Compactification of the moduli of abelian varieties and

Morphisms of $SQ_{g,K}$ to Alexeev's Moduli

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(Hokkaido University)

2012 November 15, Hokkaido University

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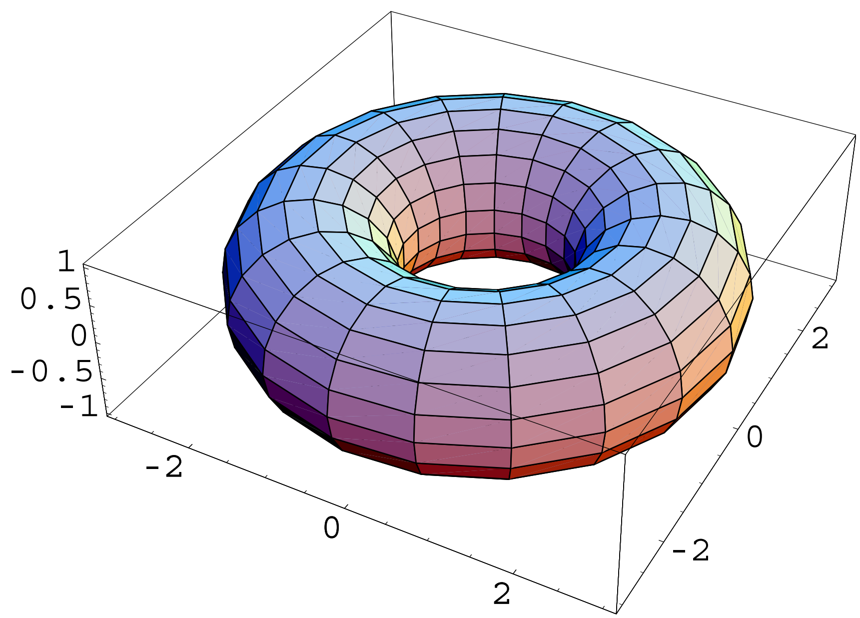
1 Hesse cubic curves

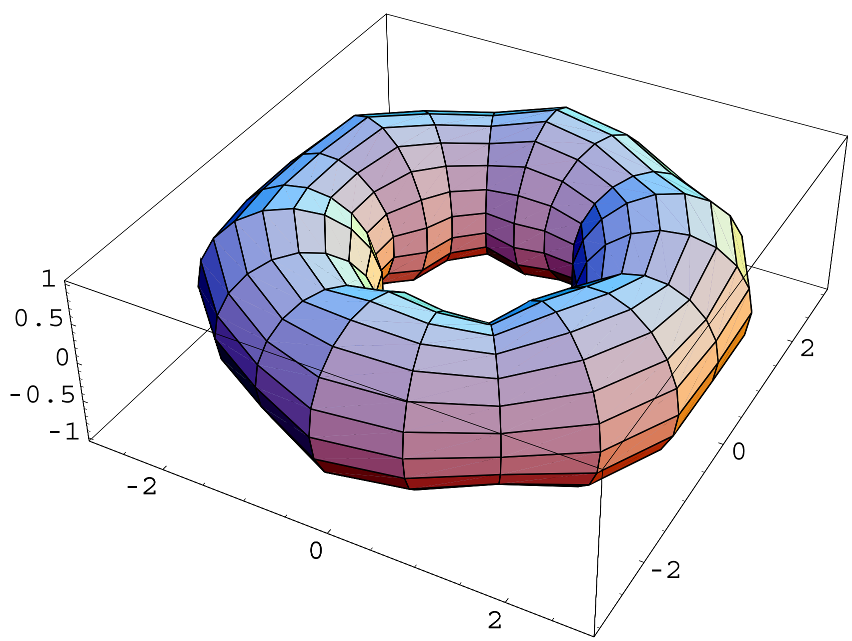
$$C(\mu) : x_0^3 + x_1^3 + x_2^3 - 3\mu\, x_0 x_1 x_2 = 0$$

$$(\mu \in \mathbf{P}^1_{\mathbf{Z}[\zeta_3, 1/3]})$$



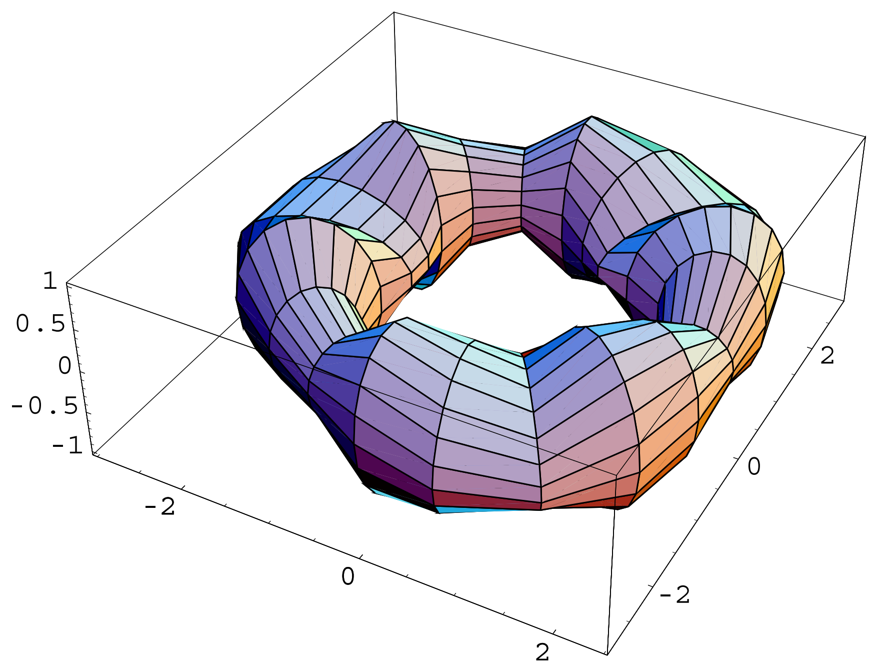
$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0$$

if μ gets closer to ∞



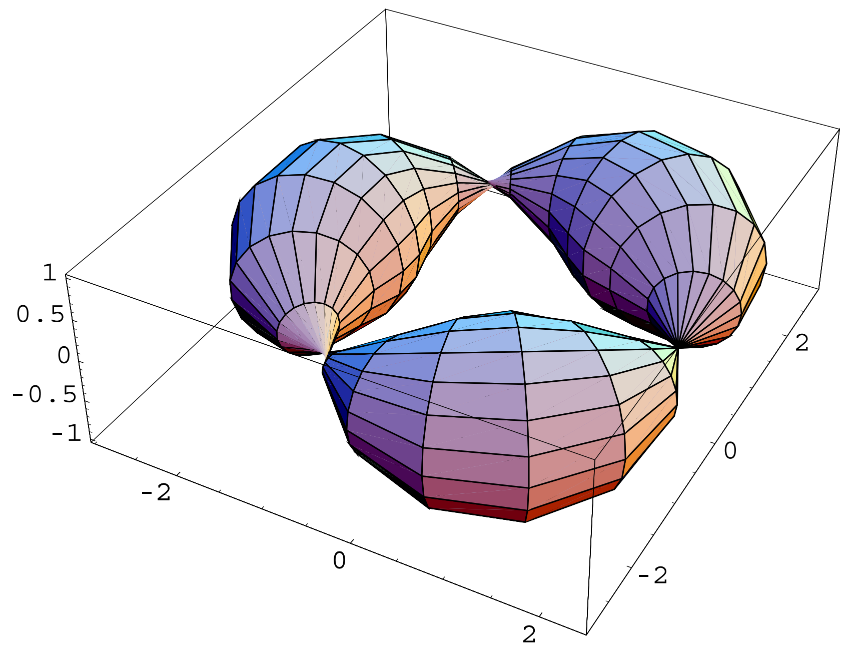
$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu \in \mathbf{Z}[\zeta_3, 1/3])$$

if μ gets much closer to ∞



$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu^3 = 1 \ \text{or} \ \infty)$$

It degenerates into 3 copies of $\mathbf{P}^1$



2 Moduli of cubic curves

Thm 1 (classical form over C) (Hesse 1849)

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$\simeq \mathbf{C} \setminus \{1, \zeta_3, \zeta_3^2\} \simeq \mathbf{H}/\Gamma(3)\ (\mathbf{H} : \text{upper half plane})$$

$$SQ_{1,3} := \overline{A_{1,3}}$$

$$= \{\text{stable cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \,\text{or}\, \infty\right\} \simeq \mathbf{P}^1$$

$$= \{\text{moduli of compact objects}\}$$

We wish to extend this to aribitrary dimension

1. over $\mathbf{Z}[\zeta_N, 1/N]$ (Today) or over $\mathbf{Z}[\zeta_N]$
2. to define a representable functor of compact obj.

