join two vertices $v(E')$ and $v(E'')$ if and only if $(E'E'')_{\mathrm{SING}} = 1$. Thus we have a finite graph with simple edges. We call this graph the *dual graph* of E, and denote it by $\Gamma_{\mathrm{SING}}(S)$ or $\Gamma(\mathrm{Irr}\, E)$.

There exists a unique divisor E_{fund}, called the *fundamental cycle* of X, which is the minimal nonzero effective divisor such that $E_{\mathrm{fund}} E_i \le 0$ for all i. Let $E_{\mathrm{fund}} := \sum_{i=1}^r m_i^{\mathrm{SING}} E_i$ and $E_0 := -E_{\mathrm{fund}}$. For the simple singularities we have $E_0 E_i = 0$ or 1 for any $E_i \in \mathrm{Irr}\, E$ (except for the case A_1, when $E_0 E_1 = 2$). Thus we can draw a new graph $\widetilde{\Gamma}_{\mathrm{SING}}$ by adding the vertex $v(E_0)$ to $\Gamma_{\mathrm{SING}}(S)$. By abuse of notation we denote $\mathrm{Irr}\, E \cup \{E_0\}$ by $\mathrm{Irr}_*\, E$. Also for a given finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$, we have a quotient singularity $(\mathbb{A}^2/G, 0)$, which is one of simple singularities so that we have a Dynkin diagram as a dual graph $\Gamma_{\mathrm{SING}}(\mathbb{A}^2/G,0)$ of the exceptional set. Also we denote $\Gamma_{\mathrm{SING}}(\mathbb{A}^2/G,0)$ by $\Gamma_{\mathrm{SING}}(G)$ and similarly $\widetilde{\Gamma}_{\mathrm{SING}}(\mathbb{A}^2/G,0)$ by $\widetilde{\Gamma}_{\mathrm{SING}}(G)$.

In the D_5 case, we have $E = E_1 + E_2 + E_3 + E_4 + E_5$ with $E_i^2 = -2$ and

$$-E_0 = E_{\mathrm{fund}} = E_1 + 2E_2 + 2E_3 + E_4 + E_5.$$

Then $E_0E_2 = E_1E_2 = E_2E_3 = E_3E_4 = E_3E_5 = 1$, and all other $E_iE_j = 0$. Hence $(m_1^{\mathrm{SING}}, \ldots, m_5^{\mathrm{SING}}) = (1, 2, 2, 1, 1)$, as indicated in Figure 2.

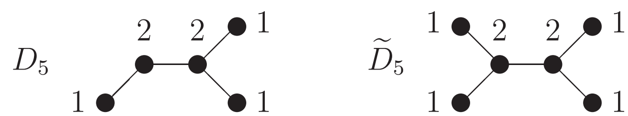

Figure 2. The Dynkin diagrams D_5 and $\widetilde{D}_5$

1.4. **McKay correspondence.** Any simple singularity is a quotient singularity by a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$, and so has a corresponding Dynkin diagram of exceptional set. McKay [McKay80] showed how one can recover the same graph purely in terms of the representation theory of G, without passing through the geometry of the germ $(\mathbb{A}^2/G, 0)$.

To be more precise, let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Clearly, G has a two-dimensional representation, which maps G injectively into $\mathrm{SL}(2,\mathbb{C})$; we call this the *natural representation* ρ_{nat}. Let $\mathrm{Irr}_* G$, respectively $\mathrm{Irr}\, G$, be the set of all equivalence classes of irreducible representations, respectively nontrivial ones. (Caution: note that this goes against the familiar notation of group theory.) Thus by definition, $\mathrm{Irr}_* G = \mathrm{Irr}\, G \cup \{\rho_0\}$, where ρ_0 is the one-dimensional trivial representation. Any representation of G over $\mathbb{C}$ is completely reducible, that is, is a direct sum of irreducible representations up to equivalence. Therefore for any $\rho \in \mathrm{Irr}_* G$, we have

$$\rho \otimes \rho_{\mathrm{nat}} = \sum_{\rho' \in \mathrm{Irr}_* G} a_{\rho,\rho'} \rho',$$

where $a_{\rho,\rho'}$ are certain nonnegative integers. In our situation, we see that $a_{\rho,\rho'} = 0$ or 1 (except for the case A_1, when $a_{\rho,\rho'} = 2$).

Let us look at the example D_5, the case of the binary dihedral group $G := \mathbb{D}_3$ of order 12. The group G is generated by σ and τ:

$$\sigma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{where } \epsilon = e^{2\pi i/6}.$$

We note that $\mathrm{Tr}(\sigma) = 1$, $\mathrm{Tr}(\tau) = 0$, hence in this case, the natural representation is ρ_2 in Table 2.

Definition 1.5. *The graph $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ is defined to be the graph consisting of vertices $v(\rho)$ for $\rho \in \mathrm{Irr}_* G$, and simple edges connecting any pair of vertices $v(\rho)$ and $v(\rho')$ with $a_{\rho,\rho'} = 1$. We denote by $\Gamma_{\mathrm{GROUP}}(G)$ the full subgraph of*

ρ	$\operatorname{Tr}\rho$	1	σ	τ
ρ_0	χ_0	1	1	1
ρ_1	χ_1	1	1	-1
ρ_2	χ_2	2	1	0
ρ_3	χ_3	2	-1	0
ρ_4	χ_4	1	-1	i
ρ_5	χ_5	1	-1	$-i$

TABLE 2. Character table of $\mathbb{D}_3$ (of type D_5)

$\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ *consisting of the vertices* $v(\rho)$ *for* $\rho \in \operatorname{Irr} G$ *and all the edges between them.*

For example, let us look at the D_5 case. Let $\chi_j := \operatorname{Tr}(\rho_j)$ be the character of ρ_j. Then from Table 2 we see that

$$\chi_2(g)\chi_{\mathrm{nat}}(g) = \chi_2(g)\chi_2(g) = \chi_0(g) + \chi_1(g) + \chi_3(g), \quad \text{for } g = 1, \sigma \text{ or } \tau.$$

Hence $\chi_2\chi_{\mathrm{nat}} = \chi_0 + \chi_1 + \chi_3$. General representation theory says that an irreducible representation of G is uniquely determined up to equivalence by its character. Therefore $\rho_2 \otimes \rho_{\mathrm{nat}} = \rho_0 + \rho_1 + \rho_3$. Hence $a_{\rho_2,\rho_j} = 1$ for $j = 0, 1, 3$ and $a_{\rho_2,\rho_j} = 0$ for $j = 2, 4, 5$. Similarly, we see that

$$\begin{aligned}
&\chi_0\chi_{\mathrm{nat}} = \chi_2, \quad \chi_1\chi_{\mathrm{nat}} = \chi_2,\\
&\chi_3\chi_{\mathrm{nat}} = \chi_2 + \chi_4 + \chi_5,\\
&\chi_4\chi_{\mathrm{nat}} = \chi_3 \quad \text{and} \quad \chi_5\chi_{\mathrm{nat}} = \chi_3.
\end{aligned}$$

In this way we obtain a graph – the extended Dynkin diagram $\widetilde{D}_5$ of Figure 3. Thus we see that there are two completely different ways to obtain the same extended Dynkin diagram $\widetilde{D}_5$ as $\widetilde{\Gamma}_{\mathrm{SING}}(\mathbb{A}^2/G, 0)$ and $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$, while D_5 as $\Gamma_{\mathrm{SING}}(\mathbb{A}^2/G, 0)$ and $\Gamma_{\mathrm{GROUP}}(G)$.

$\widetilde{D}_5$

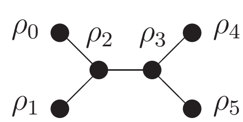

FIGURE 3. $\widetilde{\Gamma}_{\mathrm{GROUP}}(\mathbb{D}_3)$

The same is true in the other cases. Namely the two graphs $\Gamma_{\mathrm{SING}}(\mathbb{A}^2/G, 0)$ and $\Gamma_{\mathrm{GROUP}}(G)$ turn out to be one of the Dynkin diagrams ADE and coincide with each other, while both $\widetilde{\Gamma}_{\mathrm{SING}}(\mathbb{A}^2/G, 0)$ and $\widetilde{\Gamma}_{\mathrm{GROUP}}(G)$ are the corresponding extended Dynkin diagram (See Figure 4). It is also interesting to note that the degrees of the characters $\deg \rho_j = \chi_j(1)$ are equal to the multiplicities of the fundamental cycle we computed in section 1.3.

This is the second observation of McKay that we are going to discuss in this article.

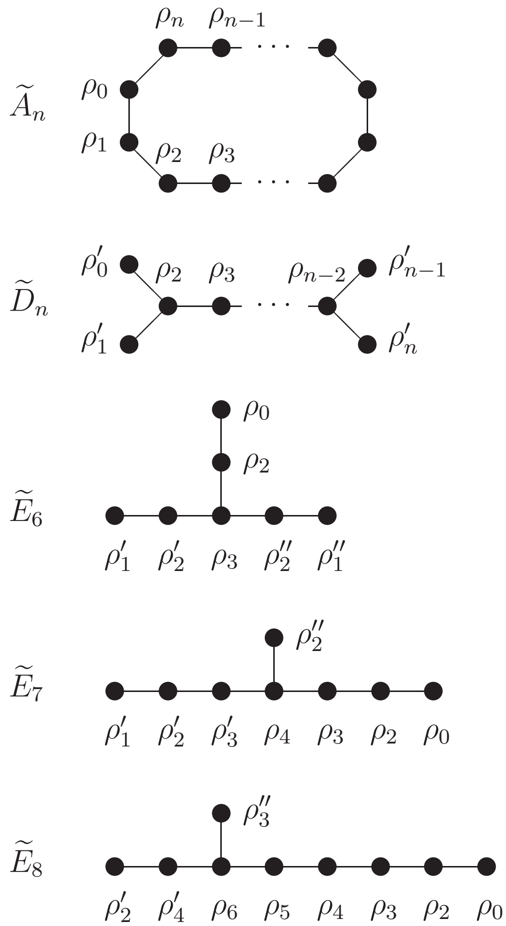

FIGURE 4. The extended Dynkin diagrams and representations

2. THE G-ORBIT HILBERT SCHEMES

2.1. **Hilbert schemes.** The Hilbert scheme of a given projective (or quasiprojective) scheme X is the scheme parametrizing all the subschemes of X. More precisely, let X be a projective scheme embedded in a projective space $\mathbb{P}^N$, and L the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to X. The *Hilbert scheme* is the scheme representing the functor

$$\mathrm{Hilb}_X\colon S \mapsto \Big\{\text{flat families } Y \text{ of subschemes of } X \text{ over } S\Big\}.$$

The Euler–Poincaré characteristic $P(m) := \sum_{q\in\mathbb{Z}}(-1)^q h^q(Y_s, L_{Y_s}^{\otimes m})$ (called the *Hilbert polynomial*) of the sheaf $L_{Y_s}^{\otimes m}$ is constant on each connected component of S. Therefore the Hilbert scheme decomposes as the disjoint union of open subsets labelled by Hilbert polynomials.

The point set Hilb_X^P classifies the subschemes of X with Hilbert polynomial equal to P. Let $U \subset X$ be an open subscheme. Then Hilb_U is an open subscheme of Hilb_X consisting of the subschemes of X whose supports are contained in U. This means that Hilb_U^P is empty or an open subscheme of Hilb_X^P for a fixed Hilbert polynomial P.

2.2. **The Hilbert scheme of n points.** In what follows we denote Hilb_X^P by $\mathrm{Hilb}^P(X)$. Write $\mathrm{S}^n(\mathbb{A}^2)$ for the nth symmetric product of the affine plane $\mathbb{A}^2$. This is by definition the quotient of the products of n copies of $\mathbb{A}^2$ by the natural permutation action of the symmetric group S_n on n letters. It is the set of formal sums of n points, in other words, the set of unordered n-tuples of points. Let $P(m) = n$ for any $m \in \mathbb{Z}$, namely let $P(m)$ be a constant polynomial. We call $\mathrm{Hilb}^n(\mathbb{A}^2) := \mathrm{Hilb}^P(\mathbb{A}^2)$ the *Hilbert scheme of n points in $\mathbb{A}^2$*. It is a quasiprojective scheme of dimension $2n$. Any $Z \in \mathrm{Hilb}^n(\mathbb{A}^2)$ is a zero-dimensional subscheme with $h^0(Z, \mathcal{O}_Z) = \dim(\mathcal{O}_Z) = n$. Suppose that Z is reduced. Then Z is the union of n distinct points. Since being reduced is an open and generic condition, $\mathrm{Hilb}^n(\mathbb{A}^2)$ contains a Zariski open subset consisting of formal sums of n distinct points.

We have a natural morphism π from $\mathrm{Hilb}^n(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)$ defined by

$$\pi\colon Z \mapsto \sum_{p \in \mathrm{Supp}(Z)} \dim(\mathcal{O}_{Z,p})p$$

which is called the *Hilbert-Chow morphism*. One of the most remarkable features of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is the following result.

Theorem 2.3 ([Fogarty68])**.** $\mathrm{Hilb}^n(\mathbb{A}^2)$ *is a smooth quasi-projective scheme, and the Hilbert-Chow morphism* $\pi\colon \mathrm{Hilb}^n(\mathbb{A}^2) \to \mathrm{S}^n(\mathbb{A}^2)$ *is a resolution of singularities of the symmetric product.*

We note that smoothness of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is peculiar to $\dim \mathbb{A}^2 = 2$.

2.4. **The G-orbit Hilbert scheme.** For any finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ of order n, we consider the Hilbert-Chow morphism π from $\mathrm{Hilb}^n(\mathbb{A}^2)$ onto $\mathrm{S}^n(\mathbb{A}^2)$. Since the morphism $\pi : \mathrm{Hilb}^n(\mathbb{A}^2) \to \mathrm{S}^n(\mathbb{A}^2)$ is G-equivariant, we have a natural morphism between G-fixed point loci. We note that the G-fixed point set of $\mathrm{S}^n(\mathbb{A}^2)$ is nothing but $\mathbb{A}^2/G$ because $n = |G|$. The G-fixed point set $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is always nonsingular, but could a priori be disconnected. There is however a unique irreducible component of $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ dominating $\mathrm{S}^n(\mathbb{A}^2)^G$, which we denote by $\mathrm{Hilb}^G(\mathbb{A}^2)$ and call it the *G-orbit Hilbert scheme*. It is a G-invariant subscheme of $\mathrm{Hilb}^n(\mathbb{A}^2)$ that parametrizes all smoothable G-invariant subschemes of length $|G|$.

Now we recall the following theorem proved in [IN99].

Theorem 2.5. *Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a finite subgroup of order n. Then there is a unique irreducible component* $\mathrm{Hilb}^G(\mathbb{A}^2)$ *of* $\mathrm{Hilb}^n(\mathbb{A}^2)^G$ *dominating* $\mathbb{A}^2/G$, *which is a minimal resolution of $\mathbb{A}^2/G$. In particular, the dualizing sheaf of* $\mathrm{Hilb}^G(\mathbb{A}^2)$ *is trivial.*

Proof. Any point of $\mathrm{S}^n(\mathbb{A}^2)^G \setminus \{0\}$ is a G-orbit of a point $0 \neq \mathfrak{p} \in \mathbb{A}^2$, which is a reduced zero-dimensional subscheme invariant under G. It follows that $\mathrm{Hilb}^G(\mathbb{A}^2)$ is isomorphic to $\mathrm{S}^n(\mathbb{A}^2)^G$ ($\simeq \mathbb{A}^2/G$) over $\mathrm{S}^n(\mathbb{A}^2)^G \setminus \{0\}$ under the Hilbert-Chow morphism. Hence $\mathrm{Hilb}^G(\mathbb{A}^2)$ is birationally equivalent to $\mathbb{A}^2/G$), so that it is a resolution of $\mathbb{A}^2/G$. Moreover by [Fujiki83], Proposition 2.6, $\mathrm{Hilb}^G(\mathbb{A}^2)$ inherits a canonical holomorphic symplectic structure from $\mathrm{Hilb}(\mathbb{A}^2)$. Since $\dim \mathrm{Hilb}^G(\mathbb{A}^2) = \dim \mathbb{A}^2/G = 2$, this implies that the dualizing sheaf of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is trivial. This proves the theorem. $\square$

We denote the natural morphism from $\mathrm{Hilb}^G(\mathbb{A}^2)$ onto $\mathbb{A}^2/G$ by the same letter π. There are two corollaries to Theorem 2.5, useful for explicit computations. We only quote these from [IN99]. See [IN99] for proofs.