   $\boxed{F := SQ_{g,K}}$ (fine moduli)
3. to relate $SQ_{g,K}$ to GIT stability, (This is new)
4. $\boxed{\text{GIT stable objects = our model PSQASes}}$:

   Projectively **S**table **Q**uasi **A**belian **S**cheme
5. to relate 3 known compactif. $\boxed{SQ_{g,K}}$, $\boxed{SQ_{g,K}^{\text{toric}}}$

   Alexeev's moduli $\boxed{\overline{A}_{g,d}}$

3 Moduli over $\mathbf{Z}[\zeta_N, 1/N]$

Thm 2 (a new version of the theorem of Hesse)

$$SQ_{1,3} = \mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]},$$

the projective fine moduli

(1) The universal cubic curve

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0$$

where $(\mu_0, \mu_1) \in SQ_{1,3} = \mathbf{P}^1$.

(2) when k is alg. closed and char. $k \neq 3$

$$SQ_{1,3}(k) = \left\{ \begin{array}{l} \text{closed orbit cubics} \\ \text{with level 3-structure } /k \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} \text{Hesse cubics} \\ \text{with level 3-str. } /k \end{array} \right\} /\text{isom.=id.}$$

$$A_{1,3}(k) = \left\{ \begin{array}{l} \text{closed orbit nonsing. cubics} \\ \text{with level 3-str. } /k \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} \text{nonsing. Hesse cubics} \\ \text{with level 3-structure } /k \end{array} \right\} /\text{isom.=id.}$$

Thm 3 (N. 1999) There exists the fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$, $N = \sqrt{|K|}$, For k closed

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{closed orb. deg. abelian sch. } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant PSQAS } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\},$$

$$A_{g,K}(k) = \left\{ \begin{array}{l} \text{(nonsingular) abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-inv. abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

4 Comparison of three compactifications

Summary $N = \sqrt{|K|}$, $\mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$, $d > 0$.

1. $SQ_{g,K}$ is a proj. fine moduli over $\mathcal{O}_N$ [N99],
2. $SQ^{\mathrm{toric}}_{g,K}$ is a proj. coarse mod. over $\mathcal{O}_N$ [N01] [N10],
3. $\overline{AP}_{g,d} = \{(P, G, D)\}$ is a proper separated coarse moduli over $\mathbf{Z}$ [Alexeev02],
4. $\dim SQ_{g,K} = \dim SQ^{\mathrm{toric}}_{g,K} = g(g+1)/2$,
5. $\dim \overline{AP}_{g,d} = g(g+1)/2 + d - 1$,
6. $\exists$ a bij. mor. $\mathrm{sq} : SQ^{\mathrm{toric}}_{g,K} \to SQ_{g,K}$ [N10]

$$(SQ^{\mathrm{toric}}_{g,K})^{\mathrm{norm}} \simeq SQ^{\mathrm{norm}}_{g,K} \tag{1}$$

$SQ_{g,K,1/N} := SQ_{g,K}$: proj. over $\mathbf{Z}[\zeta_N, 1/N]$ (1999)

$\overline{AP}_{g,N}$: by Alexeev, over Z, dim. excessive by $N-1$ (2002)

$\overline{A}_{g,N}$: by Olsson, over Z, proper separated (2008)

Thm 4 $\exists$ a finite Galois morph. over $\mathcal{O}_N$, $N = \sqrt{|K|}$,

$$\text{sqap} : SQ^{\text{toric}}_{g,K} \times (\mathrm{P}^{N-1} \setminus H_{g,K}) \to \overline{AP}_{g,N} \otimes \mathcal{O}_N$$

$$(P, \phi, \tau) \times v \mapsto (P, \mathrm{Aut}^{\dagger 0}(P), \mathrm{Div}(\phi^*(v)))$$

such that for any fixed $v \in \mathrm{P}^{N-1} \setminus H_{g,K}$

$$(P, \phi, \tau) \mapsto (P, \mathrm{Aut}^{\dagger 0}(P), \mathrm{Div}(\phi^*(v)))$$

is a closed immersion of $SQ^{\text{toric}}_{g,K}$.

5 Tate curve and PSQAS

R:DVR, $L = \mathrm{Frac}(R) = R[1/q]$, q uniformizer.

Tate curve : $\mathrm{G}_m(L)/w \mapsto qw$

Hesse cubics at ∞ : $\mathrm{G}_m(L)/w \mapsto q^3 w$

Rewrite Tate curve as $\mathrm{G}_m(L)/w^n \mapsto q^{mn} w^n \ (n \in \mathrm{Z})$

Hesse cubics at ∞ : $\mathrm{G}_m(L)/w^n \mapsto q^{3mn} w^n \ (n \in \mathrm{Z})$

The general case : B pos. def. symmetric

$$\mathrm{G}_m(L)^g/w^x \mapsto q^{B(x,y)} b(x,y) w^x,$$

$$b(x,y) \in L^\times \quad (x \in X, y \in Y)$$

The usual Tate curve over CDVR R

$$X : x_0x_2^2 = x_1^3 - x_0x_1^2 + qx_0^3$$

Or $\quad X : y^2 = x^3 - x^2 + q$

The generic fibre $\quad X_\eta : y^2 = x^3 - x^2 + q \quad (q \neq 0)$

The fibre $X_0 : y^2 = x^2(x-1)$ for $q = 0$: a limit of X_q

$$X_0 \setminus \{0, 0\} = \mathrm{G}_m,$$

To compactify the moduli, need to find all nice limits !!

The general case : B pos. def. symmetric

The generic fibre:

$$\mathrm{G}_m(L)^g/w^x \mapsto q^{B(x,y)}b(x,y)w^x,$$

$$b(x,y) \in L^\times \quad (x \in X, y \in Y)$$

PSQAS is the closed fibre of it

6 Review of Theta functions

An elliptic curve, $w = e^{2\pi i z}$, $q = e^{2\pi i \tau/6}$

$$E(\tau) = \mathrm{C}/(\mathrm{Z} + \mathrm{Z}\tau) = \mathrm{C}^*/w \mapsto wq^6, \quad q = e^{2\pi i \tau/6}$$

Theta function $\theta_k(\tau, z) = \sum_{m \in \mathrm{Z}} q^{(k+3m)^2} w^{k+3m}$.

The map Θ embeds $E(\tau)$ into P^2.

$$\Theta : E(\tau) \ni z \;\mapsto\; [x_0, x_1, x_2] = [\theta_0, \theta_1, \theta_2] \in \mathrm{P}^2$$

To compactify the moduli

we find the limit of the image of Θ as $q \to 0$

General case will lead us to the next definition

Before it,recall again $w = e^{2\pi iz}$, $q = e^{2\pi i\tau/6}$

$$\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$

$$\theta_k(\tau, z + \frac{\tau}{3}) = (qw)^{-1}\theta_{k+1}(\tau, z),$$

$$[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)$$

σ, τ are the liftings to GL(3),

$z \mapsto z + \frac{1}{3}$ is lifted to $\sigma(\theta_k) = \zeta_3^k \theta_k$

$z \mapsto z + \frac{\tau}{3}$ is lifted to $\tau(\theta_k) = \theta_{k+1}$

$G(3) :=$ the group $\langle \sigma, \tau \rangle$

The image of Θ is a Hesse cubic.