Corollary 2.6. *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$*, and I an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ (to be exact, the subscheme defined by I belonging to $\mathrm{Hilb}^G(\mathbb{A}^2)$). Then as G-modules $\mathcal{O}_{\mathbb{A}^2}/I \simeq \mathbb{C}[G] \simeq \sum_\rho (\deg \rho)\rho$, the group algebra (the regular representation).*

Corollary 2.7. *Let I be an ideal of $\mathcal{O}_{\mathbb{A}^2}$ with $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$. Any G-invariant function vanishing at the origin is contained in I.*

3. Theorems

3.1. **A link from $\mathrm{Hilb}^G(\mathbb{A}^2)$ to McKay correspondence.** For any finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ of order n, the *G-orbit Hilbert scheme* $\mathrm{Hilb}^G(\mathbb{A}^2)$ of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is a minimal resolution of the simple singularity $S_G := \mathbb{A}^2/G$. Let $\pi : \mathrm{Hilb}^G(\mathbb{A}^2) \to S_G$ be the natural morphism.

Let $S := \mathbb{C}[x,y]$. Let $\mathfrak{m}$ be the ideal of S generated by x and y, $\mathfrak{n}$ the ideal of S generated by all the G-invariant polynomials vanishing at the origin. A point $\mathfrak{p}$ of $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a G-invariant zero-dimensional subscheme Z of $\mathbb{A}^2$. Since $\mathbb{A}^2$ is affine, we can associate to it the G-invariant ideal $I := I_Z$ of $S = \mathbb{C}[x,y]$ defining Z with $S/I \simeq \mathbb{C}[G]$, the regular representation of G. Thus often we write $I \in \mathrm{Hilb}^G(\mathbb{A}^2)$ instead of $Z \in \mathrm{Hilb}^G(\mathbb{A}^2)$ or $\mathfrak{p} \in \mathrm{Hilb}^G(\mathbb{A}^2)$. We also have the exact sequence

$$0 \to I \to S \to \Gamma(Z, \mathcal{O}_Z) \to 0.$$

Any point of the exceptional set E of π is a G-invariant zero-dimensional subscheme Z of $\mathbb{A}^2$ with support the origin. In other words, the ideal I of S defining Z is an infinite-dimensional G-module contained in $\mathfrak{m}$. By deriving a finite-dimensional G-module from it naturally, we will give the McKay correspondence as described in section 1. We shall introduce the McKay quiver (in the quotient of the coinvariant algebra) and subquivers so as to understand the natural stratification of $\mathrm{Hilb}^G(\mathbb{A}^2)$. Then the corresponding extended

Dynkin diagram emerges naturally (See Figure 6) by extending the McKay quiver in the polynomial ring. We also have a natural irreducible decomposition of a certain universal coherent sheaf on $\mathrm{Hilb}^G(\mathbb{A}^2)$. This somewhat sharpens the McKay correspondence (Theorem 3.9).

3.2. **A stratification of $\mathrm{Hilb}^G(\mathbb{A}^2)$ by $\mathrm{Irr}\, G$.** Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. In what follows we assume that G is not cyclic because A_n-case is much easier. As in section 1.4, we write $\mathrm{Irr}\, G$ for the set of all the equivalence classes of nontrivial irreducible G-modules, and $\mathrm{Irr}_*\, G$ for the union of $\mathrm{Irr}\, G$ and the trivial one-dimensional G-module.

Let E the exceptional set of the minimal resolution, $\mathrm{Irr}\, E$ the set of irreducible components of E.

Any $I \in X$ contained in E is a G-invariant ideal of S which contains $\mathfrak{n}$ by Corollary 2.7. For any ρ, ρ', and $\rho'' \in \mathrm{Irr}\, G$, we define

$$\begin{aligned} V(I) &:= I/(\mathfrak{m}I + \mathfrak{n}), \\ E(\rho) &:= \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset \rho \right\}, \\ P(\rho, \rho') &:= \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset \rho \oplus \rho' \right\}, \\ Q(\rho, \rho', \rho'') &:= \left\{ I \in \mathrm{Hilb}^G(\mathbb{A}^2); V(I) \supset \rho \oplus \rho' \oplus \rho'' \right\}. \end{aligned}$$

Any nonempty stratum will be described in a simple manner by using Dynkin diagrams with directed edges, called McKay subquivers. See subsection 4.8, Figures 7-12.

Definition 3.3. *Two irreducible G-modules ρ and ρ' are said to be* adjacent *if $\rho \otimes \rho_{\mathrm{nat}} \supset \rho'$ or vice versa.*

The Dynkin diagram $\Gamma(\mathrm{Irr}\, G)$ *of* $\mathrm{Irr}\, G$ *is the graph whose vertices are* $\mathrm{Irr}\, G$*, with ρ and ρ' joined by a simple edge if and only if ρ and ρ' are adjacent.*

Then the following theorem was proved in [IN99].

Theorem 3.4. *Let G be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$*. Then*

(1) *the map $\rho \mapsto E(\rho)$ is a bijective correspondence between* $\mathrm{Irr}\, G$ *and* $\mathrm{Irr}\, E$*;*
(2) *$E(\rho)$ is a smooth rational curve with $E(\rho)^2 = -2$ for any $\rho \in \mathrm{Irr}\, G$;*
(3) *$P(\rho, \rho') \neq \emptyset$ if and only if ρ and ρ' are adjacent. In this case $P(\rho, \rho')$ is a single (reduced) point, at which $E(\rho)$ and $E(\rho')$ intersect transversally;*
(4) *$P(\rho, \rho) = Q(\rho, \rho', \rho'') = \emptyset$ for any $\rho, \rho', \rho'' \in \mathrm{Irr}\, G$.*

By Theorem 3.4, (3), $\Gamma(\mathrm{Irr}\, G)$ is the same as the dual graph $\Gamma(\mathrm{Irr}\, E)$ of E, in other words, the Dynkin diagram of the singularity $(\mathbb{A}^2/G, 0)$.

In what follows we assume that G is not cyclic. We define nonnegative integers $d(\rho)$ for any $\rho \in \mathrm{Irr}\, G$ as follows. Since G is not cyclic, $\Gamma(\mathrm{Irr}\, G)$ is star-shaped with a unique centre. For any $\rho \in \mathrm{Irr}\, G$, we define $d(\rho)$ to be the distance from the vertex ρ to the centre, where $d(\rho) = 0$ for the centre ρ. It is obvious that $d(\rho) = d(\rho') \pm 1$ if ρ and $\rho' \in \mathrm{Irr}\, G$ are adjacent.

Let $S = \mathbb{C}[x, y]$. For any positive integer k let S_k be the subspace of homogeneous polynomials of degree k in S. We say that a G-submodule W of $\mathfrak{m}/\mathfrak{n}$ is *homogeneous* of degree k if it is generated over $\mathbb{C}$ by homogeneous polynomials of degree k. The G-module $\mathfrak{m}/\mathfrak{n}$ splits as a direct sum of irreducible homogeneous G-modules. If W is a direct sum of homogeneous G-submodules, then we denote the homogeneous part of W of degree k by $S_k(W)$. For any G-module W in some $S_k(\mathfrak{m}/\mathfrak{n})$, we write $S_j \cdot W$ for the G-submodule of $S_{k+j}(\mathfrak{m}/\mathfrak{n})$ generated over $\mathbb{C}$ by the products of $S_j(\mathfrak{m}/\mathfrak{n})$ and W. We denote by $W[\rho]$ the ρ factor of W, that is, the sum of all the copies of ρ in W; and similarly, we denote by $[W : \rho]$ the multiplicity of $\rho \in \operatorname{Irr} G$ in a G-module W.

Definition 3.5. *The quotient algebra $S/\mathfrak{n}$ is called the coinvariant algebra of G, denoted by* $\operatorname{Coinv}(G)$ *(or denoted by* $\operatorname{Coinv}($*the type of* $S_G)$ *with the notation of Table 1). Let h be the Coxeter number of the simple singularity S_G. Then we define the very positive part $S^\dagger := S^\dagger(\mathfrak{m}/\mathfrak{n})$ of $S/\mathfrak{n}$ to be*

$$S^\dagger := \sum_{k > \frac{h}{2} + d(\rho), \rho \in \operatorname{Irr} G} S_k(\mathfrak{m}/\mathfrak{n})[\rho].$$

We also define the McKay quiver of G by

$$S_{\mathrm{McKay}}(G) = \sum_{\rho \in \operatorname{Irr} G} S_{\frac{h}{2} \pm d(\rho)}(\mathfrak{m}/\mathfrak{n})[\rho] + S^\dagger/S^\dagger,$$

which we also denote by $S_{\mathrm{McKay}}($*the type of* $S_G)$. *We also define $V_k(\rho) = S_k(\mathfrak{m}/\mathfrak{n})[\rho]$ and $V(\rho) = S_{\mathrm{McKay}}(G)[\rho]$.*

The following Lemma 3.6 and Lemma 3.7 describe the structure of the McKay quiver $S_{\mathrm{McKay}}(G)$. See [IN99] for the proofs.

Lemma 3.6. *Let φ_i be three generators of G-invariants, $d_i = \deg \varphi_i$ and h the Coxeter number of the simple singularity S_G. Then $d_1 + d_2 = d_3 + 2$ and $d_3 = h$, and moreover $S_k(\mathfrak{m}/\mathfrak{n}) = 0$ for $k \geq d_3$.*

Lemma 3.7. *Assume that G is not cyclic. Let h be the Coxeter number of the simple singularity S_G. Then as G-modules, we have*

(1) $\mathfrak{m}/\mathfrak{n} = \sum_{\rho \in \operatorname{Irr} G} 2(\deg \rho)\rho$;
(2) $S_{\mathrm{McKay}}(G) \simeq \sum_{\rho \in \operatorname{Irr} G} 2\rho$;
(3) $V_{\frac{h}{2} - d(\rho)}(\rho) \simeq V_{\frac{h}{2} + d(\rho)}(\rho) = \rho$ *if* $d(\rho) \geq 1$ *and* $V_{\frac{h}{2}}(\rho) = \rho^{\oplus 2}$ *if* $d(\rho) = 0$;
(4) $S_{\frac{h}{2} - k}(\mathfrak{m}/\mathfrak{n}) \simeq S_{\frac{h}{2} + k}(\mathfrak{m}/\mathfrak{n})$ *for any* k.

Using this notation we see by Lemma 3.7, (2) that $V(\rho) = V_{\frac{h}{2} - d(\rho)}(\rho) + V_{\frac{h}{2} + d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ and that the subset $E(\rho)$ is $\mathbb{P}(V(\rho))$, the set of all nontrivial G-submodules of $\rho^{\oplus 2}$. This is isomorphic to the projective line, or a smooth rational curve by Schur's lemma. This proves Theorem 3.4, (2).

3.8. **The ideals $\mathfrak{n}_X$ and $\pi^*\overline{\mathfrak{m}}$.** Since $(\mathbb{A}^2/G) \times_{(\mathbb{A}^2/G)} X \simeq X$, X is a closed subscheme of $(\mathbb{A}^2/G) \times X$, which is defined by an ideal I_X of $O_{\mathbb{A}^2/G} \otimes O_X$. Let $\mathfrak{n}_X$ be the ideal of $O_{\mathbb{A}^2} \otimes O_X$ generated by I_X. Let Z_{univ} be the universal subscheme of $\mathbb{A}^2 \times X$, and I_{univ} the ideal sheaf of $O_{\mathbb{A}^2 \times X}$ defining Z_{univ}. Since we have a commutative diagram

$$\begin{array}{ccc} Z_{\mathrm{univ}} & \xrightarrow{pr_2} & X \\ {\scriptstyle pr_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathbb{A}^2 & \xrightarrow{\phi} & \mathbb{A}^2/G, \end{array}$$

the morphism $\phi \times \mathrm{id}_X : \mathbb{A}^2 \times X \to \mathbb{A}^2/G \times X$ sends Z_{univ} into $(\mathbb{A}^2/G) \times_{(\mathbb{A}^2/G)} X \simeq X$. This implies that $\mathfrak{n}_X = I_X O_{\mathbb{A}^2} \otimes O_X \subset I_{\mathrm{univ}}$. Now we define

$$\mathcal{V} := V(I_{\mathrm{univ}}) = I_{\mathrm{univ}}/\mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X.$$

Let $\overline{\mathfrak{m}}$ be the maximal ideal of $O_{\mathbb{A}^2/G}$ of the unique singular point. We note $\mathfrak{n} = \Gamma(\mathbb{A}^2, \phi^*\overline{\mathfrak{m}})$. We also note $\pi^*(\overline{\mathfrak{m}}) \cap I_{\mathrm{univ}} = \{0\}$. In fact, suppose $\pi^*(F) \in I_{\mathrm{univ}}$. Then $\pi^*(F) = 0$ on Z_{univ}. Since Z_{univ} is surjective over $\mathbb{A}^2/G$, $F = 0$. It follows $\pi^*(\overline{\mathfrak{m}}) \cap I_{\mathrm{univ}} = \{0\}$.

Since $\pi^*\overline{\mathfrak{m}} := \overline{\mathfrak{m}}O_X$ is the defining ideal of E_{fund} by [Artin66], we see that $\mathcal{V}$ is a finite $O_{E_{\mathrm{fund}}}$-module. See section 6 for the proof. However we prove a little stronger

Theorem 3.9. *The $O_{\mathbb{A}^2 \times_{\mathbb{C}} X}$-module $\mathcal{V}$ is a finite O_E-module; as an $O_{\mathbb{A}^2 \times_{\mathbb{C}} X}$-module with G-action, we have an isomorphism*

$$\mathcal{V} \simeq \bigoplus_{\rho \in \mathrm{Irr}\, G} \rho \otimes_{\mathbb{C}} O_{E(\rho)}(-1).$$

In section 6 we will prove Theorem 3.9. In order to prepare for the proof of Theorem 3.9, we recall in section 4 the proof of Theorem 3.4 and related constructions in the case of D_5 in full detail. In the cases D_n ($n \neq 5$) and E_6 we give only a sketch of proofs of Theorem 3.9. Though we give almost no proof for E_7 because we need more notation, we remark that we can prove it in the same manner. It remains to check E_8.

4. The Simple Singularity D_5

In this section we explain the case of D_5 in full detail.

4.1. **The binary dihedral group $\mathbb{D}_3$.** The simple singularity D_5 is the quotient singularity of $\mathbb{A}^2$ by the binary dihedral group $G := \mathbb{D}_3$ of order 12, which is generated by σ and τ:

$$\sigma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\epsilon := e^{2\pi i/6}$. We have $\sigma^6 = \tau^4 = 1$, $\sigma^3 = \tau^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. The ring of G-invariants in $\mathbb{C}[x, y]$ is generated by three elements $A_6 := x^6 + y^6$,

$A_8 := xy(x^6 - y^6)$ and $A_4 := x^2y^2$. The quotient $\mathbb{A}^2/G$ is isomorphic to the hypersurface $4A_4^4 + A_8^2 - A_4A_6^2 = 0$.

k	S_k	$S_k(S/\mathfrak{n})$
0	ρ_0	ρ_0
1	ρ_2	ρ_2
2	$\rho_1 + \rho_3$	$(\rho_1) + \rho_3$
3	$\rho_2 + \rho_4 + \rho_5$	$(\rho_2 + \rho_4 + \rho_5)$
4	$[\rho_0] + 2\rho_3$	$(2\rho_3)$
5	$2\rho_2 + \rho_4 + \rho_5$	$(\rho_2 + \rho_4 + \rho_5)$
6	$[\rho_0] + 2\rho_1 + 2\rho_3$	$(\rho_1) + \rho_3$
7	$3\rho_2 + \rho_4 + \rho_5$	ρ_2

TABLE 3. Irreducible decompositions of S and $\mathrm{Coinv}(D_5)$

k	$V_k(\rho) \ \subset S_k(S/\mathfrak{n})$	equiv.class
0	$\{1\}_{\rho_0}$	ρ_0
1	$\{x, y\}_{\rho_2}$	ρ_2
2	$\{xy\}_{\rho_1} \oplus \{x^2, y^2\}_{\rho_3}$	$(\rho_1) + \rho_3$
3	$\{x^2y, -xy^2\}_{\rho_2}$	$(\rho_2 + \rho_4 + \rho_5)$
	$\oplus\{x^3 + iy^3\}_{\rho_4} \oplus \{x^3 - iy^3\}_{\rho_5}$	
4	$\{y^4, x^4\}_{\rho_3} \oplus \{x^3y, -xy^3\}_{\rho_3}$	$(2\rho_3)$
5	$\{y^5, -x^5\}_{\rho_2}$	$(\rho_2 + \rho_4 + \rho_5)$
	$\oplus\{xy(x^3 - iy^3)\}_{\rho_4} \oplus \{xy(x^3 + iy^3)\}_{\rho_5}$	
6	$\{x^6 - y^6\}_{\rho_1} \oplus \{xy^5, -x^5y\}_{\rho_3}$	$(\rho_1) + \rho_3$
7	$\{xy^6, x^6y\}_{\rho_2}$	ρ_2

TABLE 4. $\mathrm{Coinv}(D_5)$

<table>
<tr><td>degree</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>equiv.class</td><td>ρ_2</td><td>$(\rho_1) + \rho_3$</td><td>$(\rho_2 + \rho_4 + \rho_5)$</td><td rowspan="3">$(2\rho_3)$</td></tr>
<tr><td>degree</td><td>7</td><td>6</td><td>5</td></tr>
<tr><td>equiv.class</td><td>ρ_2</td><td>$(\rho_1) + \rho_3$</td><td>$(\rho_2 + \rho_4 + \rho_5)$</td></tr>
</table>

TABLE 5. Dual pairs of $\mathrm{Coinv}(D_5)$

4.2. **The McKay quiver, tables and figures.** We consider the case of D_5. By an elementary computation we have the irreducible decomposition of the coinvariant algebra $\mathrm{Coinv}(D_5)$ in Table 3. As Tables 3-5 indicate, $\mathrm{Coinv}(D_5)$ consists of dual pairs. The McKay quiver $S_{\mathrm{McKay}}(D_5)$ of $\mathrm{Coinv}(D_5)$, that is, the irreducible factors in the parentheses in Tables 3 - 5 consist of dual pairs, exactly one pair for each equivalence class $\rho \in \mathrm{Irr}\, G$ except ρ_3, while $V_4(\rho_3) = 2\rho_3$ is self-dual.

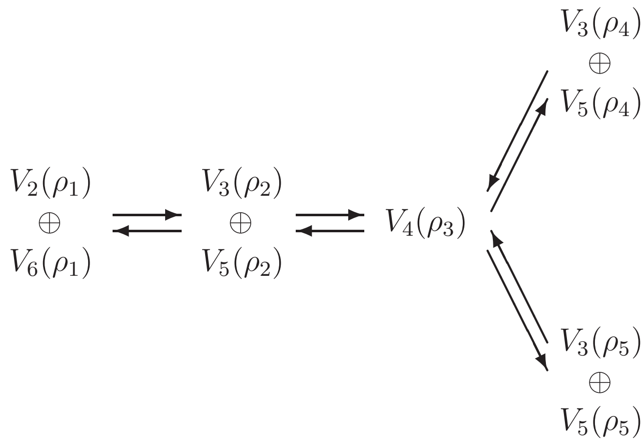

FIGURE 5. The McKay quiver of D_5

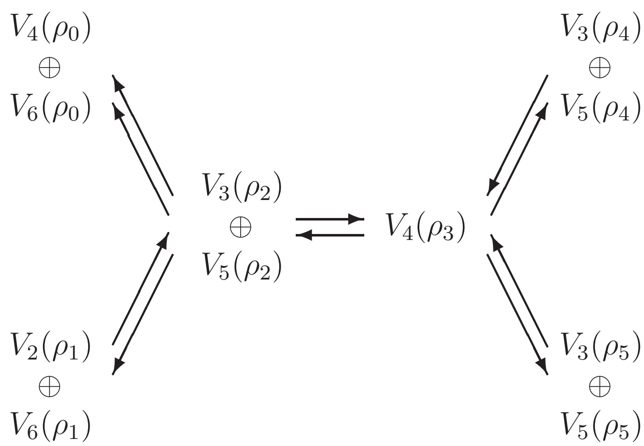

FIGURE 6. The extended McKay quiver of D_5

The coinvariant algebra $\mathrm{Coinv}(D_5)$ admits a quiver structure induced from multiplication of the symmetric algebra. This induces a quiver structure on $S_{\mathrm{McKay}}(D_5)$, which we call the McKay quiver of D_5. It yields naturally a corresponding Dynkin diagram of the minimal resolution $X := \mathrm{Hilb}^G(\mathbb{A}^2)$, as we see later. In other words, the McKay quiver of D_5 gives the Dynkin diagram D_5 of the exceptional set E of X by replacing each pair of arrows by an edge. To describe the strata of E precisely we also introduce the subquivers $\mathrm{Quiv}(\rho)$ and $\mathrm{Quiv}(\rho,\rho')$, which are Dynkin diagrams with certain directed edges. See Figures 7-16. These enable us to specify which of the G-invariants A_4, A_6 and A_8 generates the ideal $\pi^*(\mathfrak{n})$ along each $E(\rho)$.

Moreover in order to obtain the extended Dynkin diagram $\widetilde{D}_5$ it seems natural to add to Figure 5 the G-submodules $V_4(\rho_0) := S_4[\rho_0]$ and $V_6(\rho_0) := S_6[\rho_0]$ as in Figure 6, where $V_4(\rho_0)$ and $V_6(\rho_0)$, the parts of Table 3 in the bracket, are generated by the generators $A_4 = x^2y^2$ and $A_6 = x^6 + y^6$ of G-invariants respectively.

We note that the arrows between ρ_2 to ρ_0 are exceptional in the sense that both the arrows between them are directed from ρ_2 to ρ_0, while in any other cases, say for ρ_1 and ρ_2, both directions are taken by arrows between them. As its consequence $\deg V_4(\rho_0) + \deg V_6(\rho_0) = 10 > 8 = 2+6 = 3+5 = 4+4$, the sum of degrees for the other pairs or the Coxeter number of D_5.

4.3. **The subset $E(\rho_1)$.**

4.3.1. We first classify $I \in E(\rho_1)$. Now we recall

$$\begin{aligned} &V_2(\rho_1) = \{xy\}, \quad V_6(\rho_1) = \{x^6 - y^6\},\\ &V_3(\rho_2) = \{x^2y, xy^2\}, \quad V_5(\rho_2) = \{x^5, y^5\}. \end{aligned}$$

For any nonzero G-submodule W of $V_2(\rho_1) \oplus V_6(\rho_1)$ with $W \neq V_6(\rho_1')$, let $I(W) := SW + \mathfrak{n}$ where $S = \mathbb{C}[x,y]$. The G-module W is generated by $xy + t(x^6 - y^6)$ for some $t \in \mathbb{C}$. Then since $S_8 \subset \mathfrak{n}$ by Lemma 3.6, we see

$$\begin{aligned} S_2W + \mathfrak{n} &= S_2 \cdot (xy + t(x^6 - y^6)) + \mathfrak{n} = S_2 \cdot xy + \mathfrak{n},\\ S_kW + \mathfrak{n} &= S_kV_2(\rho_1) + \mathfrak{n} \quad \text{for any } k \geq 2 \end{aligned}$$

By Table 4, we see in $S/\mathfrak{n}$

$$\begin{aligned} &S_2V_2(\rho_1) = \{x^3y, x^2y^2, xy^3\} = \{x^3y, -xy^3\} = V_4(\rho_3),\\ &S_3V_2(\rho_1) = \{x^4y, xy^4\} = V_5(\rho_4) \oplus V_5(\rho_5),\\ &S_4V_2(\rho_1) = V_6(\rho_3),\ S_5V_2(\rho_1) = V_7(\rho_2). \end{aligned}$$

It follows that

$$S_1W + S_5W + \mathfrak{n} = S_1V_2(\rho_1) + S_5V_2(\rho_1) + \mathfrak{n}.$$

Hence we see

$$\begin{aligned} I(W)/\mathfrak{n} &= W + \sum_{k=1}^{5} S_kW + \mathfrak{n}/\mathfrak{n} = W + \sum_{k=1}^{5} S_kV_2(\rho_1) + \mathfrak{n}/\mathfrak{n}\\ &\simeq W + \rho_2 + \rho_3 + (\rho_4 + \rho_5) + \rho_3 + \rho_2 \simeq \sum_{\rho \in \mathrm{Irr}\, G} \deg(\rho)\rho. \end{aligned}$$

Thus we have $S/I(W) \simeq S/\mathfrak{n} \ominus I(W)/\mathfrak{n} \simeq \sum_{\rho \in \mathrm{Irr}_*\, G} \deg(\rho)\rho$. Hence $I(W)$ belongs to $\mathrm{Hilb}^{|G|}(\mathbb{A}^2)^{G-\mathrm{inv}}$.

4.3.2. Next we prove that $I(V_2(\rho_1))$ belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$. For this purpose we consider local deformations of $I(V_2(\rho_1))$ in $\mathrm{Hilb}^{|G|}(\mathbb{A}^2)^{G-\mathrm{inv}}$. Let $I = I(V_2(\rho_1))$. Then $I = xyS + A_6S$ because $\mathfrak{n}$ is generated by $A_4 = x^2y^2$, $A_6 = x^6 + y^6$ and $A_8 = xy(x^6 - y^6)$. General deformation theory says that G-equivariant local deformations of I are captured by the ρ_0-part of $\mathrm{Hom}_S(I/I^2, S/I)$. We see

$$(S/I)[\rho_1] = \{xy, x^6 - y^6\} = V_2(\rho_1) + V_6(\rho_1), (S/I)[\rho_0] = \{1\} = \mathbb{C}.$$

Since I/I^2 is generated by the elements xy and $A_6 = x^6+y^6$, the G-module $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$ is spanned by the elements ψ_1 and ψ_2 :

$$\psi_1(xy) = x^6 - y^6, \psi_1(A_6) = 0, \psi_2(xy) = 0, \psi_2(A_6) = 1.$$

In fact, let $\phi \in \mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$. Let $a = xy$ be a generator of $V_2(\rho_1)$. Then we may assume $\phi(a) = s(x^6 - y^6)$. Then we have $(x^6 - y^6)\phi(a) = \phi(A_8) + s_0\phi(A_6^2 - 4A_4^3) = \phi(A_8)$ because $A_jA_k \in I^2$, while $(x^6 - y^6)\phi(a) = s(A_6^2 - 4A_4^3) = 0$ in S/I because $A_k \in I$. Hence $\phi(A_8) = 0$. Moreover, $\phi(A_4) = \phi(xya) = sA_8 = 0$. Since $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$ is two-dimensional and I is generated by a and A_6, it follows that letting $\phi(A_6) = t$, then $\phi = s\psi_1 + t\psi_2$, whence ψ_1 and ψ_2 span $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$.

Thus the tangent space $T_{[I]}(\mathrm{Hilb}^{|G|}(\mathbb{A}^2))^{G-\mathrm{inv}}$ at the point $[I]$ is spanned by $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$. In other words, s and t are local (regular) parameters of $\mathrm{Hilb}^G(\mathbb{A}^2)$ at I. This implies the following. Let $\mathbb{C}[[s,t]]$ be the formal power series ring of two variables s and t, and $R = \mathbb{C}[[s,t]][x,y]$. Now we define $\mathcal{I}$ to be the ideal of R generated by the elements

$$xy + t(x^6 - y^6), A_6 + s, A_4 + tA_8, A_8 + s^2t + 4t^4A_8^3.$$

Let $\phi_1(s,t)$ be a power series with initial term $-s^2t$ satisfying the equation

$$\phi_1 + s^2t + 4t^4\phi_1^3 = 0,$$

which comes from $A_8^2 = A_4A_6^2 - 4A_4^4$. The power series ϕ_1 is uniquely determined by this property. Then $\mathcal{I}$ is also generated by the elements

$$xy + t(x^6 - y^6), A_6 + s, A_4 + t\phi_1(s,t), A_8 - \phi_1(s,t).$$

We note that

$$S/I = \bigoplus_{k=1}^{5}\{x^k, y^k\} \oplus \{x^6 - y^6\}.$$

By the upper semi-continuity $R/\mathcal{I}$ is generated over $\mathbb{C}[[s,t]]$ by S/I (regarded as a G-submodule of S). Then it is almost clear that $R/\mathcal{I}$ is a free $\mathbb{C}[[s,t]]$-module with the same basis S/I because of the forms of the four generators of $\mathcal{I}$. Hence $R/\mathcal{I}$ gives a $\mathbb{C}[[s,t]]$-flat family of zero-dimensional subschemes deforming $Z := \mathrm{Spec}(S/I)$. It is obvious that the support of the subscheme over $\mathbb{C}((s,t))$ is away from the origin (if necessary by restricting the deformation to a small disc of the (s,t) space near the origin in the complex topology). In other words, Z is deformed into a G-invariant reduced subscheme of $\mathbb{A}^2$. Therefore Z, hence I, belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$.

4.3.3. Since I belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$, any deformation I' of I over a connected base belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$ because $\mathrm{Hilb}^G(\mathbb{A}^2)$ is connected. In particular, $I(W)$ defined above and all I' we are going to construct in this (sub)section belong to $\mathrm{Hilb}^G(\mathbb{A}^2)$.

4.3.4. Next we consider the case where $W = V_6(\rho_1)$. Let $W_\infty = V_6(\rho_1)$ and $W_t = (xy + t(x^6 - y^6))S + \mathfrak{n}$. Then $\lim_{t\to\infty} W_t = W_\infty$ in $\mathbb{P}^1 = \mathbb{P}(V(\rho_1)) = \mathbb{P}(V_2(\rho_1) \oplus V_6(\rho_1))$. Now we compute $\lim_{t\to\infty} I(W_t)$ in $\mathrm{Hilb}^G(\mathbb{A}^2)$. For any t, hence for $t = \infty$ we have

$$I(W_t)/\mathfrak{n} = W_t + \sum_{k=1}^{5} S_k W_t + \mathfrak{n}/\mathfrak{n} = W_t + \sum_{k=1}^{5} S_k V_2(\rho_1) + \mathfrak{n}/\mathfrak{n},$$

$$I(W_\infty)/\mathfrak{n} = V_6(\rho_1) + \sum_{k=1}^{5} S_k V_2(\rho_1) + \mathfrak{n}/\mathfrak{n} \simeq \sum_{\rho\in\mathrm{Irr}\,G} \deg(\rho)\rho.$$

Since $I(W_\infty)$ is a limit of $I(W_t)$, to be more precise, we have a flat family over $\mathbb{P}^1$ deforming $I(W_t)$ into $I(W_\infty)$, the limit $I(W_\infty)$ belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$.