7 Heisenberg groups $G(K)$, $G(3)$

$G(3) = \langle \sigma, \tau \rangle$ acts on V, order $|G(3)| = 27$,

$$V \;= Rx_0 + Rx_1 + Rx_2,$$

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathrm{Z}/3\mathrm{Z})$$

ζ_3 is a primitive cube root of 1, $R \ni \zeta_3, 1/3$

- $x_0^3 + x_1^3 + x_2^3$, $x_0x_1x_2 \in S^3V$ only are $G(3)$-invariant
- $G(3)$ determines x_i "uniquely" ($\because$ V:$G(3)$-irred,)
- x_i are classical theta over C

General case will lead us to the next definition

In terms of theta, $w = e^{2\pi i z}$, $q = e^{2\pi i \tau/6}$

$$\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$

$$\theta_k(\tau, z + \frac{\tau}{3}) = (qw)^{-1} \theta_{k+1}(\tau, z),$$

$$[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)$$

σ, τ are the liftings to GL(3),

$z \mapsto z + \frac{1}{3}$ is lifted to $\sigma(\theta_k) = \zeta_3^k \theta_k$

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$G(3) :=$ the group $\langle \sigma, \tau \rangle$

8 Definition of PSQAS

R : DVR, q a uniformizer of R,

$k(0) = R/m$, $k(\eta) = R[1/q]$: the fraction field of R

Suppose (G_η, L_η) : abelian variety over $k(\eta)$

(G, L) is the (connected) Néron model of (G_η, L_η)

Let $\lambda(L_\eta) : G_\eta \to {}^tG_\eta = \mathrm{Pic}^0(G_\eta)$

$({}^tG_\eta, {}^tL_\eta)$ dual AV, ${}^tG_\eta = \mathrm{Pic}^0(G_\eta)$.

$({}^tG, {}^tL)$: the (connected) Néron model of $({}^tG_\eta, {}^tL_\eta)$

Suppose G_0 a split torus over $k(0)$,

Then $({}^tG_0, {}^tL_0)$ is a split torus over $k(0)$

For the Tate curve over CDVR R

The generic fibre $G_\eta : y^2 = x^3 - x^2 + q \quad (q \neq 0)$

The fibre $X_0 : y^2 = x^2(x-1)$ for $q = 0$: a limit of X_q

$$X_0 \setminus \{0,0\} = \mathrm{G}_m,$$

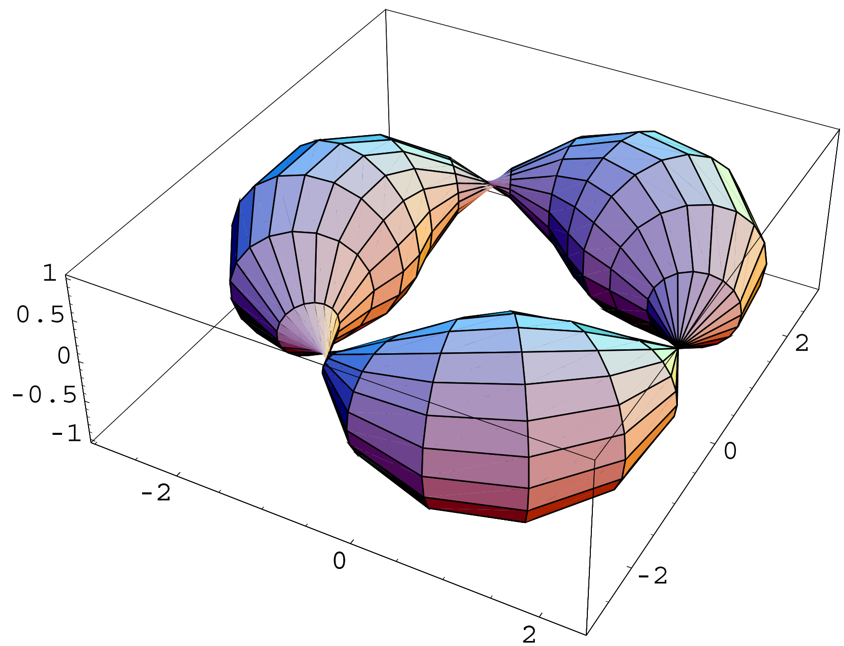
This is the key assumption G_0 a split torus

$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu^3 = 1 \ \text{or} \infty)$$

It degenerates into 3 copies of P^1

$\mu = \infty$, $x_0 x_1 x_2 = 0$ contains $G_m \times Z/3Z$

This is the key assumption G_0 a split torus



Definition of PSQAS

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Suppose G_0 a split torus over $k(0)$,

Then $({}^tG_0, {}^tL_0)$ is a split torus over $k(0)$

Let $X = \mathrm{Hom}(G_0, \mathbf{G}_m), \quad Y = \mathrm{Hom}({}^tG_0, \mathbf{G}_m)$.

Hence $X \simeq \mathbf{Z}^g$, $Y \simeq \mathbf{Z}^g$,

$\lambda(L_\eta)$ extends, $\exists$ a surjection $G_0 \to {}^tG_0$

Hence Y : a sublattice of X, $[X : Y] < \infty$.

$K_\eta := \ker \lambda(L_\eta)$, $N := |K_\eta|$.

K:=the closure of K_η. May assume Over $\mathbf{Z}[\zeta_N, 1/N]$

$K \simeq (X/Y) \oplus (X/Y)^\vee$,

This finite group helps us to take up the necessary data

From G and K we can construct

- $G(K)$: Heisenberg group scheme

$$1 \longrightarrow \mu_N \longrightarrow G(K) \longrightarrow K \longrightarrow 0 \text{ (exact)}$$

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),$$

- $R[X/Y] = \oplus_{x \in X/Y} R\, v(x)$ (group alg. of X/Y)

$$v(0) = 1,\ v(x + y) = v(x)v(y)$$

- $G(K)$ acts on $R[X/Y]$ by

$$(a, z, \alpha) \cdot v(x) = a\alpha(x)v(z + x)$$

$$a, b \in \mu_N;\ z, x \in (X/Y);\ \alpha, \beta \in (X/Y)^{\vee}$$

Facts. G : conn. Néron model of G_η,

$$K_\eta := \ker(\lambda(L_\eta)) \simeq (X/Y) \oplus (X/Y)^\vee,$$

- $V := H^0(G, L)$: finite R-free, $G(K)$-irreducible
- $V = H^0(G, L) \simeq R[X/Y]$ as $G(K)$-module
- $H^0(G, L) \ni \exists \theta_x \overset{G(K)\text{-isom}}{\longleftrightarrow} v(x) \in R[X/Y]$ gp alg

θ_x can be thought as "classical theta"

Idea: Find the limit of the image $[\theta_x]_{x \in X/Y}$

Let G_{for} : the formal completion of G along G_0

Key Fact:

$$G_{\text{for}} \simeq (\mathrm{G}^{g}_{m,R})_{\text{for}}$$

Fourier expansion of θ_x $(x \in X/Y)$ on G_{for} :

$$\theta_x = \sum_{y \in Y} a(x+y) w^{x+y}$$

$a(x+y)$: Fourier coeff. of θ_x

called Faltings-Chai's degeneration data of (G, L)

- $B(x,y) := \mathrm{val}_q(a(x+y)a(x)^{-1}a(y)^{-1})$ **is pos. def.**

generalized Tate curves

The general case : B pos. def. symmetric

The generic fibre:

$$\mathrm{G}_m(L)^g/w^x \mapsto q^{B(x,y)} b_0(x,y) w^x,$$

$$b_0(x,y) \in L^\times \quad (x \in X, y \in Y)$$

PSQAS is the closed fibre of a gener. Tate curve

We construct a canonical gen. of Tate curves.

$$\widetilde{R} := R[a(x)w^x\vartheta, x \in X], \quad \vartheta\text{:deg one}$$

$\mathrm{Proj}(\widetilde{R})$: locally of finite type over R

$\mathcal{X}$: the formal completion of $\mathrm{Proj}(\widetilde{R})$

The Quotient $\mathcal{X}/Y$ is a degenerating family of AV

$(\mathcal{X}/Y, O_{\mathcal{X}/Y}(1))$ is a generalization of Tate curves

Grothendieck (EGA) guarantees

$\exists$ a projective R-scheme $(Z, O_Z(1))$

s.t. the formal completion Z_{for} of Z

$$Z_{\mathrm{for}} \simeq \mathcal{X}/Y, \quad (Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$$

(the stable reduction theorem)

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

Projectively Stable Quasi Abelian Scheme

$G(K)$ acts on $(Z, O_Z(1))$

Summary Let R be CDVR over $\mathbf{Z}[\zeta_N, 1/N]$

- There is a natural choice of $\theta_x \in H^0(G, L)$
- $a(x+y),\ y \in Y$ is Fourier coeff of $\theta_x, x \in X/Y$
- all $a(x)$ recover the given G_η over $k(\eta) := \mathrm{Frac}(R)$
- There is an extention $\mathcal{X}/Y$ of G_η to R so that