We note that

$$V(I(W_\infty)) = V_6(\rho_1) + S_1 V_2(\rho_1) = V_6(\rho_1) + S_3(\rho_2).$$

Hence $I(W_\infty)$ belongs to the subset $P(\rho_1, \rho_2)$ of $\mathrm{Hilb}^G(\mathbb{A}^2)$.

4.3.5. Now we shall prove the converse. Let $I \in \mathrm{Hilb}^{|G|}(\mathbb{A}^2)^{G-\mathrm{inv}}$. Suppose $S/I \simeq \mathbb{C}[G]$ and $\rho_1 \subset V(I)$. Then $\mathfrak{n} \subset I$ by (the same reason as) Corollary 2.7. This implies that a nonzero G-submodule W of $V_2(\rho_1) \oplus V_6(\rho_1)$ is contained in $V(I)$, and in I as part of generators of I. If $W \neq V_6(\rho_1)$, then I contains $I(W)$ defined above. Since $I(W)/\mathfrak{n} \simeq I/\mathfrak{n}$, we have $I = I(W)$. Suppose $W = W_\infty = V_6(\rho_1)$, or equivalently $V_6(\rho_1) \subset V(I)$. Since W is part of generators of I, $V_5(\rho_2)$ is not contained in I. There are only $V_3(\rho_2)$ and $V_5(\rho_2)$ in $S(\mathfrak{m}/\mathfrak{n})[\rho_2]$. Hence $V(I) \supset \{x^2y + ty^5, xy^2 - tx^5\}$ for some t, whence $t\{x^6, -y^6\} + \mathfrak{n} \subset S_1 V(I) + \mathfrak{n} \subset I$ because $x^2y^2 \in \mathfrak{n}$. If $t \neq 0$, this implies that $\{x^6, -y^6\} = V_6(\rho_1)$ is not part of generators of I, which contradicts the assumption $W = V_6(\rho_1)$. It follows that $t = 0$, and $V_3(\rho_2)$ is contained in I. Let $I(W) = I(W_\infty) = I(V_6(\rho_1)) := V_6(\rho_1)S + V_3(\rho_2)S + \mathfrak{n}$. Then $I(W_\infty) \subset I$. As we saw above, $I(W_\infty) \in \mathrm{Hilb}^G(\mathbb{A}^2)$ and $I(W_\infty)/\mathfrak{n} \simeq \mathbb{C}[G]$. Hence $I(W_\infty)/\mathfrak{n} \simeq I/\mathfrak{n}$, whence $I = I(W_\infty)$.

Thus we have proved that if $S/I \simeq \mathbb{C}[G]$ and $V(I) \supset \rho_1$, then $I = I(W)$ for some nonzero G-submodule $W \subset V_2(\rho_1) \oplus V_6(\rho_1)$.

4.3.6. To summarize the above, we define for a nonzero G-submodule W of $V_2(\rho_1)+V_6(\rho_1)$,

$$I_1(W)=\begin{cases} V_6(\rho_1)S+V_3(\rho_2)S+\mathfrak{n} & \text{if} \quad W=V_6(\rho_1),\\ SW+\mathfrak{n} & \text{otherwise,}\end{cases}$$
$$=W+\sum_{k=1}^{5}S_kV_2(\rho_1)+\mathfrak{n}.$$

The above classification of I shows that

$$E(\rho_1)=\{I_1(W);\rho_1\simeq W\subset V_2(\rho_1)\oplus V_6(\rho_1)\},$$
$$P(\rho_1,\rho_2)=\{I_1(V_6(\rho_1))\}=\{V_6(\rho_1)S+V_3(\rho_2)S+\mathfrak{n}\}.$$

We note that $E(\rho_1)$ is identified with the set of all nonzero G-submodules of $V_2(\rho_1)\oplus V_6(\rho_1)$, which is (at least set-theoretically) isomorphic to $\mathbb{P}^1$ by Schur's lemma. Moreover

$$\lim_{W\to V_6(\rho_1)} I_1(W)=I_1(V_6(\rho_1))=V_6(\rho_1)S+V_3(\rho_2)S+\mathfrak{n}.$$

The family $I(W)$, $W\in\mathbb{P}(V_2(\rho_1)\oplus V_6(\rho_1))\simeq\mathbb{P}^1$ is flat over $\mathbb{P}^1$ because $\mathcal{I}$ with $\mathcal{I}_0=I_1(V_6(\rho_1))$ is $\mathbb{C}[[s,t]]$-flat (See subsection 4.4). Hence we have an injective morphism $\phi:\mathbb{P}^1\to\mathrm{Hilb}^G(\mathbb{A}^2)$ with injective homomorphism of tangent spaces, whence $E(\rho_1)$, the image of ϕ, is a smooth rational curve. Since $\mathrm{Hilb}^G(\mathbb{A}^2)$ is a smooth surface with trivial dualizing sheaf by Theorem 2.5, this proves that the self-intersection of $E(\rho_1)$ is -2.

4.4. **The subset $E(\rho_2)$.**

4.4.1. First we consider the tangent space $T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$ at $I:=I_1(V_6(\rho_1))$. Recall $V(I)=V_6(\rho_1)+V_3(\rho_2)$. Since $(S/I)[\rho_1]=V_2(\rho_1)$ and $(S/I)[\rho_2]=V_1(\rho_2)\oplus V_5(\rho_2)$, the tangent space $T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$ at I is spanned as before by the elements ψ_1 and ψ_2 :

$$\psi_1(x^6-y^6)=xy,\psi_1(V_3(\rho_2))=0,$$
$$\psi_2(V_6(\rho_1))=0,\psi_2(x^2y)=y^5,\psi_2(-xy^2)=-x^5$$

Because if $\psi_2(x^2y)=ax+by^5$ and $\psi_2(-xy^2)=ay-bx^5$ for nonzero a, then deformations by ϕ_2 of I contain S, a contradiction. Hence $a=0$.

We define $\mathcal{I}$ to be the ideal of $\mathbb{C}[[s,t]][x,y]$ generated by the elements

$$B:=x^6-y^6+sxy,$$
$$C_1:=x^2y+ty^5-\frac{st}{2}x,C_2:=-xy^2-tx^5-\frac{st}{2}y,$$
$$2A_4+tA_6,A_6-\phi_2,A_8+sA_4,$$

where $\phi_2 \in \mathbb{C}[[s,t]]$ is the unique power series with initial term $-\frac{1}{2}s^2t$ satisfying the equation

$$\phi_2 + \frac{s^2t}{2} + \frac{t^3}{2}\phi_2^2 = 0.$$

We note $2A_4 + tA_6 = yC_1 - xC_2$, $A_8 + sA_4 = xyB$ and $A_8 - \frac{st}{2}A_6 = x^5C_1 + y^5C_2$. The ideal $\mathcal{I}$ is therefore generated by the elements

$$B, C_1, C_2, A_4 + \frac{t}{2}\phi_2, A_6 - \phi_2, A_8 - \frac{st}{2}\phi_2. \tag{1}$$

One can show that $\mathcal{I}$ defines a $\mathbb{C}[[s,t]]$-flat family of subschemes, which is a local universal deformation of $\mathrm{Spec}(S/I)$ or simply I.

4.4.2. One can read from this that $s = 0$ yields a new deformation of subschemes. This leads us to the following definitions.

For W, any nonzero G-submodule of $V(\rho_2) := V_3(\rho_2) \oplus V_5(\rho_2)$, we define

$$I_2(W) = \begin{cases} V_6(\rho_1)S + V_3(\rho_2)S + \mathfrak{n} & \text{if } \quad W = V_3(\rho_2), \\ V_5(\rho_2)S + S_1V_3(\rho_2)S + \mathfrak{n} & \text{if } \quad W = V_5(\rho_2), \\ SW + \mathfrak{n} & \text{otherwise.} \end{cases}$$

Suppose W to be any nonzero G-submodule of $V(\rho_2)$ such that $W \neq V_3(\rho_2)$ and $W \neq V_5(\rho_2)$. Then we see $S_3W + S_4W + \mathfrak{n} = V_6(\rho_3) + V_7(\rho_2) + \mathfrak{n}$ as $S_8 \subset \mathfrak{n}$. Since $W \neq V_3(\rho_2)$ and $W \neq V_5(\rho_2)$, we infer

$$\begin{aligned} SW + \mathfrak{n} &= W + S_1W + S_2W + V_6(\rho_3) + V_7(\rho_2) + \mathfrak{n} \\ &= W + S_1V_3(\rho_2) + V_5(\rho_4) + V_5(\rho_5) \\ &\qquad + V_6(\rho_1) + V_6(\rho_3) + V_7(\rho_2) + \mathfrak{n}, \end{aligned}$$

whence $I_2(W)/\mathfrak{n} \simeq \mathbb{C}[G]$. This implies that $I_2(W) \in \mathrm{Hilb}^G(\mathbb{A}^2)$. In the same manner as above one can prove that $I_2(W)$ belongs to $\mathrm{Hilb}^G(\mathbb{A}^2)$ for $W = V_3(\rho_2)$ or $W = V_5(\rho_2)$. We also see that

$$\begin{aligned} I_2(W) &= W + S_1W + S_2W + V_6(\rho_3) + V_7(\rho_2) + \mathfrak{n} \\ &= W + S_1V_3(\rho_2) + V_5(\rho_4) + V_5(\rho_5) \\ &\qquad + V_6(\rho_1) + V_6(\rho_3) + V_7(\rho_2) + \mathfrak{n} \end{aligned}$$

for any nonzero G-submodule of $V(\rho_2)$.

4.4.3. Moreover in the same manner as before one can check that if I satisfies $S/I \simeq \mathbb{C}[G]$ and $V(I)$ contains ρ_2, then $I = I_2(W)$ for some nonzero G-submodule W of $V_3(\rho_2) \oplus V_5(\rho_2)$.

This proves

$$\begin{aligned} E(\rho_2) &= \{I_2(W); \rho_2 \simeq W \subset V_3(\rho_2) \oplus V_5(\rho_2)\} \simeq \mathbb{P}^1, \\ P(\rho_1, \rho_2) &= \{I_2(V_3(\rho_2))\} = \{V_6(\rho_1)S + V_3(\rho_2)S + \mathfrak{n}\}, \\ P(\rho_2, \rho_3) &= \{I_2(V_5(\rho_2))\} = \{V_5(\rho_2)S + S_1V_3(\rho_2)S + \mathfrak{n}\}. \end{aligned}$$

The subset $E(\rho_2)$ is proved as before to be a smooth rational curve with self-intersection -2. We also see that

$$\begin{aligned} I_2(V_3(\rho_2)) &= \lim_{W\to V_3(\rho_2)} I_2(W) = V_3(\rho_2)S + V_6(\rho_1)S + \mathfrak{n},\\ I_2(V_5(\rho_2)) &= \lim_{W\to V_5(\rho_2)} I_2(W) = V_5(\rho_2)S + S_1V_3(\rho_2)S + \mathfrak{n}. \end{aligned}$$

4.4.4. Now we focus on the intersection point $P(\rho_1,\rho_2)$ of two rational curves $E(\rho_1)$ and $E(\rho_2)$. The computation of limits shows that

$$\begin{aligned} I_1(V_6(\rho_1)) = \lim_{W\to V_6(\rho_1)} I_1(W) &= \{V_6(\rho_1) + S_1V_2(\rho_1)\}S + \mathfrak{n}\\ &= \{V_6(\rho_1) + V_3(\rho_2)\}S + \mathfrak{n},\\ I_2(V_3(\rho_2)) = \lim_{W\to V_3(\rho_2)} I_2(W) &= \{V_3(\rho_2)S + S_1V_5(\rho_2)\}S + \mathfrak{n}\\ &= \{V_3(\rho_2) + V_6(\rho_1)\}S + \mathfrak{n}. \end{aligned}$$

The reason why $I_1(V_6(\rho_1)) = I_2(V_3(\rho_2))$ holds true is just the fact

$$S_1V_2(\rho_1) = V_3(\rho_2), S_1V_5(\rho_2) = V_6(\rho_1) \mod \mathfrak{n},$$

where $V_2(\rho_1)$ is the dual partner of $V_6(\rho_1)$, while $V_3(\rho_2)$ is the dual partner of $V_5(\rho_2)$. Since $S_1 \simeq \rho_{\text{nat}}$, the natural representation of G, this is part of the McKay rule of irreducible decompositions by tensoring with ρ_{nat}, relevant to ρ_1 and ρ_2 : $\rho_1\rho_{\text{nat}} = \rho_2$, $\rho_2\rho_{\text{nat}} = \rho_1 + \cdots$.

4.5. **The subset $E(\rho_3)$.** The subset $E(\rho_3)$ is computed in the same manner as before. Let W be a nonzero irreducible G-submodule of $V_4(\rho_3)$. Then we define in the same manner as before

$$\begin{aligned} I_3(W) &= \begin{cases} (S_1V_3(\rho_2)[\rho_3] + V_5(\rho_2))S + \mathfrak{n} & \text{if} \quad W = S_1V_3(\rho_2)[\rho_3],\\ (S_1V_3(\rho_4) + V_5(\rho_4))S + \mathfrak{n} & \text{if} \quad W = S_1V_3(\rho_4),\\ (S_1V_3(\rho_5) + V_5(\rho_5))S + \mathfrak{n} & \text{if} \quad W = S_1V_3(\rho_5),\\ SW + \mathfrak{n} & \text{otherwise}, \end{cases}\\ &= W + \sum_{k=5}^{7} S_k(\mathfrak{m}/\mathfrak{n}) + \mathfrak{n} \quad \text{for any } W. \end{aligned}$$

Then one can prove in the same manner as before

$$\begin{aligned} E(\rho_3) &= \{I_3(W); \rho_3 \simeq W \subset V_4(\rho_3)\},\\ P(\rho_2,\rho_3) &= \{I_3(S_1V_3(\rho_2))\} = \{S_1V_3(\rho_2)S + V_5(\rho_2)S + \mathfrak{n}\},\\ P(\rho_3,\rho_4) &= \{I_3(S_1V_3(\rho_4))\} = \{S_1V_3(\rho_4)S + V_5(\rho_4)S + \mathfrak{n}\},\\ P(\rho_3,\rho_5) &= \{I_3(S_1V_3(\rho_5))\} = \{S_1V_3(\rho_5)S + V_5(\rho_5)S + \mathfrak{n}\}. \end{aligned}$$

We have also similar formulae of limits

$$\begin{aligned} P(\rho_2,\rho_3) &= I_3(S_1V_3(\rho_2)[\rho_3]) = S_1V_3(\rho_2) + \sum_{k=5}^{7} S_k + \mathfrak{n} \\ &= I_2(V_5(\rho_2)) = V_5(\rho_2)S + S_1V_3(\rho_2)S + \mathfrak{n}, \\ P(\rho_3,\rho_4) &= I_3(S_1V_3(\rho_4)) = \lim_{W\to S_1V_3(\rho_4)} I_3(W), \\ P(\rho_3,\rho_5) &= I_3(S_1V_3(\rho_5)) = \lim_{W\to S_1V_3(\rho_5)} I_3(W). \end{aligned}$$

The first formula is true because $SS_1W + \mathfrak{n} = \sum_{k=5}^{7} S_k + \mathfrak{n}$ for any general irreducible G-submodule W of $V_4(\rho_3)$. Hence, the reason why $I_2(V_5(\rho_2)) = I_3(S_1V_3(\rho_2))$ holds true is just the fact

$$\begin{aligned} S_1V_3(\rho_2) = \{x^3y, -xy^3\} + \{x^2y^2\} = \{x^3y, -xy^3\} \simeq \rho_3 \mod \mathfrak{n}, \\ S_1\{x^4, y^4\} = V_5(\rho_2) + V_5(\rho_4) + V_5(\rho_5) \simeq \rho_2 + \rho_4 + \rho_5, \end{aligned}$$

where $V_3(\rho_2)$ is the dual partner of $V_5(\rho_2)$, while $\{x^3y, -xy^3\}$ is the dual partner of $\{x^4, y^4\}$ with respect to the natural pairing (See [IN99], Lemma 11.5). This reminds us of the McKay rule for representations explained in subsection 1.4.