(a) it is a canonical generalization of Tate curves,

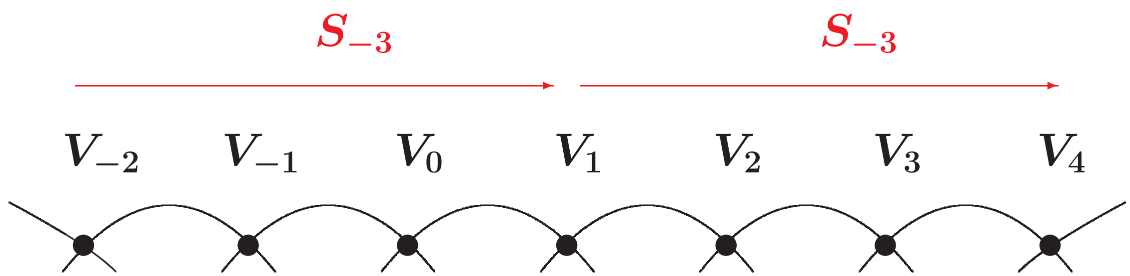
(b) $G(K)$ acts on $(\mathcal{X}/Y, O_{\mathcal{X}/Y}(1))$

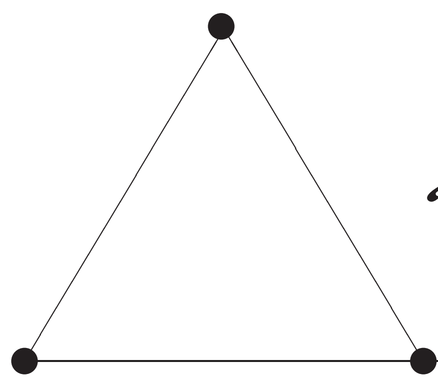
(c) hence $G(K)$ acts on $(Z, O_Z(1))$

(d) the closed fibre $(Z_0, O_{Z_0}(1))$ is a PSQAS.

Exam 1 $\quad g = 1,\ X = \mathbb{Z},\ Y = 3\mathbb{Z}.$

$$\mathcal{X} = \mathrm{Proj}(\widetilde{R}), \quad a(x) = q^{x^2},\ (x \in X)$$





$\mathcal{X}_0/Y$

Recall

Thm 5 Over $\mathbf{Z}[\zeta_3, 1/3]$

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$SQ_{1,3} := \overline{A_{1,3}}$$

$$= \{\text{stable cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \,\text{or}\, \infty\right\} \simeq \mathrm{P}^1.$$

Hesse cubics are PSQASes in dimension one, level 3.

We wish to extend this to arbitrary dimension

1. over $\mathbf{Z}[\zeta_N, 1/N]$ or over $\mathbf{Z}[\zeta_N]$
2. to define a representable functor of compact obj.

   $\boldsymbol{F := SQ_{g,K}}$ (fine moduli)
3. to relate to GIT stability, that is,

   to aim at $\boldsymbol{F(k)}$ =GIT stable objects for $\boldsymbol{k}$ alg. closed

$SQ_{g,K,1/N} := SQ_{g,K}$: proj. over $\mathbf{Z}[\zeta_N, 1/N]$ (1999)

$\overline{AP}_{g,N}$: over **Z**, dim. excessive by $N-1$ (2002)

Olsson : over **Z**, nonseparated nonproper stack (2008)

Olsson uses the same model as ours (Alexeev-Nakamura's model)

We prefer to separated moduli.

It is easy to construct nonseparated stack moduli.

9 Separatedness of the moduli

There are difficulties never seen in dimension one

- Classical level structure = base of n-divison points,
- Singular limits of Abelian varieties are very reducible
- Classical level str. gives non-separated moduli
- We need to prove in any dimension,

Lemma. (Valuative Lemma for Separatedness)

R : DVR, $L = \mathrm{Frac}(R)$, $X, Y \in F(R)$.

If $X_L \simeq Y_L$, then $X \simeq Y$. In other words,

Isom. over L implies isom. over R.

- **separated = Hausdorff, (e.g. if X projective, then separated)**
- X**: non-separated = non Hausdorff,**
- **If non-Hausdorff, then** $\exists\ P_n \in X\ (n = 1, 2, \cdots)$, $P = \lim P_n$, $Q = \lim P_n$. **But** $P \neq Q$
- **This really happens in geometry.**

Example R : DVR, q : uniformizer of R, $L = R[1/q]$,

E, E' : elliptic curves over R

$$E : y^2 = x^3 - q^6, \quad E' : Y^2 = X^3 - 1$$

Let us consider $P_n := E_L, Q_n := E'_L$

$P_n = Q_n$, i.e. $E_L \simeq E'_L$

because

$$E_L : (y/q^3)^2 = (x/q)^3 - 1,$$

$$E'_L : Y^2 = X^3 - 1$$

Example R : DVR, q : uniformizer of R, $L = R[1/q]$,

E, E' : elliptic curves over R

$$E : y^2 = x^3 - q^6, \quad E' : Y^2 = X^3 - 1$$

Let us consider $P_n := E_L, Q_n := E'_L$

$P := E_0 = \lim E_L$, $Q := E'_0 = \lim E'_L$

$P_n = Q_n$, i.e. $E_L \simeq E'_L$ But $P \neq Q$

$$P := E_0 : y^2 = x^3, \quad Q := E'_0 : Y^2 = X^3 - 1$$

To overcome the difficulty of level str/n-div. pts :

- Non-abelian Heisenberg gp. $G := G(K)$
- New level str. = Framing of irred. reps. of G
- To prove Val. Lemma for Separatedness, we use

Schur's Lemma over R. Let $|G| = N$,
R : a ring over $\mathrm{Z}[\zeta_N, 1/N]$, V : free R-mod.
V : irr. G-mod. of wt one, $(\Rightarrow G \subset GL(V \otimes R))$
Let $h \in \mathrm{GL}(V \otimes R)$. If $gh = hg$ for $\forall\ g \in G$,
then h is scalar.

Summary

- **Separatedness of the moduli**

  follows from $G(K)$-Irreducibility of $V = H^0(X, L)$,

  $(X, L) = (Z_0, O_{Z_0}(1))$: any PSQAS, level $N \geq 3$

  if $K \simeq \ker(\lambda(L) : G_\eta \to G_\eta^t$ (dual)).

We re-start with

Thm 6 **Over $\mathbf{Z}[\zeta_3, 1/3]$**

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\text{ isom.}$$

$$\overline{A_{1,3}} := \{\text{stable cubics with 9 inflection pts}\}/\text{ isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \text{ or } \infty\right\} \simeq \mathbf{P}^1.$$

We convert it into $G(3)$-equivariant theory

$G(3)$: Heisenberg group of level 3

10 Heisenberg groups $G(K)$, $G(3)$

$G(3) = \langle \sigma, \tau \rangle$ acts on V, order $|G(3)| = 27$,

$$V = Rx_0 + Rx_1 + Rx_2,$$

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathbf{Z}/3\mathbf{Z})$$

ζ_3 is a primitive cube root of 1, $R \ni \zeta_3, 1/3$

Fact

- $x_0^3 + x_1^3 + x_2^3$, $x_0x_1x_2 \in S^3V$ only are $G(3)$-invariant
- $G(3)$ determines x_i "uniquely" ($\because$ V:$G(3)$-irred,)
- x_i are classical theta over C

Summary $G(K)$: Heisenberg gp. *e.g.* $G(3)$

- $G(K)$ chooses a basis of $V = H^0(X, L)$, X:PSQAS
- $G(K)$ chooses a basis of $H^0(G, L)$, G:Néron model
- $G(K)$ determines Faltings-Chai degeneration data
- $G(K)$ extends G_η to define $(Z, O_Z(1))$, $Z = \mathcal{X}/Y$
- Separatedness of the moduli

  follows from $G(K)$-Irreducibility of $V = H^0(X, L)$,

  X : any PSQAS, level $N \geq 3$

11 The space of closed orbits

X	the set of geometric objects
G	the group of isomorphisms
x, x' are isom.	G-orbits are the same $O(x) = O(x')$
X_{ps}	the set of properly-stable objects
X_{ss}	the set of semistable objects
$X_{ss}//G$	"compact moduli"

Exam 2 Action on $\mathbb{C}^2$ of $G = \mathbb{G}_m (= \mathbb{C}^*)$,

$$\mathbb{C}^2 \ni (x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbb{G}_m)$$

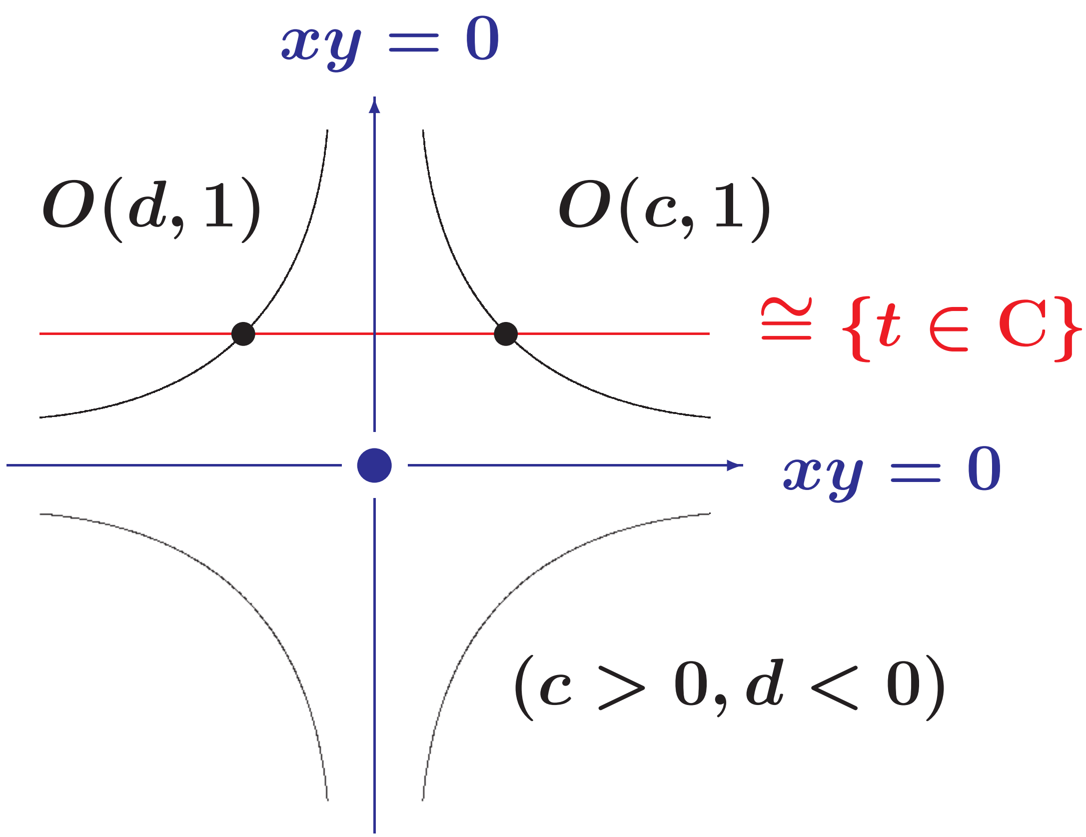
What is the quotient of $\mathbb{C}^2$ by G ?

- Simple answer : the set of G-orbits (×)
- Answer : Spec(the ring of all G-invariant poly.)(○)
- $t := xy$ is the unique G-inv. !

$$\mathbb{C}^2 // G := \operatorname{Spec} \mathbb{C}[t] = \{t \in \mathbb{C}\}$$

But this is different from "the set of G-orbits".

- $\mathbb{C}^2 // G = \{t \in \mathbb{C}\}$ is the set of all closed orbits.



- $t = 0$ is a point of $\mathbb{C} = \mathbb{C}^2//G = \operatorname{Spec} \mathbb{C}[t]$.
- But $\{xy = 0\}$ consists of three G-orbits

$$\mathbb{C}^* \times \{0\}, \quad \{0\} \times \mathbb{C}^*, \quad \{(0,0)\}$$

- $\{(0,0)\}$ is the only closed orbit in $\{xy = 0\}$

Def 7 The same notation as before. Let $p \in X$.

(1) semistable if $\exists$ G-inv. homog. poly. F, $F(p) \neq 0$,

(2) Kempf-stable (= closed orbit)

if the orbit $O(p)$ is closed in X_{ss},

(3) properly-stable if (2) and Stab(p) finite.

Rem stable $\Longrightarrow$ closed orbit $\Longrightarrow$ semistable

Thm 8 (Seshadri,Mumford) G : reductive, acting on a scheme X, (e.g. $G = \mathrm{G}_m$). Let X_{ss} = the set of semistable points. Then

- $X_{ss}//G :=$ Spec(all G-inv.) = the set of closed orbits.
- $X_{ss}//G$ is a scheme, $X_{ps}//G$ is also a scheme,
- $X_{ss}//G$ compactifies $X_{ps}//G$.

Rem The set of points with closed orbits is not an algebraic subscheme.

Thus we consider only those objects with closed orbits

As its consequence we will see

- Abelian varieties have closed orbits (Kempf), and
- our PSQASes have closed orbits,

  Conversely

- Any degenerate abelian scheme with closed orbit is one of our PSQASes
- There is a simple characterization of our PSQASes,
- This characterization enables us to compactify the moduli of abelian varieties.

12 Stable curves of Deligne-Mumford

Def 9 C is a stable curve of a genus g if

(1) connected projective reduced with finite autom.,

(2) the singularities of C are like $xy = 0$

(3) $\dim H^1(O_C) = g$

Let $\overline{M_g}$: moduli of stable curves of genus g,

M_g : moduli of nonsing. curves of genus g.

Thm 10 $\overline{M_g}$ compactifies M_g

(Deligne-Mumford 1969)

Definition of stable curves is irrelevant to GIT stability

Nevertheless

Thm 11 The following are equivalent

(1) C is a stable curve (moduli-stable)

(2) any Hilbert point of $\Phi_{|mK|}(C)$ is GIT-stable

(3) any Chow point of $\Phi_{|mK|}(C)$ is GIT-stable

---

(1)$\Leftrightarrow$(2) Gieseker 1982 (before Mumford 1977)

(1)$\Leftrightarrow$(3) Mumford 1977 (suggested by Gieseker 1982)

13 Stability of cubic curves

CUBIC CURVES	STABILITY	STAB GP.
smooth elliptic	stable	finite
3-gon	closed orbit	2-dim
a line+a conic (transv.)	semistable	1-dim
irred. with a node	semistable	finite
others	unstable	1-dim

Thm 12 **For a cubic C, the following cond. are equiv.**

(1) C has a closed SL(3)-orbit in $(S^3V)_{ss}$

(2) C is a Hesse cubic curve, that is, $G(3)$-invariant

(3) C is either smooth elliptic or a 3-gon

14 Stability in higher-dim.

Thm 13 (Kempf) (A, L) an abelian variety,

$V = H^0(A, L)$ very ample, w:=Hilbert point of (A, L).

Then $\mathrm{SL}(V)w$ is closed in P_{ss} : the semistable locus of a big proj. space.

Thm 14 (N.1999)

(X, L) : PSQAS of level $G(K)$,

$V = H^0(X, L)$ very ample. Then

any Hilbert point of (X, L) has a closed $\mathrm{SL}(V)$-orbit.

Thm 15 (N.1999)

Assume (X, L) is a limit of abelian varieties A with $\ker(\lambda(L)) = K$, $\lambda(L) : A \to A^t$ (dual)

Then the following are equivalent:

(1) X has a closed SL(V)-orbit (GIT-stable)

(2) X is invariant under $G(K)$ ($G(K)$-stable)

(3) X is one of our PSQASes (moduli-stable)

To be more precise,

Thm 16 **(N.1999)**

Assume (X, L) is a limit of AV A's with $\ker(\lambda(L)) = K$

Then the following are equivalent:

(1) The m-th Hilbert point of X has a closed SL(V)-orbit in $\mathrm{P}(\overset{M}{\wedge} S^m V)_{ss}$ (GIT-stable)

(2) X is invariant under $G(K)$ ($G(K)$-stable)

(3) X is one of our PSQASes (moduli-stable)

where $M := \dim H^0(X, mL)$.