4.6. **The subsets $E(\rho_4)$ and $E(\rho_5)$.** Since ρ_4 and ρ_5 are one-dimensional, we can argue in the same manner as ρ_1. Then we define

$$\begin{aligned} I_4(W) &= \begin{cases} SW + S_1V_3(\rho_4)S + \mathfrak{n} & \text{if} \quad W = V_5(\rho_4), \\ SW + \mathfrak{n} & \text{otherwise}, \end{cases} \\ &= W + S_1V_3(\rho_4) + V_5(\rho_2) + V_5(\rho_5) + S_6 + S_7 + \mathfrak{n} \end{aligned}$$

for any nonzero G-submodule W of $V(\rho_4) := V_3(\rho_4) + V_5(\rho_4)$, and

$$\begin{aligned} I_5(W) &= \begin{cases} SW + S_1V_3(\rho_5)S + \mathfrak{n} & \text{if} \quad W = V_5(\rho_5), \\ SW + \mathfrak{n} & \text{otherwise}, \end{cases} \\ &= W + S_1V_3(\rho_5) + V_5(\rho_2) + V_5(\rho_4) + S_6 + S_7 + \mathfrak{n} \end{aligned}$$

for any nonzero G-submodule W of $V(\rho_5) := V_3(\rho_5) + V_5(\rho_5)$. In the same manner as before we see

$$\begin{aligned} E(\rho_4) &= \{I_4(W); \rho_4 \simeq W \subset V(\rho_4)\} \simeq \mathbb{P}^1, \\ E(\rho_5) &= \{I_5(W); \rho_5 \simeq W \subset V(\rho_5)\} \simeq \mathbb{P}^1, \\ P(\rho_3,\rho_4) &= \{I_4(V_5(\rho_4))\}, P(\rho_3,\rho_5) = \{I_5(V_5(\rho_5))\}. \end{aligned}$$

By subsection 4.5

$$\begin{aligned} P(\rho_3,\rho_4) = I_4(V_5(\rho_4)) = I_3(S_1V_3(\rho_4)), \\ P(\rho_3,\rho_5) = I_5(V_5(\rho_5)) = I_3(S_1V_3(\rho_5)). \end{aligned}$$

These formulae come from the relations

$$S_1V_3(\rho_4) \simeq \rho_3, S_1V_3(\rho_5) \simeq \rho_3 \mod \mathfrak{n},$$
$$S_1\{x^4, y^4\} = V_5(\rho_2) + V_5(\rho_4) + V_5(\rho_5),$$

which reminds us of the McKay rule for representations in subsection 1.4.

4.7. **Versal deformations.**

4.7.1. Let $I = I_2(V_5(\rho_2)) = I_3(S_1V_3(\rho_2)) = V_5(\rho_2)S + S_1V_3(\rho_2)S + \mathfrak{n}$. Let $R = \mathbb{C}[[s,t]][x,y]$, and $\lambda = \frac{s^2t}{1+t^2}$. We define a deformation $\mathcal{I}_3$ of I as the ideal of R generated by the elements

$$y^5 + sx^2y + \lambda x, -x^5 - sxy^2 + \lambda y,$$
$$x^3y + ty^4 + stx^2, -xy^3 + tx^4 + sty^2,$$
$$A_6 + 2sA_4, A_8 - 2\lambda A_4, A_8 + stA_6 + 2tA_4^2, 2A_4^2 - tA_8, A_4 - t\lambda.$$

It turns out that $\mathcal{I}_3$ is generated by the elements

$$y^5 + sx^2y + \lambda x, -x^5 - sxy^2 + \lambda y,$$
$$x^3y + ty^4 + stx^2, -xy^3 + tx^4 + sty^2,$$
$$A_6 + 2st\lambda, A_8 - 2t\lambda^2, A_4 - t\lambda,$$

which gives a $\mathbb{C}[[s,t]]$-flat deformation of the subscheme defined by I.

4.7.2. Next let $I = I_3(S_1V_3(\rho_4)) = I_4(V_5(\rho_4)) = V_5(\rho_4)S + S_1V_3(\rho_4) + \mathfrak{n}$. Then we define a $\mathbb{C}[[s,t]]$-(co)flat deformation $\mathcal{I}_4$ of I as the ideal of R generated by the elements

$$y^4 - ix^3y + sy^4 + istx^2, x^4 + ixy^3 + sx^4 + isty^2,$$
$$x^4y - ixy^4 + t(x^3 + iy^3),$$
$$(1+s)A_8 + 2iA_4^2, (1+s)A_6 + 2istA_4,$$
$$A_8 - 2iA_4^2 + tA_6, (2+2s)A_4^2 - iA_8 + istA_6.$$

It turns out that the ideal $\mathcal{I}_4$ is generated by the elements

$$y^4 - ix^3y + sy^4 + istx^2, x^4 + ixy^3 + sx^4 + isty^2,$$
$$x^4y - ixy^4 + t(x^3 + iy^3),$$
$$A_4 + \frac{st^2}{2+s}, A_6 - \frac{2is^2t^3}{(1+s)(2+s)}, A_8 + \frac{2is^2t^4}{(1+s)(2+s)^2},$$

which gives a $\mathbb{C}[[s,t]]$-flat deformation of the subscheme defined by I.

4.7.3. Let $I = I_3(S_1V_3(\rho_5)) = I_4(V_5(\rho_5)) = V_5(\rho_5)S + S_1V_3(\rho_5) + \mathfrak{n}$. We can construct a versal deformation $\mathcal{I}_5$ of I in the same manner as ρ_4. In fact, we define $\mathcal{I}_5$ by replacing i in the definition of $\mathcal{I}_4$ by $-i$. Then $\mathcal{I}_5$ gives a $\mathbb{C}[[s,t]]$-flat family of deformations of the subscheme defined by I.

4.8. **The subquivers of the McKay quiver.**

4.8.1. Now we give a simple algorithm for describing $E(\rho)$ for $\rho \in \operatorname{Irr} G$. First we note that the very positive part $S^\dagger$ is contained in any $I \in E$, the exceptional divisor. It remains to describe precisely $I + S^\dagger / S^\dagger \subset S_{\mathrm{McKay}}(G)$. This is done by using the subquivers of the McKay quiver as follows.

Now we take ρ_1 as an example. The curve $E(\rho_1)$ consists of all $I_1(W)$, $W \in \mathbb{P}(V_2(\rho_1) \oplus V_6(\rho_1))$, where

$$\begin{aligned} I_1(W) &= W + \sum_{k=1}^{5} S_k V_2(\rho_1) \\ &= W + V_3(\rho_2) + S_1 V_3(\rho_2) + V_5(\rho_4) + V_5(\rho_5) + S^\dagger. \end{aligned}$$

We define $\operatorname{Quiv}(\rho_1)$, a subquiver of $S_{\mathrm{McKay}}(G)$ associated with ρ_1, to be the sum of all G-submodules V of $S_{\mathrm{McKay}}(G)$ in Figure 7 equipped with arrows (= quiver structure) of $S_{\mathrm{McKay}}(G)$.

$$\begin{tikzcd} & & & V_5(\rho_4) \\ \begin{array}{c} V_2(\rho_1) \\ \oplus \\ V_6(\rho_1) \end{array} \arrow[r, shift left] & \begin{array}{c} V_3(\rho_2) \\ \oplus \\ \{0\} \end{array} \arrow[l, shift left] \arrow[r, shift left] & S_1 V_3(\rho_2) \arrow[l, shift left, "0"] \arrow[ur] \arrow[dr] & \\ & & & V_5(\rho_5) \end{tikzcd}$$

FIGURE 7. The original $\operatorname{Quiv}(\rho_1)$

We denote it simply by the following :

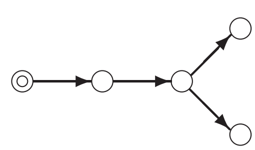

FIGURE 8. $\operatorname{Quiv}(\rho_1)$

It is easy to recover the original $\operatorname{Quiv}(\rho_1)$ from Figure 8 because between the subspaces of $\operatorname{Quiv}(\rho_1)$ corresponding to the vertices there are no nonzero arrows of $S_{\mathrm{McKay}}(G)$ with reverse directions. For instance, in $\operatorname{Quiv}(\rho_1)$ the arrow from $V_2(\rho_1) \oplus V_6(\rho_1)$ to $V_3(\rho_2)$ is precisely the disjoint union of the arrow from $V_2(\rho_1)$ to $V_3(\rho_2)$ and the reverse arrow from $\{0\}$ ($\subset V_5(\rho_2)$) to $V_6(\rho_1)$, or equivalently, just the arrow from $V_2(\rho_1)$ to $V_3(\rho_2)$.

The G-submodule $(I_1(W)/\mathfrak{n} + S^\dagger) \cap S_{\mathrm{McKay}}(G)$ is the sum of W and all G-submodules of $\operatorname{Quiv}(\rho_1)$ inequivalent to ρ_1. Therefore any $I_1(W) \in E(\rho_1)$

is given by

$$I_1(W) = W + \sum_{V \subset \mathrm{Quiv}(\rho_1), V \not\simeq \rho_1} V + S^\dagger \quad \text{for} \quad W \in \mathbb{P}(V_2(\rho_1) \oplus V_6(\rho_1)),$$

where the summation ranges over all irreducible G-submodules V of $\mathrm{Quiv}(\rho_1)$ inequivalent to ρ_1.

4.8.2. This is generalized in the other cases in the obvious manner. The subquivers $\mathrm{Quiv}(\rho_i)$ $(i = 2, 3, 4, 5)$ are given in Figure 9.

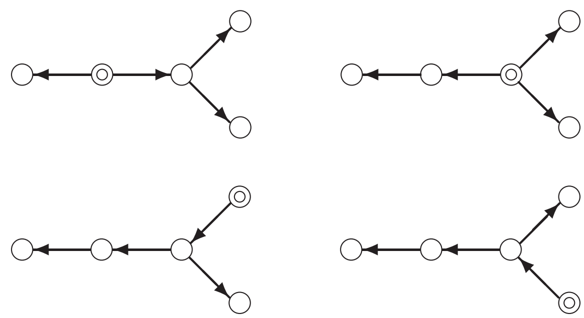

FIGURE 9. $\mathrm{Quiv}(\rho)$ for D_5

Any $I(W) \in E(\rho)$ is given by

$$I(W) = W + \sum_{V \subset \mathrm{Quiv}(\rho), V \not\simeq \rho} V + S^\dagger \quad \text{for} \quad W \in \mathbb{P}(V_{h-d(\rho)}(\rho) + V_{h+d(\rho)}(\rho)),$$

where the summation ranges over all irreducible G-submodules V of $\mathrm{Quiv}(\rho)$ inequivalent to ρ. The first diagram in Figure 9 is denoted $\mathrm{Quiv}(\rho_2)$, which means the subquiver diagram in Figure 10. We note that the quiver $S_{\mathrm{McKay}}(G)$ restricted to $S_1V_3(\rho_2)$ automatically reduces to zero to the direction of $V_5(\rho_2)$.

With these diagrams, the point $P(\rho_1, \rho_2)$ is understood as the first diagram of Figure 11 which can be embedded into both of $\mathrm{Quiv}(\rho_1)$ and $\mathrm{Quiv}(\rho_2)$, to which it may be natural to assign the second graph in Figure 11. Thus the four intersection points $P(\rho, \rho')$ are described by the subquivers in Figure 12.

$$\begin{tikzcd}
 & & & V_5(\rho_4) \\
\begin{matrix}\{0\}\\ \oplus \\ V_6(\rho_1)\end{matrix} \arrow[r, shift left] & \begin{matrix}V_3(\rho_2)\\ \oplus \\ V_5(\rho_2)\end{matrix} \arrow[l, shift left] \arrow[r, shift left] & S_1V_3(\rho_2) \arrow[l, shift left, "0"] \arrow[ur] \arrow[dr] & \\
 & & & V_5(\rho_5)
\end{tikzcd}$$

FIGURE 10. The original $\mathrm{Quiv}(\rho_2)$

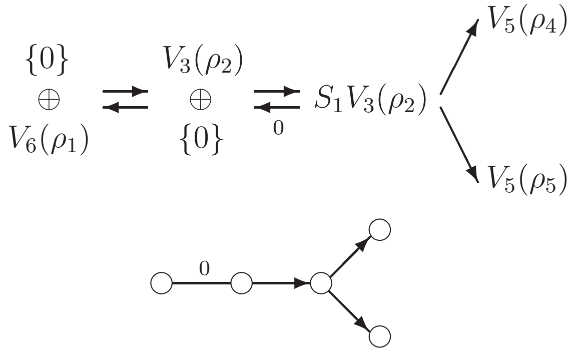

FIGURE 11. Quiv(ρ_1, ρ_2)

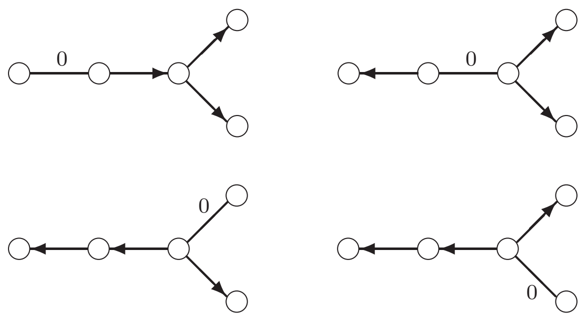

FIGURE 12. Quiv(ρ, ρ') for D_5

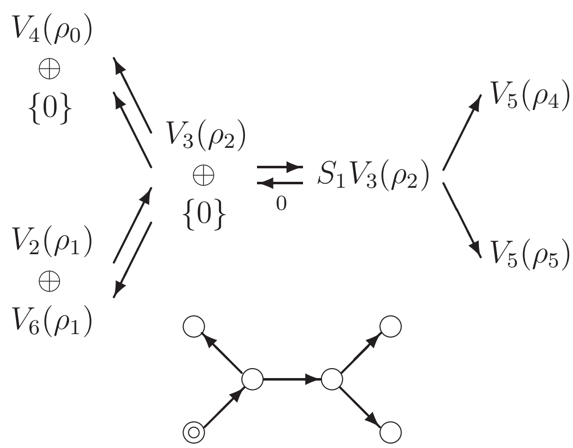

FIGURE 13. The extended Quiv(ρ_1)

4.9. **The extended McKay quiver revisited.** The subquiver Quiv(ρ_1) extends itself in the extended McKay quiver by adding an arrow from $V_3(\rho_2)$ to $V_4(\rho_0)$ as Figure 13 indicates.

This implies the following. By Table 4, $V_3(\rho_2) = S_1V_2(\rho_1)$ and $V_4(\rho_0) \subset S_1V_3(\rho_2)$. The subquiver Quiv$(\rho_1)$ does not generate $V_6(\rho_0)$ in $I_1(W)$, which explains why $V_6(\rho_0)$ is always included as a generator of $I_1(W)$. This is true over a Zariski open subset of $\mathrm{Hilb}^G(\mathbb{A}^2)$ containing $\operatorname{Spec}\mathbb{C}[s]$ ($\subset E(\rho_1)$) in subsection 4.3.