Thm 17 For cubics the following are equiv:

(1) it has a closed SL(3)-orbit (GIT-stable)

(2) it is a Hesse cubic, that is, $G(3)$-inv. ($G(3)$-stable)

(3) it is smooth ell. or a 3-gon. (moduli-stable)

Thm 18 Let X be a degenerate AV. The following are equiv. under natural assump.:

(1) it has a closed SL(V)-orbit (GIT-stable)

(2) X is $G(K)$-inv ($G(K)$-stable)

(3) it is a PSQAS (p.20) (moduli-stable)

Thus we see

- **Abelian varieties have closed orbits (Kempf), and**
- **our PSQASes have closed orbits,**

  Conversely

- **Any degenerate abelian scheme with closed orbit is one of our PSQASes**
- X **is our PSQAS iff** X **is** $G(K)$**-stable,**
- **This characterization will compactify the moduli of abelian varieties.**

The characterization of PSQASes will compactify the moduli of abelian varieties. We recall

"Closed orbit" is not a Zariski open/closed condition.

Exam 3

Let $G := \{(s,t,u) \in (\mathrm{G}_m)^3; stu = 1\}$

$$C_{a,b,c} \;:\; ax_0^3 + bx_1^3 + cx_2^3 - x_0x_1x_2 = 0.$$

G acts on A^3 : $(a,b,c) \mapsto (sa,tb,uc)\mathrm{A}^3$

Closed $(\mathrm{G}_m)^2$-orbit iff $abc \neq 0$ or $(a,b,c) = (0,0,0)$.

15 Moduli over $\mathbf{Z}[\zeta_N, 1/N]$

Thm 19 (a new version of the theorem of Hesse)

$$SQ_{1,3} = \mathbf{P}^1_{\mathbf{Z}[\zeta_3, 1/3]},$$

the projective fine moduli

(1) The universal cubic curve

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0$$

where $(\mu_0, \mu_1) \in SQ_{1,3} = \mathbf{P}^1$.

(2) when k is alg. closed and char. $k \neq 3$

$$SQ_{1,3}(k) = \left\{ \begin{array}{l} \text{closed orbit cubics} \\ \text{with level 3-structure } /k \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} \text{Hesse cubics} \\ \text{with level 3-str. } /k \end{array} \right\} /\text{isom.=id.}$$

$$A_{1,3}(k) = \left\{ \begin{array}{l} \text{closed orbit nonsing. cubics} \\ \text{with level 3-str. } /k \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} \text{nonsing. Hesse cubics} \\ \text{with level 3-structure } /k \end{array} \right\} /\text{isom.=id.}$$

Thm 20 (N. 1999) There exists the fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$, $N = \sqrt{|K|}$, For k closed

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{closed orb. deg. abelian sch. } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant PSQAS } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\},$$

$$A_{g,K}(k) = \left\{ \begin{array}{l} \text{(nonsingular) abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-inv. abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

Summary $G(K)$: **Heisenberg gp.** *e.g.* $G(3)$

(A) $H^0(X, L)$ **is** $G(K)$**-irred for** X**: PSQAS**

- **(A) implies Stability of** X **with** L **very ample,**
- **(A) implies Separatedness of the moduli,**
- **(A) gives a simple characterization of PSQASes,**
- $G(K)$ **finds a compact separated moduli** $SQ_{g,K}$

16 The Second Compactification over $\mathbf{Z}[\zeta_N, 1/N]$

Recall Grothendieck (EGA) guarantees

$$\exists \text{ a projective } R\text{-scheme } (Z, O_Z(1))$$

s.t. the formal completion Z_{for} of Z

$$Z_{\mathrm{for}} \simeq \mathcal{X}/Y, \quad (Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$$

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

The normalization Z^{norm} of Z with Z_0^{norm} reduced

gives a bit different central fiber

$(Z_0^{\mathrm{norm}}, O_{Z_0^{\mathrm{norm}}}(1))$, we call it TSQAS.

Thm 21 (N. 2010) over $\mathbf{Z}[\zeta_N, 1/N]$,

∃ another cano. compactif. $SQ^{\text{toric}}_{g,K}$

:coarse moduli of TSQASes with level-$G(K)$ str.

∃ cano. bij. birat. morphism

$$\mathrm{sq}: SQ^{\text{toric}}_{g,K} \to SQ_{g,K}$$

$$(P, \phi, \tau) \mapsto (Q, \phi_Q, \tau_Q), \quad Q := \mathrm{Proj}(\mathrm{Sym}(\phi))$$

when any generic fibre of P is an abelian var.

Corollary

The normalizations of $SQ^{\text{toric}}_{g,K}$ and $SQ_{g,K}$ are isom.

Recall $(P, \phi, \tau) \in SQ^{\mathrm{toric}}_{g,K}$

- P:TSQAS=modified PSQAS,
- $\phi : P \to \mathrm{P}^{N-1} = \mathrm{P}(k[H^\vee])$ is a finite morphism
- $L = \phi^*(O_{\mathrm{P}^{N-1}}(1))$,
- $H^0(P, L) \overset{\phi^*}{\simeq} k[H^\vee] = H^0(O_{\mathrm{P}^{N-1}}(1))$
- τ : a compatible action of $G(K)$ on the pair (P, L)
- τ on P = translation by K when $P = A$: AV

$(Q, \phi_Q, \tau_Q) \in SQ_{g,K}$

- Q:PSQAS,
- $\phi_Q : Q \to \mathrm{P}^{N-1} = \mathrm{P}(k[H^\vee])$ is a closed immersion
- $L_Q = \phi^*(O_{\mathrm{P}^{N-1}}(1))$,
- $H^0(Q, L_Q) \simeq H^0(P, L) \overset{\phi^*}{\simeq} k[H^\vee] = H^0(O_{\mathrm{P}^{N-1}}(1))$
- τ_Q : a compatible action of $G(K)$ on the pair (Q, L_Q)
- τ_Q on Q = translation by K when $Q = A$: AV

Definition of sq : For $(P, L, \phi, \tau) \in SQ^{\text{toric}}_{g,K}(T)$

Suppose (P, L, ϕ, τ) is a T-TSQAS

such that any generic fibre is AV.

Then let $\boxed{Q = \phi(P) := \text{Proj}(\text{Sym}(\phi))}$

Can define (Q, L_Q, ϕ_Q, τ_Q) T-PSQAS, Then

the morphism sq is

$$\boxed{\text{sq}(P, L, \phi, \tau) = (Q, L_Q, \phi_Q, \tau_Q) \in SQ_{g,K}(T)}$$

17 Comparison of three compactifications

Summary $N = \sqrt{|K|}$, $\mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$, $d > 0$.

1. $SQ_{g,K}$ is a proj. fine moduli over $\mathcal{O}_N$ [N99],
2. $SQ_{g,K}^{\text{toric}}$ is a proj. coarse mod. over $\mathcal{O}_N$ [N01] [N10],
3. $\overline{AP}_{g,d} = \{(P, G, D)\}$ is a proper separated coarse moduli over $\mathbf{Z}$ [Alexeev02],
4. $\dim SQ_{g,K} = \dim SQ_{g,K}^{\text{toric}} = g(g+1)/2$,
5. $\dim \overline{AP}_{g,d} = g(g+1)/2 + d - 1$,
6. $\exists$ a canonical bij. birat. morphism [N10]

$$\mathrm{sq} : SQ_{g,K}^{\text{toric}} \to SQ_{g,K}$$

Alexeev's moduli $\overline{AP}_{g,d} = \{(P, G, D)\}$

- P is semi-normal proj. with L ample line bundle
- G semi-abelian acting on P with extra cond.
- $D \in H^0(P, L)$ a Cartier divisor
- D contains no G-orbits
- $\dim \overline{AP}_{g,d} = \dim A_g + d - 1.$

k alg. closed

$SQ_{1,K}$, $K = (\mathrm{Z}/3\mathrm{Z})^2$, Roughly

$SQ_{1,K}(k) = \{C$ a nonsing. cubic or a 3-gon cubic$\}$

$\overline{AP}_{1,3}(k) = \{(C, G, D)\}$

C nonsingular elliptic or a 3-gon,

or a conic plus a line, rational with a node

$G = C$ (elliptic) or G_m, $D \in H^0(C, L)$, degree $D = 3$.

To define a morphism from $SQ_{1,K}$ to $\overline{AP}_{1,3}$

is equivalent to the following

For a given

a flat family over T

$$(C, \phi, \tau) \in SQ_{1,K}(T)$$

always ! construct (G, D) so that

$$(C, G, D) \in \overline{AP}_{1,3}(T)$$

Problem: Construct G and Find D

For almost all $v \in k[\mathbf{Z}/3\mathbf{Z}]$,

$$(P, \phi, \tau) \times v$$

$$\mapsto (P, \mathrm{Aut}^{\dagger 0}(P), \mathrm{Div}(\phi^*(v)))$$

Need to prove

Any T-TSQAS has a flat group scheme action

This is done in general

Thm 22 If (P, L) is an S-flat TSQAS, then

$\mathrm{Aut}^{\dagger 0}_S(P)$ is S-flat semi-abelian group scheme

Thm 23 $\exists$ a finite Galois morph. over $\mathcal{O}_N$, $N = \sqrt{|K|}$,

$$\text{sqap} : SQ^{\text{toric}}_{g,K} \times (\mathrm{P}^{N-1} \setminus H_{g,K}) \to \overline{AP}_{g,N} \otimes \mathcal{O}_N$$

$$(P, \phi, \tau) \times \mapsto (P, \mathrm{Aut}^{\dagger 0}(P), \mathrm{Div}(\phi^*(v))$$

such that for any fixed $v \in \mathrm{P}^{N-1} \setminus H_{g,K}$

$$(P, \phi, \tau) \mapsto (P, \mathrm{Aut}^{\dagger 0}(P), \mathrm{Div}(\phi^*(v))$$

is an injective morphism of $SQ^{\text{toric}}_{g,K}$ extending an injective immersion of $A^{\text{toric}}_{g,K}$.

- $\mathrm{P}^{N-1} = \mathrm{P}(\mathcal{O}_N[H^\vee]^\vee)$, $v \in \mathcal{O}_N[H^\vee]$.
- $H_{g,K}$ is a hypersurf. of P^{N-1} of deg. known.
- $\dim SQ^{\text{toric}}_{g,K} + N - 1 = \dim \overline{AP}_{g,N}$.

$SQ_{g,K,1/N} := SQ_{g,K}$: over $\mathbf{Z}[\zeta_N, 1/N]$

$\overline{AP}_{g,N}$: Alexeev, over Z, no level str.

$\overline{A}_{g,N}$: Olsson, over Z, no level str.

18 The shape of PSQASes — Delaunay decompositions

”Limits of theta functions are described by the Delaunay decomposition.”

PSQAS is a geometrization of limit of thetas

PSQAS is a generalization of 3-gons.

which is described by the Delaunay decomposition.

PSQAS : a generalization of Tate curve, R:DVR

Tate curve : $\mathrm{G}_m(R)/w \mapsto qw$

Hesse cubics at ∞ : $\mathrm{G}_m(R)/w \mapsto q^3 w$

Rewrite Tate curve as :

$$\mathrm{G}_m(R)/w^n \mapsto q^{mn} w^n \ (m \in \mathrm{Z})$$

Hesse cubics at ∞ : $\mathrm{G}_m(R)/w^n \mapsto q^{3mn} w^n \ (m \in \mathrm{Z})$

The general case : B pos. def. symmetric

$$\mathrm{G}_m(R)^g/w^x \mapsto q^{B(x,y)} b(x,y) w^x,$$

$$b(x,y) \in R^\times \quad (x \in X, y \in Y)$$

Let $X = \mathbb{Z}^g$, B a positive symmetric on $X \times X$.

$$\|x\| = \sqrt{B(x,x)} : \text{a distance of } X \otimes \mathbb{R} \text{ (fixed)}$$

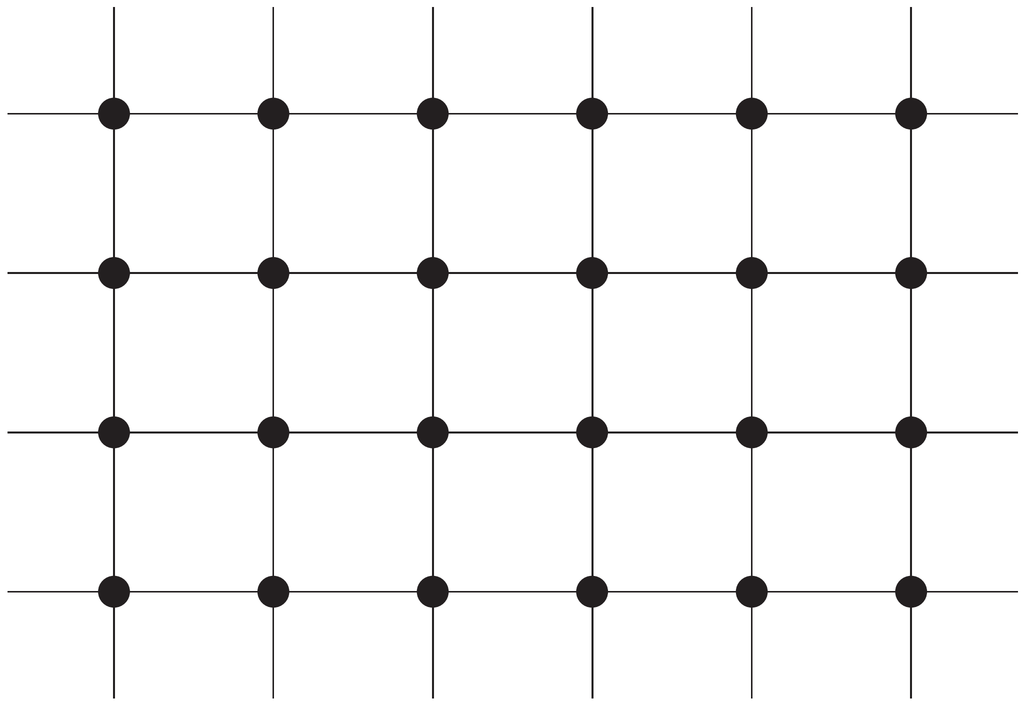
Def 24 Let $\alpha \in X_{\mathbb{R}}$. a Delaunay cell $D(\alpha)$: the convex closure of points of X closest to α.

Exam 4 1-dim. $B(x,y) = 2xy$, $X/Y = \mathbb{Z}/n\mathbb{Z}$, then PSQAS Z_0 is an n-gon of $\mathbb{P}^1$

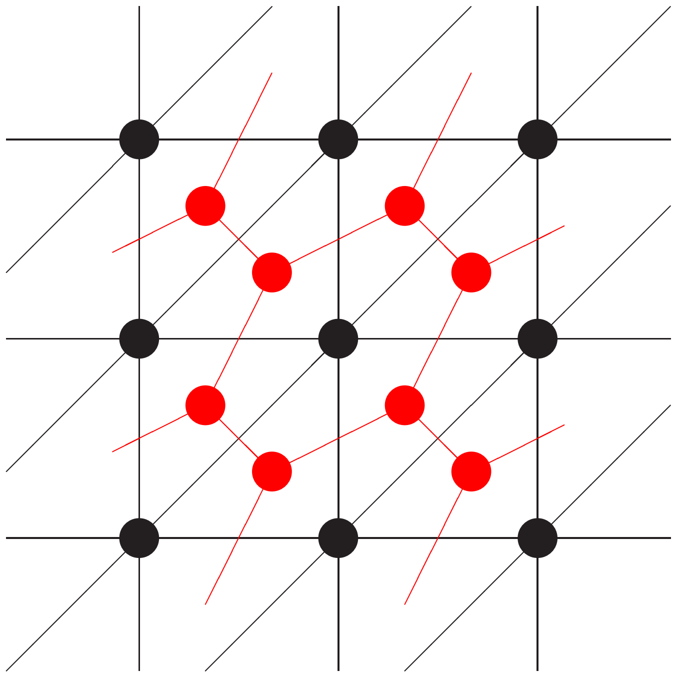
- **All Delaunay cells for a B form a Delaunay decomp.**
- **Each PSQAS (its scheme struture) and its decomposition into torus orbits (its stratification) are described by Delaunay decomp.**
- **Each pos. symm. B defines a Delaunay decomp.**
- **Different B can yield the same Delaunay decomp. and the same PSQAS.**