Similarly, the subquiver $\mathrm{Quiv}(\rho_2)$ extends itself by adding an arrow from $V_3(\rho_2)\oplus V_5(\rho_2)$ to $V_4(\rho_0)\oplus V_6(\rho_0)$ as well. This means that the image of the arrow spans a one-dimensional subspace of $V_4(\rho_0)\oplus V_6(\rho_0)$, whose one-dimensional complement is necessary as a part of the generators of $I_2(W)$.

In the other cases, the subquiver $\mathrm{Quiv}(\rho_k)$ ($k=3,4,5$) extends itself by adding an arrow from $V_5(\rho_2)$ to $V_6(\rho_0)$ as well. This shows that $V_4(\rho_0)$ is always necessary as a part of the generators of $I_k(W)$.

5. The Simple Singularity E_6

5.1. **The binary tetrahedral group $\mathbb{T}$.** The simple singularity E_6 is the quotient singularity of $\mathbb{A}^2$ by the binary tetrahedral group $G:=\mathbb{T}$, which is the subgroup of $\mathrm{SL}(2,\mathbb{C})$ of order 24 generated by $\mathbb{D}_2=\langle\sigma,\tau\rangle$ and μ:

$$\sigma=\begin{pmatrix} i, & 0\\ 0, & -i\end{pmatrix},\quad \tau=\begin{pmatrix} 0, & 1\\ -1, & 0\end{pmatrix},\quad \mu=\frac{1}{\sqrt{2}}\begin{pmatrix}\epsilon^7, & \epsilon^7\\ \epsilon^5, & \epsilon\end{pmatrix},$$

where $\epsilon=e^{2\pi i/8}$ [Slodowy80], p. 74. The group G acts on $\mathbb{A}^2$ from the right by $(x,y)\mapsto(x,y)g$ for $g\in G$ and $\mathbb{D}_2$ is a normal subgroup of G with the following exact sequence:

$$1\to\mathbb{D}_2\to G\to\mathbb{Z}/3\mathbb{Z}\to 1.$$

See Table 6 for the character table of G [Schur07] where $h=12$ and $\omega=(-1+\sqrt{3}i)/2$.

ρ	1	2	3	4	5	6	7	d	$(\frac{h}{2}\pm d)$
	1	-1	τ	μ	μ^2	μ^4	μ^5		
$(\sharp)$	1	1	6	4	4	4	4		
ρ_0	1	1	1	1	1	1	1	(2)	–
ρ_2	2	-2	0	1	-1	-1	1	1	$(5,7)$
ρ_3	3	3	-1	0	0	0	0	0	$(6,6)$
ρ_2'	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω	1	$(5,7)$
ρ_1'	1	1	1	ω^2	ω	ω^2	ω	2	$(4,8)$
ρ_2''	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2	1	$(5,7)$
ρ_1''	1	1	1	ω	ω^2	ω	ω^2	2	$(4,8)$

Table 6. Character table of $\mathbb{T}$

k	S_k	V_k
0	ρ_0	0
1	ρ_2	ρ_2
2	ρ_3	ρ_3
3	$\rho_2' + \rho_2''$	$\rho_2' + \rho_2''$
4	$\rho_1' + \rho_1'' + \rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
5	$\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
6	$[\rho_0] + 2\rho_3$	$(2\rho_3)$
7	$2\rho_2 + \rho_2' + \rho_2''$	$(\rho_2 + \rho_2' + \rho_2'')$
8	$[\rho_0] + \rho_1' + \rho_1'' + 2\rho_3$	$(\rho_1' + \rho_1'') + \rho_3$
9	$\rho_2 + 2\rho_2' + 2\rho_2''$	$\rho_2' + \rho_2''$
10	$\rho_1' + \rho_1'' + 3\rho_3$	ρ_3
11	$2\rho_2 + 2\rho_2' + 2\rho_2''$	ρ_2
12	$2\rho_0 + \rho_1' + \rho_1'' + 3\rho_3$	0

TABLE 7. Irreducible decompositions of S and $\mathrm{Coinv}(E_6)$

$$
\begin{tikzcd}
 & & V_6(\rho_0) \oplus V_8(\rho_0) & & \\
 & & V_5(\rho_2) \oplus V_7(\rho_2) \arrow[u, shift left] \arrow[u, shift right] \arrow[d, shift right] & & \\
\begin{matrix} V_4(\rho_1') \\ \oplus \\ V_8(\rho_1') \end{matrix} \arrow[r, shift left] & \begin{matrix} V_5(\rho_2') \\ \oplus \\ V_7(\rho_2') \end{matrix} \arrow[l, shift left] \arrow[r, shift left] & V_6(\rho_3) \arrow[l, shift left] \arrow[u, shift right] \arrow[r, shift right] & \begin{matrix} V_5(\rho_2'') \\ \oplus \\ V_7(\rho_2'') \end{matrix} \arrow[l, shift right] \arrow[r, shift right] & \begin{matrix} V_4(\rho_1'') \\ \oplus \\ V_8(\rho_1'') \end{matrix} \arrow[l, shift right]
\end{tikzcd}
$$

FIGURE 14. The extended McKay quiver of E_6

5.2. **Symmetric tensors modulo $\mathfrak{n}$.** Let S_m be the space of homogeneous polynomials in x and y of degree m. The G-modules S_m and $S_m(S/\mathfrak{n})$ via ρ_{nat} decompose into irreducible G-submodules. We define a G-submodule of $\mathfrak{m}/\mathfrak{n}$ by $V_i(\rho_j) := S_i(\mathfrak{m}/\mathfrak{n})[\rho_j]$ the sum of all copies of ρ in $S_i(\mathfrak{m}/\mathfrak{n})$, which we always choose as a homogeneous G-submodule of S_i. For a G-module W we define $W[\rho]$ to be the sum of all the copies of ρ in W. It is known by [Klein], p. 51 that there are G-invariant polynomials A_6, A_8, A_6^2 and A_{12} respectively of degrees 6, 8, 12 and 12. We may assume that $A_6 = T$, $A_8 = W$ and $A_{12} = U := \varphi^3 + \psi^3$. Let $V_6(\rho_0) = S_6[\rho_0] = \{A_6\}$ and $V_8(\rho_0) = S_8[\rho_0] = \{A_8\}$,

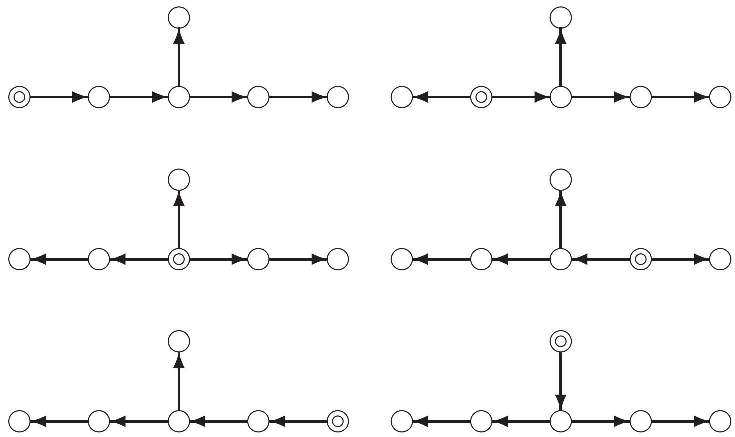

FIGURE 15. $\mathrm{Quiv}(\rho)$ for E_6

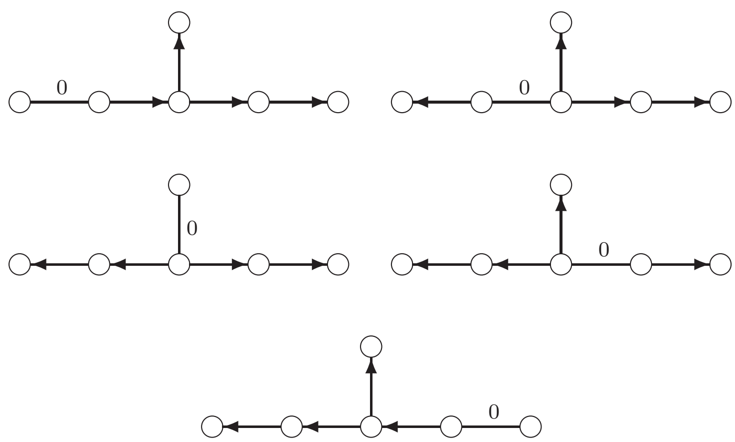

FIGURE 16. $\mathrm{Quiv}(\rho, \rho')$ for E_6

where $U^2 = 4W^3 - 27T^4$. See [IN99], subsection 14.3 for the notation. We caution that A_{12} in [IN99] is different from the present one.

The decomposition of S_k and $S_k(\mathfrak{m}/\mathfrak{n})$ for small values of k are given in Table 7. The factors of $S_k(\mathfrak{m}/\mathfrak{n})$ in the parentheses are those in $S_{\mathrm{McKay}}(G)$. We see by Table 7 that $V_{6\pm d(\rho)}(\rho) \simeq \rho^{\oplus 2}$ if $d(\rho) = 0$, or ρ if $d(\rho) \geq 1$. We also see that $S_{6-k}(\mathfrak{m}/\mathfrak{n}) \simeq S_{6+k}(\mathfrak{m}/\mathfrak{n})$ for any k. Thus Lemma 3.7 for E_6 follows from Table 7 immediately. The (extended) McKay quiver and subquivers of E_6 are given in Figures 14-16. See also Figure 4.

6. Proof of Theorem 3.9

Now we prove Theorem 3.9 mainly for D_5.

6.1. **The sheaf $\mathcal{V}$.** Let Z_{univ} be the universal subscheme of $\mathbb{A}^2$ of G-orbits parameterized by $\mathrm{Hilb}^G(\mathbb{A}^2)$, and $X = \mathrm{Hilb}^G(\mathbb{A}^2)$. The natural morphism

$\pi : X \to \mathbb{A}^2/G$ is known by Theorem 2.5 to be the minimal resolution. We define I_{univ} to be the ideal sheaf of $O_{\mathbb{A}^2\times X}$ defining Z_{univ} in $\mathbb{A}^2 \times X$. Then we have an exact sequence $0 \to I_{\mathrm{univ}} \to O_{\mathbb{A}^2\times X} \to O_{Z_{\mathrm{univ}}} \to 0$.

We define I_X to be the defining ideal of X as a subscheme of $X \simeq (\mathbb{A}^2/G) \times_{(\mathbb{A}^2/G)} X$ of $(\mathbb{A}^2/G) \times X$, $\mathfrak{n}_X := I_X O_{\mathbb{A}^2\times X}$ and

$$V(I_{\mathrm{univ}}) := I_{\mathrm{univ}}/\mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X,$$

which we denote by $\mathcal{V}$. The sheaf $\mathcal{V}$ is a finite $O_{\mathbb{A}^2} \otimes O_X$-module supported by $\{0\}\times E$ because $\mathfrak{m}O_{\mathbb{A}^2} = O_{\mathbb{A}^2}$ outside the origin and $\{0\}\times X \cap \mathrm{Supp}(Z_{\mathrm{univ}})_{\mathrm{red}} = \{0\} \times E$. It is clear that $\mathfrak{m}\mathcal{V} = \mathfrak{n}_X\mathcal{V} = 0$ from the definition of $\mathcal{V}$.

Let $\overline{\mathfrak{m}}$ be the maximal ideal of the unique singular point of $\mathbb{A}^2/G$. By the definition of $\mathfrak{n}_X$, $\phi^*F = \pi^*F \mod \mathfrak{n}_X$ for any $F \in \overline{\mathfrak{m}}$.

We prove next $\pi^*(\overline{\mathfrak{m}})\mathcal{V} = 0$. Let $H \in I_{\mathrm{univ}}$ and $a \in \pi^*\overline{\mathfrak{m}}$. Then there are some $F_k \in O_X$ and $A_k \in \overline{\mathfrak{m}}$ such that $a = \sum_k F_k\pi^*(A_k)$. Since $\mathfrak{n}_X \subset I_{\mathrm{univ}}$ (See subsection 3.8), we have $aH \equiv \sum_k F_k\phi^*(A_k)H$ in $I_{\mathrm{univ}}/\mathfrak{n}_X$. However $\sum_k F_k\phi^*(A_k)H \in \mathfrak{m}I_{\mathrm{univ}}$, which proves $aH = 0$ in $\mathcal{V}$. It follows $\pi^*(\overline{\mathfrak{m}})\mathcal{V} = 0$. Since $\pi^*\overline{\mathfrak{m}}$ is the ideal of O_X defining the fundamental cycle E_{fund} of E, this implies that $\mathcal{V}$ is a finite $O_{E_{\mathrm{fund}}}$-module.

Since $E(\rho)$ is a subscheme of E_{fund}, $\mathcal{V} \otimes O_{E(\rho)}$ is a finite $O_{E(\rho)}$-module and we have a natural homomorphism

$$\mathcal{V} \to \sum_{\rho\in\mathrm{Irr}\,G} \mathcal{V} \otimes O_{E(\rho)}. \tag{2}$$

We prove this is an isomorphism in subsections 6.2 and 6.3.

6.2. **Freeness outside** $\mathrm{Sing}(E)$**.** First we prove that (2) is an isomorphism at a nonsingular point of E. Let $S = \mathbb{C}[x, y]$.

Let $I \in E(\rho) \setminus \mathrm{Sing}(E)$. Then the ideal I is generated by a nonzero irreducible G-submodule W of $V(\rho) = V_{h-d(\rho)}(\rho) + V_{h+d(\rho)}(\rho)$ and $\mathfrak{n}$ by section 4 or by [IN99], Theorem 10.7.

First we consider the case D_5.

6.2.1. Let $\rho = \rho_1$. Then $W \neq V_6(\rho_1)$. As $\mathrm{Hilb}^G(\mathbb{A}^2)$ is nonsingular and two-dimensional, the tangent space $T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$ is exactly two-dimensional. By subsection 4.3

$$\mathrm{Hom}_S(I/I^2, S/I)[\rho_0] = \mathrm{Hom}_{\mathbb{C}}(W, V_6(\rho_1)) \oplus \mathrm{Hom}_{\mathbb{C}}(V_6(\rho_0), V_0(\rho_0)).$$

We note that I is generated by W and A_6 by subsection 4.9. Since $T_{[I]}(E(\rho_1)) = \mathrm{Hom}_{\mathbb{C}}(W, V_6(\rho_1))$, the parameter t of $\mathrm{Hom}_{\mathbb{C}}(V_6(\rho_0), V_0(\rho_0))$ gives a defining equation of $E(\rho_1)$.

By subsection 4.3 the ideal I_{univ} of Z_{univ} is over $E(\rho_1) \setminus \mathrm{Sing}(E)$ generated by $xy+s(x^6-y^6)$ and A_6+t. Since $A_6+t \in \mathfrak{n}_X$, the quotient $\mathcal{V}$ is $S\otimes\mathbb{C}[s, t]/t$-free of rank one, hence O_E-free of rank one.