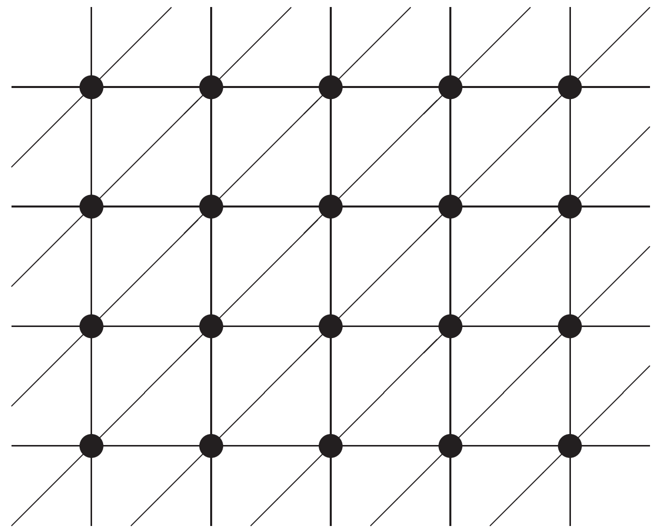
Exam 5 $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Z_0 := \mathcal{X}_0/Y$ is a union of $\mathrm{P}^1 \times \mathrm{P}^1$



Exam 6 $$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$





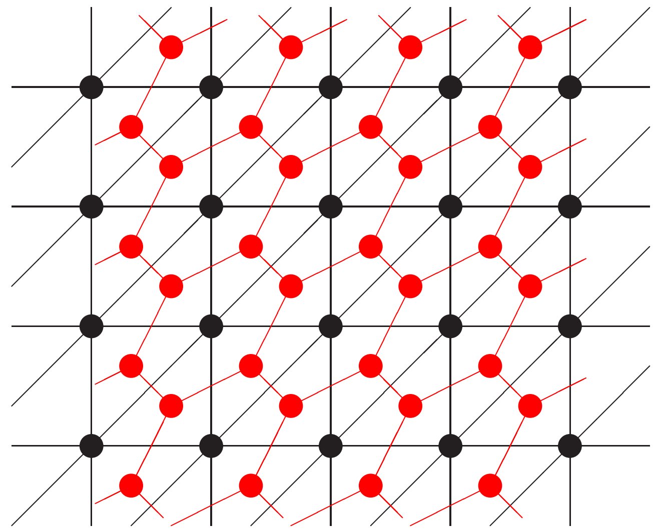
1. **This (mod Y) is a PSQAS.**

   It is a union of P^2, each triangle stands for P^2,

2. **each line segment is a P^1, two P^2 intersect along P^1**

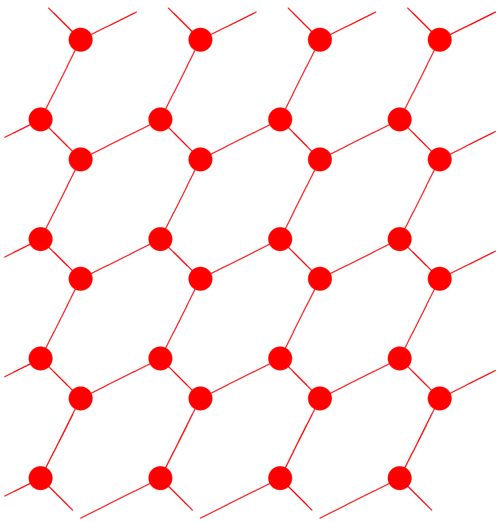
3. **six P^2 meet at a point,**

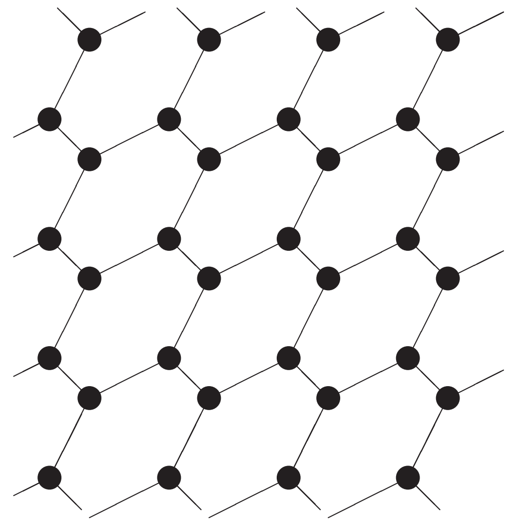
   locally $k[x_1, \cdots, x_6]/(x_i x_j, |i-j| \geq 2)$



Red one is the decomp. dual to the Delaunay decomp.

called Voronoi decomp.





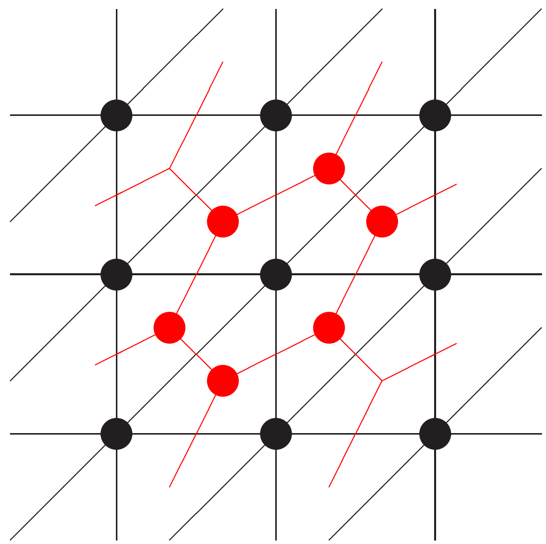
Voronoi decomposition

Def 25 D : for Delaunay cells

$$V(D) := \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R};\, D = D(\lambda)\}$$

We call it a Voronoi cell

$$\overline{V(0)} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R};\, \|\lambda\| \leqq \|\lambda - q\|, (\forall q \in X)\}$$



This is a crystal of mica.

$$\textbf{For } \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get $\overline{V(0)}$, a cube (salt),

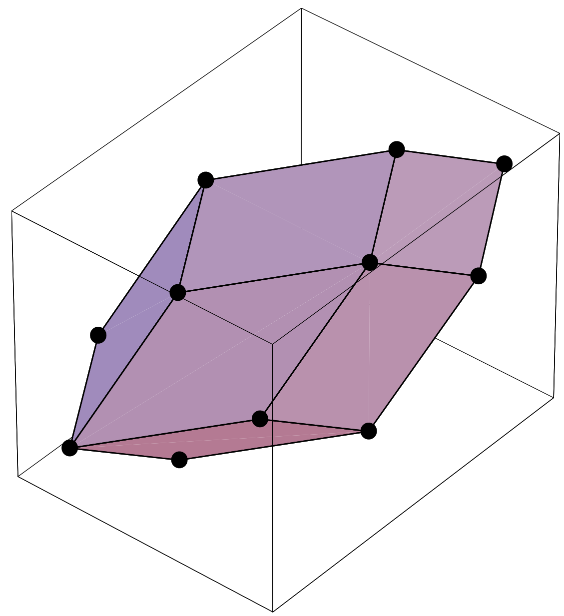
$$\textbf{For } \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

then we get a hexagonal pillar (calcite),

and then

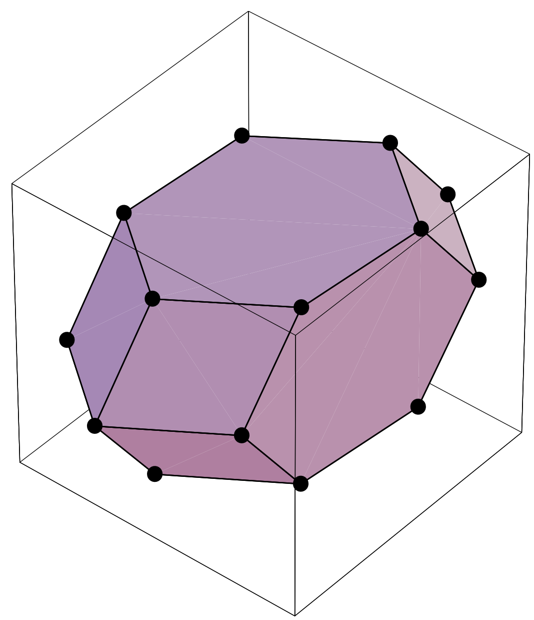
$$\boldsymbol{B} = \begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

A Dodecahedron (Garnet)



$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Apophyllite $KCa_4(Si_4O_{10})_2F \cdot 8H_2O$



$$B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

A Trunc. Octahed. — Zinc Blende ZnS