6.2.2. Let us consider next the case $\rho = \rho_4$. In this case, $I = I_4(W)$, $W \neq V_5(\rho_4)$. Then I is generated by W and A_4 by subsection 4.9. Let $a = x^3 + iy^3 + s_0(x^4y - ixy^4) \in W$ and take $\phi \in \mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$. We may assume $\phi(a) = s(x^4y - ixy^4) \in V_5(\rho_4)$. Then we see $A_8 + 2iA_4^2 + s_0A_4A_6 = (x^4y + ixy^4)a$. Since $A_jA_k \in I^2$, we have $\phi(A_8) = (x^4y + ixy^4)\phi(a) = sA_4A_6 = 0$. Similarly, $\phi(A_6 + s_0A_8 - 2is_0A_4^2) = (x^3 - iy^3)\phi(a) = s(A_8 - 2iA_4^2) = 0$. Hence we have $\phi(A_6) = \phi(A_8) = 0$. It follows that letting $\phi(A_4) = t$, then s and t are local (regular) parameters of $\mathrm{Hilb}^G(\mathbb{A}^2)$ at I. Thus we see

$$\mathrm{Hom}_S(I/I^2, S/I)[\rho_0] = \mathrm{Hom}_{\mathbb{C}}(W, V_3(\rho_4)) \oplus \mathrm{Hom}_{\mathbb{C}}(V_4(\rho_0), V_0(\rho_0)).$$

Thus we have generators $A := A_4 + t$ and $B := x^3 + iy^3 + sxy(x^3 - iy^3)$ of I_{univ}, whose derivation span $T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2))$. Moreover $A_6 + sA_8 - 2isA_4^2 = (x^3 - iy^3)B \in I_{\mathrm{univ}}$, and $A_8 + 2iA_4^2 + sA_4A_6 = (x^3 + iy^3)xyB \in I_{\mathrm{univ}}$. Thus I_{univ} is generated by A and B, whence over $E(\rho_4) \setminus \mathrm{Sing}(E) \simeq \mathrm{Spec}\,\mathbb{C}[s,t]/(t)$,

$$I_{\mathrm{univ}} = (x^3 + iy^3 + sxy(x^3 - iy^3), A_4 + t).$$

Since $\pi^*(\overline{\mathfrak{m}})$ is the defining ideal of E_{fund}, we have $\pi^*(\overline{\mathfrak{m}}) = (t)$. Therefore $\mathcal{V}$ is $S \otimes \mathbb{C}[s,t]/t$-free of rank one.

6.2.3. The above arguments are easily generalized to any one-dimensional irreducible representation of D_n, E_6 and E_7 by using [IN99], sections 13-15. We note that there is no one-dimensional irreducible representation for E_8.

6.2.4. Next we consider ρ_2 in the D_5-case. Let $I = I_2(W)$, $W \neq V_3(\rho_2), V_5(\rho_2)$. Then we note that $I_6(\rho_1) \subset SW + \mathfrak{n}$ and that $V_4(\rho_0) + V_6(\rho_0)/W$ is a part of generators of I/I^2 by the property of $\mathrm{Quiv}(\rho_2)$ mentioned in subsection 4.9. Thus I/I^2 is generated by $b_1 = x^2y + s_0y^5$, $b_2 = -xy^2 - s_0x^5$ and A_k. The condition $W \neq V_3(\rho_2), V_5(\rho_2)$ is just $s_0 \neq 0, \infty$.

We compute now $T_{[I]}(\mathrm{Hilb}^G(\mathbb{A}^2)) = \mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$, relying on these facts. It is clear that $(S/I)[\rho_2] \simeq V_5(\rho_2) \oplus V_1(\rho_2)$. We define ψ_1 and ψ_2 to be the elements of $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$

$$\psi_1(b_1) = y^5, \psi_1(b_2) = -x^5, \psi_2(b_1) = -x, \psi_2(b_2) = -y.$$

We prove that ψ_1 and ψ_2 span $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$. So we take an element ϕ of $\mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$ and let $\phi(b_1) = sy^5 - tx, \phi(b_2) = -sx^5 - ty$.

We shall prove $\phi(A_k) = 0$ $(k = 4, 6, 8)$. We note $\phi(A_k) \in \mathbb{C} = (S/I)[\rho_0]$, a constant. First $\phi(A_8) = x^5\phi(b_1) + y^5\phi(b_2) = -tA_6 = 0$. Secondly, $\phi(2A_4 + s_0A_6) = y\phi(b_1) - x\phi(b_2) = sA_6 = 0$. Thirdly, we see

$$xy\phi(A_6) = x^5\phi(b_1) - y^5\phi(b_2) - 2s_0xy\phi(A_4^2) = -t(x^6 - y^6) = 0$$

because $(x^6 - y^6) \in V_6(\rho_1) \subset SW + \mathfrak{n} = I_2(W) = I$. Meanwhile, $xy \in V_2(\rho_1)$ which is nonzero in S/I. Hence $\phi(A_6) = 0$. Hence we have $\phi(A_k) = 0$ for all A_k. It follows that $\phi = s\psi_1 + t\psi_2$.

Since I/I^2 is generated by b_1, b_2 and A_k, this proves

$$\mathrm{Hom}_S(I/I^2, S/I)[\rho_0] = \mathrm{Hom}_{\mathbb{C}}(W, V_5(\rho_2) \oplus V_1(\rho_2)).$$

Since $T_{[I]}(E(\rho_2)) = \mathrm{Hom}_{\mathbb{C}}(W, V_5(\rho_2))$, the parameter t of $\mathrm{Hom}_{\mathbb{C}}(W, V_1(\rho_2))$ gives the equation of $E(\rho_2)$ locally along $E(\rho_2)\setminus \mathrm{Sing}(E)$. Now we prove that the ideal I_{univ} is generated by the elements

$$B_1 := x^2y + sy^5 - tx, B_2 := -xy^2 - sx^5 - ty, A := A_4 - \eta,$$

where $\eta := \pi^* A_4$ is a power series of t with initial term t^2 satisfying $s^2\eta^2 - \eta + t^2 = 0$. In fact, Noetherian property shows that I_{univ} is generated by B_1, B_2 and some elements with initial terms being A_k. We see $2A_4 + sA_6 = yB_1 - xB_2$, $sA_8 + 2tA_4 = -xy(yB_1 + xB_2)$ and $A_8 - tA_6 = x^5B_1 + y^5B_2$. Hence all of these belong to I_{univ}. Since $\pi^*(\overline{\mathfrak{m}}) \cap I_{\mathrm{univ}} = \{0\}$, we have $\pi^*(2A_4 + sA_6) = \pi^*(A_8 - tA_6) = 0$, and therefore the ideal $\pi^*\overline{\mathfrak{m}} = (\pi^*A_4, \pi^*A_6, \pi^*A_8)$ of O_X is generated by π^*A_4 because s is invertible over $E(\rho_2) \setminus \mathrm{Sing}(E)$. We note $E(\rho_2) \setminus \mathrm{Sing}(E) \simeq \mathrm{Spec}\,\mathbb{C}[s, s^{-1}]$. It is easy to infer from $A_8^2 = A_4A_6^2 - 4A_4^4$ that η satisfies $s^2\eta^2 - \eta + t^2 = 0$.

Since $-t^2 = s^2\eta^2 - \eta$ and $A_4 - \eta \in \mathfrak{n}_X$, we have in $\mathcal{V}$

$$\begin{aligned} tB_1 &= tx^2y + sty^5 + (s^2A_4^2 - A_4)x \\ &= -xyB_1 - sy^4B_2 = 0, \\ tB_2 &= xyB_2 + sx^4B_1 = 0. \end{aligned}$$

Hence $\mathcal{V}$ is an O_E-module with B_1 and B_2 generators. By section 4.4 that $V(I)$ is of rank two for any $I \in E(\rho_2)\setminus \mathrm{Sing}(E)$, hence for each $s \in \mathbb{C}$, $s \neq 0$. This implies that $\mathcal{V} \otimes S[s, s^{-1}, t]/(t)$ is $S[s, s^{-1}]$-free of rank two.

6.2.5. As the final case of D_5, we consider ρ_3. Let $I = I_3(W)$, $W \neq S_1V_3(\rho_k)$ $(k = 2, 4, 5)$. Then we see

$$\mathrm{Hom}_S(I/I^2, S/I)[\rho_0] = \mathrm{Hom}_{\mathbb{C}}(W, V_4(\rho_3)/W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_2(\rho_3)).$$

The proof in this case is however rather tricky. Let $b_1 = y^4 + is_0x^3y$ and $b_2 = x^4 - is_0xy^3$. By the condition $W \neq S_1V_3(\rho_k)$, we have $s_0 \neq \pm 1, \infty$. Let $\phi \in \mathrm{Hom}_S(I/I^2, S/I)[\rho_0]$. Since $(S/I)[\rho_3] = \{x^3y, -xy^3\} \oplus V_2(\rho_3)$, we may assume $\phi(b_1) = isx^3y + tx^2$ and $\phi(b_2) = -isxy^3 + ty^2$. First we note that $\phi(b_i) \in (S_4 + S_2)(\mathfrak{m}/I)$, whence $\phi(S_2b_i) \in (S_6 + S_4)(\mathfrak{m}/I) = S_4(\mathfrak{m}/I)$. Hence we see $\phi(y^2b_1) = isx^3y^3 + tA_4 = tA_4 = 0$, $\phi(x^2b_2) = -isx^3y^3 + tA_4 = tA_4 = 0$. Similarly $\phi(x^2b_1) = isx^5y + tx^4 = tx^4$, $\phi(xyb_1) = tx^3y$, $\phi(xyb_2) = txy^3$ and $\phi(y^2b_2) = ty^4$.

Then $\phi(xy^3b_1 - x^3yb_2) = -\phi(A_8) + 2is_0\phi(A_4^2) = -\phi(A_8)$, while $\phi(xy^3b_1 - x^3yb_2) = xy\phi(y^2b_1) - xy\phi(x^2b_2) = 0$. Hence $\phi(A_8) = 0$. Similarly $\phi(A_6) = \phi(y^2b_1 + x^2b_2) = 2tA_4 = 0$. We also see that $(1 - s_0^2)\phi(x^2y^4) = \phi(x^2b_1 - is_0xyb_2) = tx^4 - is_0txy^3 = tb_2 \in W \subset I$. Hence $(1 - s_0^2)\phi(x^2y^4) = 0$ in S/I, whence $\phi(x^2y^4) = 0$ because $1 - s_0^2 \neq 0$. Similarly we see $\phi(x^4y^2) = 0$. It follows that $\{x^2, y^2\}\phi(A_4) = 0$. Hence $\phi(A_4) = 0$ because $0 \neq \{x^2, y^2\} \subset (S/I)[\rho_2]$. This proves $\phi(A_k) = 0$ for any A_k. Hence $\mathrm{Hom}_S(I/I^2, S/I) = \mathrm{Hom}_{\mathbb{C}}(W, V_4(\rho_3)/W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_2(\rho_3))$.

We know by the subquiver $\mathrm{Quiv}(\rho_3)$ that I is generated by W and A_4. We have B_1, B_2 and A as generators of I_{univ} as follows :

$$B_1 = y^4 + isx^3y + tx^2, B_2 = x^4 - isxy^3 + ty^2, A = A_4 - \pi^*(A_4),$$

where $\pi^*(A_4) = \frac{t^2}{1-s^2}$. We note $A_6 + 2tA_4$ and $A_8 - 2isA_4^2 \in I_{\mathrm{univ}}$. Since t is the parameter of $\mathrm{Hom}_{\mathbb{C}}(W, V_2(\rho_3))$, $E(\rho_3)$ is defined by $t = 0$.

Now we prove in $S[s,t]$

$$\begin{aligned} tB_1 &= y^2B_2 + isxyB_1 - (1-s^2)x^2A,\\ tB_2 &= x^2B_1 - isxyB_2 - (1-s^2)y^2A. \end{aligned}$$

Hence $tB_1 = tB_2 = 0$. This shows that $\mathcal{V}$ is $\mathbb{C}[s, \frac{1}{1-s^2}]$-free of rank two, where $E(\rho_3) \setminus \mathrm{Sing}(E) = \mathrm{Spec}\,\mathbb{C}[s, \frac{1}{1-s^2}]$.

This completes the proof of freeness of $\mathcal{V}$ over $E \setminus \mathrm{Sing}(E)$ in the D_5-case.

6.2.6. It is clear that one can generalize the above arguments to D_n for the other n. To settle the E_6 and E_7-cases, we need to discuss three-dimensional or four-dimensional representations ρ. Since the discussion below on the three-dimensional representation ρ_3 of E_6 shows the general features of the arguments for the proof sufficiently, we take up only ρ_3 of E_6 in order to avoid the overwhelming notation for E_7.

6.2.7. We consider the E_6-case. Let G be the binary tetrahedral group $\mathbb{T}$, and ρ_3 the unique three-dimensional representation of G. Let A_6, A_8 and A_{12} be the homogeneous generators of the ring of G-invariants :

$$A_6 = T = p_1p_2p_3, A_8 = W = \varphi\psi, A_{12} = U = \varphi^3 + \psi^3,$$

where $U^2 = 4W^3 - 27T^4$. See [IN99], subsection 14.3 for the notation. We note $\varphi^3 - \psi^3 = 3(2\omega+1)T^2$ and that both φ^3 and ψ^3 are G-invariants.

Then any point $I = I_3(W) \in E(\rho_3) \setminus \mathrm{Sing}(E)$ is given by an irreducible G-submodule of $V_6(\rho_3)$ with $W \neq S_1V_5(\rho_2), S_1V_5(\rho_2'), S_1V_5(\rho_2'')$ under the notation of section 5. Then we see

$$\mathrm{Hom}_S(I/I^2, S/I) = \mathrm{Hom}_{\mathbb{C}}(W, V_6(\rho_3)/W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_4(\rho_3)).$$

Moreover a versal deformation I_{univ} of I is generated by six elements

$$B_1, B_2, B_3, A_6 - \pi^*(A_6), A_8 - \pi^*(A_8), A_{12} - \pi^*(A_{12}),$$

where with the notation of [IN99], subsection 14.3,

$$\begin{aligned} B_1 &= p_1(\varphi + \omega\psi) + sp_1\varphi + tp_2p_3 + up_1,\\ B_2 &= \omega p_2(\varphi + \omega^2\psi) + s\omega p_2\varphi - tp_3p_1 + up_2,\\ B_3 &= \omega^2 p_3(\varphi + \psi) + s\omega^2 p_3\varphi + tp_1p_2 + up_3, \end{aligned}$$

where s, t and u are parameters. We note that

$$\begin{aligned} V_6(\rho_3) &= \{p_1\varphi, \omega p_2\varphi, \omega^2 p_3\varphi\} \oplus \{p_1\psi, \omega^2 p_2\psi, \omega p_3\psi\},\\ V_4(\rho_3) &= \{p_2p_3, -p_3p_1, p_1p_2\}, V_2(\rho_3) = \{p_1, p_2, p_3\}. \end{aligned}$$

Then we see $p_1B_1 - \omega^2 p_2B_2 + \omega p_3B_3 = \omega(1-\omega)(\psi^2 - \omega u\varphi)$. Hence $\psi^2 - \omega u\varphi \in I_{\mathrm{univ}}$, whence $\psi^3 - \omega uW \in I_{\mathrm{univ}}$. Similarly we see $p_1B_1 - p_2B_2 + p_3B_3 = (\omega-1)sW - 3tT$, which belongs to I_{univ}. We also have $(1-\omega^2)\{(1+s)\varphi^2 - \omega u\psi\} = p_1B_1 - \omega p_2B_2 + \omega^2 p_3B_3 \in I_{\mathrm{univ}}$. Hence $(1+s)\varphi^3 - \omega uW \in I_{\mathrm{univ}}$. From $\pi^*\overline{\mathfrak{m}} \cap I_{\mathrm{univ}} = \{0\}$ it follows that

$$\pi^*\psi^3 = \omega u\pi^*W, (1+s)\pi^*\varphi^3 = \omega u\pi^*W,$$
$$(\omega-1)s\pi^*W = 3t\pi^*T, \pi^*\varphi^3 - \pi^*\psi^3 = 3(2\omega+1)\pi^*T^2.$$

It follows that

$$\pi^*W = \frac{\omega^2u^2}{1+s}, \pi^*\varphi^3 = \frac{u^3}{(1+s)^2}, \pi^*\psi^3 = \frac{u^3}{1+s},$$
$$3\pi^*T = (1-\omega^2)\frac{u^2s}{(1+s)t}, (1-\omega)su = t^2.$$

Though the relation $(1-\omega)su = t^2$ looks singular at $P(\rho_2, \rho_3)$, the point $s = t = 0$ of $\mathrm{Hilb}^G(\mathbb{A}^2)$, it is not singular at all because s and t/s are regular parameters at $P(\rho_2, \rho_3)$.

The condition $W \neq S_1V_5(\rho_2), S_1V_5(\rho_2'), S_1V_5(\rho_2'')$ implies $s(1+s) \neq 0$, $s \neq \infty$. The parameters s, t and u are related by $(1-\omega)su = t^2$. Hence $\pi^*\overline{\mathfrak{m}} = (\pi^*T) = (u^2/t) = (t^3)$ along $E(\rho_3) \setminus \mathrm{Sing}(E)$, whence $E_{\mathrm{fund}} = 3E(\rho_3)$ there.

We see $\mod \mathfrak{n}_X$

$$\begin{aligned}
tB_1 &= tp_1(\varphi + \omega\psi) + stp_1\varphi + t^2p_2p_3 + tup_1 \\
&= tp_1(\varphi + \omega\psi) + stp_1\varphi + t^2p_2p_3 - 3\omega(1+s)p_1T \\
&= -\omega^2(s-\omega)p_3B_2 + (s-\omega^2)p_2B_3, \\
tB_2 &= \omega(s-\omega)p_1B_3 - (s-\omega^2)p_3B_1, \\
tB_3 &= -\omega(s-\omega)p_2B_1 + (s-\omega^2)p_1B_2.
\end{aligned}$$

This proves $tB_i = 0$ in $\mathcal{V}$. Hence $\mathcal{V}$ is $S[s, \frac{1}{s(1+s)}]$-free of rank three. This completes the proof of freeness of $\mathcal{V}$ over $E(\rho_3) \setminus \mathrm{Sing}(E)$ for E_6.

6.2.8. We explain very briefly the most complicated case of E_7, that is, the ρ_4-case. The finite group G involved is the binary octahedral group $\mathbb{O}$, and the invariant ring of G is generated by homogeneous polynomials of degree 8, 12, and 18, where we note that 18 is also the Coxeter number of E_7.

Any point $I = I_4(W) \in E(\rho_4) \setminus \mathrm{Sing}(E)$ is given by an irreducible G-submodule of $V_9(\rho_4)$ with $W \neq S_1V_8(\rho_2''), S_1V_8(\rho_3), S_1V_8(\rho_3')$ under the notation of section 5. Then we see

$$\mathrm{Hom}_S(I/I^2, S/I) = \mathrm{Hom}_{\mathbb{C}}(W, V_9(\rho_4)/W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V_7(\rho_4)).$$

The versal deformation I_{univ} is generated over $E(\rho_4) \setminus \mathrm{Sing}(E)$ by five elements $B_1, B_2, B_3, B_4, A := A_8 - \pi^*(A_8)$, where $A_8 = W$ is the same as W of

E_6 and the elements B_i are of the form

$$B_i = B_{i1} + sB_{i2} + tB_{i3} + uB_{i4} + vB_{i5}$$

such that

$$V_9(\rho_4) = \{B_{i1}, B_{i2}; i = 1, 2, 3, 4\},$$
$$V_{13-2k}(\rho_4) = \{B_{ik}; i = 1, 2, 3, 4\} \quad (k = 3, 4, 5).$$

See [IN99], Table 13 for ρ_4-factors of $S_{\mathrm{McKay}}(G)$. Moreover over $E(\rho_4) \setminus \mathrm{Sing}(E)$, u (resp. v) is a unit multiple of t^2 (or resp. t^3), while $\pi^*(A_8)$ is a unit multiple of t^4, and $\pi^*\overline{\mathfrak{m}} = (A_8) = (t^4)$. Then we can prove $tB_i = 0$ in $\mathcal{V}$ in the same manner as before. In fact, the proof goes roughly as follows. The term tB_1 is the sum of B_{ik}, whose last term tvB_{i5} is a multiple of t^4. Hence tvB_{i5} can be replaced mod $\mathfrak{n}_X$ by a unit multiple of A_8B_{i5}. Then we see that $tB_1 - tvB_{i5} +$ (the unit multiple of A_8B_{i5}) is a sum of B_j over $\mathfrak{m}$. In other words, tB_1 is a sum of B_j and A over $\mathfrak{m}$. Since ρ_4 is irreducible, this implies that tB_i is also a sum of B_j and A over $\mathfrak{m}$ for any i. Thus we can prove that $\mathcal{V}$ is $O_{E(\rho_4)}$-free over $E(\rho_4) \setminus \mathrm{Sing}(E)$.

We can write down precisely the versal deformations of $I \in E(\rho) \setminus \mathrm{Sing}(E)$ similarly for any ρ of E_7. By this, we can prove freeness of $\mathcal{V}$ along $E\setminus\mathrm{Sing}(E)$. We omit the details of E_7-case because we need more notation.

6.3. **Isomorphism at $I(\rho, \rho')$.** In this subsection we prove that (2) is an isomorphism at any singular point $I := I(\rho, \rho')$ of E.

6.3.1. First we consider the pair $\rho = \rho_1$ and $\rho' = \rho_2$ in the D_5-case. Then I_{univ} is given in (1). It is clear that $\mathcal{V}$ is generated by those elements whose specializations at $s = t = 0$ are just the generators of I belonging to $V_6(\rho_1) + V_3(\rho_2)$. Let $B = x^6 - y^6 + sxy$, $C_1 = x^2y + ty^5 - \frac{st}{2}x$ and $C_2 = -xy^2 - tx^5 - \frac{st}{2}y$. The ideal $\pi^*(\overline{\mathfrak{m}})$ is generated by ϕ_2, hence by s^2t. We first see

$$tB = -(yC_1 + xC_2) = 0 \quad \text{in } \mathcal{V}.$$

Next we prove $sC_1 = 0$ in $\mathcal{V}$. We compute $\mod \mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X$:

$$\begin{aligned}
sC_1 &= sx^2y + sty^5 - \frac{s^2t}{2}x \\
&= xB - x(x^6 - y^6) + sty^5 - \frac{s^2t}{2}x \\
&= -xA_6 + 2xy^6 + sty^5 - \frac{s^2t}{2}x \quad (\because xB \in \mathfrak{m}I_{\mathrm{univ}}) \\
&= -x(A_6 + \frac{s^2t}{2} + \frac{t^3}{2}A_6^2) + 2xy^6 + sty^5 + \frac{t^3}{2}xA_6^2 \\
&= 2xy^6 + sty^5 + 2txA_4^2 = -2y^4C_2 = 0,
\end{aligned}$$

and similarly $sC_2 = 0$. This proves

$$\mathcal{V} = O_{E(\rho_1)}B + O_{E(\rho_2)}C_1 + O_{E(\rho_2)}C_2.$$

Since $\mathcal{V}$ is $O_{E(\rho)}$-free of rank $\deg\rho$ over $E(\rho)\setminus\mathrm{Sing}(E)$, the upper-semicontinuity shows that $O_{E(\rho_1)}B$ is $O_{E(\rho_1)}$-free of rank $\deg\rho_1$ $(=1)$ at $s=t=0$, while $O_{E(\rho_2)}C_1+O_{E(\rho_2)}C_2$ is $O_{E(\rho_2)}$-free of rank $\deg\rho_2$ $(=2)$ at $s=t=0$. This proves that (2) is an isomorphism at $I(\rho_1,\rho_2):s=t=0$.

6.3.2. If $\rho=\rho_2$ and $\rho'=\rho_3$ in the D_5-case, then I_{univ} is generated by those elements whose specializations at $s=t=0$ belong to $V_5(\rho_1)+S_1V_3(\rho_2)$.

Let $R=\mathbb{C}[[s,t]][x,y]$. By subsection 4.7, the versal deformation I_{univ} of I is given by $\mathcal{I}_3$, which is, as the ideal of R, generated by the elements

$$B_1:=y^5+sx^2y+\lambda x, B_2:=-x^5-sxy^2+\lambda y,$$
$$C_1:=x^3y+ty^4+stx^2, C_2:=-xy^3+tx^4+sty^2,$$
$$A_6+2sA_4, A_8-2\lambda A_4, A_4-t\lambda,$$

where $\lambda=\frac{s^2t}{1+t^2}$. Let $A:=A_4-t\lambda$. We will check $sC_1=sC_2=0$ and $tB_1=tB_2=0$ in $\mathcal{V}$. In fact, in $S[[s,t]]$ we have

$$tB_1=yC_1-xA,\ tB_2=-xC_2-yA,$$
$$sC_1=(1+t^2)xB_1-txyC_1+y^2C_2,$$
$$sC_2=(1+t^2)yB_2-txyC_2+x^2C_1.$$

It follows that

$$\mathcal{V}=O_{E(\rho_1)}B+O_{E(\rho_2)}C_1+O_{E(\rho_2)}C_2.$$

Since $\mathcal{V}$ is $O_{E(\rho)}$-free of rank $\deg\rho$ over $E(\rho)\setminus\mathrm{Sing}(E)$, the upper-semicontinuity shows that $O_{E(\rho_2)}B_1+O_{E(\rho_2)}B_2$ is $O_{E(\rho_2)}$-free of rank $\deg\rho_2$ at $s=t=0$, while $O_{E(\rho_3)}C_1+O_{E(\rho_3)}C_2$ is $O_{E(\rho_3)}$-free of rank $\deg\rho_3$ at $s=t=0$. This proves that (2) is an isomorphism at $I(\rho_2,\rho_3):s=t=0$.

6.3.3. In this subsubsection we consider one of the most complicated case of E_6 where $\rho=\rho_2$ and $\rho'=\rho_3$ with $\deg(\rho_2)=2$ and $\deg(\rho_3)=3$. For the notation see Figure 4. Let $I:=P(\rho_2,\rho_3)$ and $W=(S_1V_5(\rho_2))[\rho_3]$. Then $I:=I(\rho_2,\rho_3)=W+\sum_{k=7}^{11}S_k+\mathfrak{n}$ and $V(I)=W\oplus V_7(\rho_2)$. Moreover

$$\mathrm{Hom}_S(I/I^2,S/I)=\mathrm{Hom}_{\mathbb{C}}(W,V_6(\rho_3/W))\oplus\mathrm{Hom}_{\mathbb{C}}(V_7(\rho_2),V_5(\rho_2)).$$

The versal deformation I_{univ} of $I(\rho_2,\rho_3)$ is generated by

$$B_1,B_2,B_3,C_1,C_2,A,$$

where B_i $(i=1,2,3)$ is the same as in subsubsection 6.2.8, and

$$C_1:=-s_2\varphi+v\gamma_1+wx, C_2:=s_1\varphi+v\gamma_2+wy,$$
$$A:=T-\pi^*(T)=T-\frac{1}{2\omega+1}sw=T-\frac{1}{1+s}s^2v^3.$$

The parameters s and v are regular parameters of $\mathrm{Hilb}^G(\mathbb{A}^2)$ at I. The other parameters are related as follows :

$$t = -(1-\omega)sv, u = (1-\omega)sv^2, w = -\frac{2\omega+1}{1+s}sv^3,$$

whence $u = -tv$, $uv = -\omega^2(1+s)w$ and $(2\omega+1)\pi^*(T) = sw$.

Then we see in $S[s,v]$,

$$\begin{aligned}
vB_1 &= -\omega^2(1+s)(xC_1 - yC_2) + \frac{1}{1-\omega}(\omega p_2 B_3 - p_3 B_2),\\
vB_2 &= -\omega^2(1+s)(xC_1 + yC_2) - \frac{1}{1-\omega}(\omega p_3 B_1 - p_1 B_3),\\
vB_3 &= -\omega^2(1+s)(yC_1 + xC_2) + \frac{1}{1-\omega}(\omega p_1 B_2 - p_2 B_1),\\
sC_1 &= \frac{1}{1-\omega}(yB_1 + yB_2 - xB_3) - (2\omega+1)xA,\\
sC_2 &= \frac{1}{1-\omega}(xB_1 - xB_2 + yB_3) - (2\omega+1)yA.
\end{aligned}$$

This proves $\mathcal{V} = \mathcal{V}\otimes O_{E(\rho_2)} + \mathcal{V}\otimes O_{E(\rho_3)}$.

6.3.4. In general, let $I \in P(\rho,\rho')$ be a singular point of E. Then as we saw above, we have an isomorphism

$$\mathcal{V}_{[I]} = \mathcal{V}_{[I]}\otimes O_{E(\rho)} \oplus \mathcal{V}_{[I]}\otimes O_{E(\rho')}$$

locally at I. This proves the global isomorphism over E

$$\mathcal{V} \simeq \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathcal{V}\otimes O_{E(\rho)}.$$

6.4. **The sheaf $\mathcal{V}\otimes O_{E(\rho)}$.** It remains to prove $\mathcal{V}\otimes O_{E(\rho)} \simeq \rho\otimes O_{E(\rho)}(-1)$.

6.4.1. Let V be a two-dimensional $\mathbb{C}$-vector space, and $\mathbb{P}(V)$ the projective line of one-dimensional subspaces of V. There is a universal family of one-dimensional subspaces of V parametrized by $\mathbb{P}(V)$, which we denote W_{univ}. This is a line bundle (an invertible sheaf) on $\mathbb{P}(V)$. There is an exact sequence of $O_{\mathbb{P}(V)}$-modules:

$$0 \to W_{\mathrm{univ}} \to V\otimes O_{\mathbb{P}(V)} \to V\otimes O_{\mathbb{P}(V)}/W_{\mathrm{univ}} \to 0.$$

This implies that $W_{\mathrm{univ}} \simeq O_{\mathbb{P}(V)}(-1)$ because the bundle is twisted linearly.

Let $V(\rho) := S_{\mathrm{McKay}}(G)[\rho] = V_{h-d(\rho)} + V_{h+d(\rho)} \simeq \rho^{\oplus 2}$, and $\mathbb{P}(V(\rho))$ the projective line of nonzero irreducible G-submodules of $V(\rho)$. Then $V(\rho) \simeq \rho\otimes V$ and $\mathbb{P}(V(\rho)) \simeq \mathbb{P}(V)$. It is obvious that $\rho\otimes W_{\mathrm{univ}}$ yields a universal family $W_{\mathrm{univ}}(\rho)$ of nonzero G-submodules of $V(\rho)$ parametrized by $\mathbb{P}(V(\rho))$ ($\simeq \mathbb{P}(V)$). We see $W_{\mathrm{univ}}(\rho) \simeq \rho\otimes O_{\mathbb{P}(V(\rho))}(-1)$.

6.4.2. Let $I \in E(\rho)$. Then by identifying $E(\rho)$ with $\mathbb{P}(V)$, on any Zariski open subset $U := \operatorname{Spec} A$ of $E(\rho)$ we have

$$I_{\mathrm{univ}} \otimes A = W_{\mathrm{univ}}(\rho) + \sum_{V \subset \mathrm{Quiv}(\rho), v \not\simeq \rho} V \otimes A + S^{\dagger} \otimes A,$$

and by subsection 6.3

$$\mathcal{V} \otimes A = W_{\mathrm{univ}}(\rho) \otimes A = W_{\mathrm{univ}}(\rho).$$

In other words, $\mathcal{V} \otimes O_{E(\rho)} \simeq W_{\mathrm{univ}}(\rho) \simeq \rho \otimes O_{E(\rho)}(-1)$.

This completes the proof of Theorem 3.9.

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