HILBERT SCHEMES OF ABELIAN GROUP ORBITS

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To the memory of Professor Nobuo Sasakura

Abstract. For G a finite abelian subgroup of $\mathrm{SL}(3,k)$ we construct a crepant resolution of the quotient variety $\mathbf{A}^3/G$ in a canonical way.

0. Introduction

Let k be any algebraically closed field and G a finite subgroup of $\mathrm{SL}(3,k)$ of order prime to the characteristic of k. The group G acts on affine 3-space $\mathbf{A}^3$ and the quotient space $\mathbf{A}^3/G$ is a normal Gorenstein variety with trivial canonical sheaf. General theories such as the theory of minimal models in birational geometry and the theory of torus embeddings do not seem to provide any natural choice of a crepant smooth resolution of $\mathbf{A}^3/G$. In this article we give a crepant smooth resolution of $\mathbf{A}^3/G$ canonical in a certain sense when G is abelian.

In fact, the G-orbit Hilbert scheme (or The Hilbert scheme of G-orbits) $\mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^3)$ introduced in [IN96] is such a resolution. The G-orbit Hilbert scheme is, by definition, the scheme parametrizing all G-invariant smoothable zero-dimensional subschemes of $\mathbf{A}^3$ of length $n := |G|$, the order of G. A smoothable zero-dimensional subscheme of $\mathbf{A}^3$ of length n is a kind of substitute for n-points in $\mathbf{A}^3$, so that a G-invariant smoothable zero-dimensional subscheme of length n is a substitute for a G-orbit in $\mathbf{A}^3$ consisting of n distinct points. Hence Hilb^G is a substitute of $\mathbf{A}^3/G$, the space of G-orbits. We will prove that for G abelian Hilb^G is a smooth torus embedding associated to a certain fan in $\mathbf{R}^3$ with apices junior elements of G (Theorem 4.2). As a corollary to it, we will see

Theorem 0.1. *For any abelian subgroup G of $\mathrm{SL}(3,k)$ of order prime to the characteristic of k, $\mathrm{Hilb}^G(\mathbf{A}^3)$ is a crepant smooth resolution of $\mathbf{A}^3/G$.*

Theorem 0.1 might sound a little unexpected because $\mathrm{Hilb}^n(\mathbf{A}^3)$ is known to be very singular. We conjecture that the same is true for any finite subgroup G of $\mathrm{SL}(3,k)$ if the order of G is prime to the characteristic of

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k. The first half of the present article is devoted to describing Hilb^G as a toric variety in arbitrary dimension. In the second half of it we will prove that if G is an abelian subgroup of $\mathrm{SL}(3,k)$, the fan associated with Hilb^G is nonsingular, that is, Hilb^G is smooth. Some examples will be given in Section 5 and Section 6. See also [Reid97] and Ito-Nakajima [INkjm98].

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1. Lattices and G-graphs

1.1. **Abelian subgroups.** Let G be a finite abelian subgroup of $\mathrm{GL}(r,k)$ of order prime to the characteristic of k and n the order of G. Without loss of generality we may assume that any element of G is a diagonal matrix.

Let μ_n be the group of n-th roots of unity and $\mathbf{G}_m$ the group of units in k. We choose and fix a primitive n-th root ζ of unity. Given a set F we denote its cardinality by $|F|$. We denote by $\mathbf{Z}_+$ (resp. $\mathbf{R}_+$) the set of all non-negative integers (resp. non-negative real numbers).

Definition 1.2. Let $N_0 := \oplus_{i=1}^r \mathbf{Z}e_i$ be a free $\mathbf{Z}$ module with e_i a $\mathbf{Z}$-basis, $M_0 := \mathrm{Hom}_{\mathbf{Z}}(N_0, \mathbf{Z}) = \oplus_{i=1}^r \mathbf{Z}f_i$ and $\Delta := \sum_{i=1}^r \mathbf{R}_+ e_i$ where $f_i(e_j) = \delta_{ij}$. In what follows we assume that any element of the group G is a diagonal $r \times r$-matrix, therefore we denote it by an r-vector consisting of r diagonal coefficients. Given a nontrivial element g of G, we write $g = (\zeta^{a_1}, \cdots, \zeta^{a_r})$ for some $0 \leq a_i < n$ and identify it with an r-vector $(a_1/n)e_1 + \cdots + (a_r/n)e_r \in N_0 \otimes_{\mathbf{Z}} \mathbf{Q}$. We define $N := N_0 + \sum_{g\in G} \mathbf{Z}g$ and $M = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$. We note $N/N_0 \simeq G$ and $M_0/M \simeq G^\vee$. In fact, we see that $e^{2\alpha(g)\pi i} \in \mu_n$ is well-defined for $\alpha \in M_0/M$ and $g \in G$, which gives a nondegenerate pairing $N/N_0 \times M_0/M \to \mu_n$. This pairing enables us to identify $G^\vee$ with M_0/M. We call this isomorphism *weight* and denote it by $\mathrm{wt} : M_0/M \to G^\vee$. We denote $a \equiv b$ for $a, b \in M_0$ if $\mathrm{wt}(a) = \mathrm{wt}(b)$, as is the same, $a - b \in M$.

1.3. **The semigroup** $S(I)$**.** Let M_0^0 (resp. M_0^+) be the semigroup generated by 0 and f_i (resp. f_i) $(1 \leq i \leq r)$.

Since by the semigroup homomorphism $f_i \mapsto x_i$ there is an obvious bijection between the set of subsemigroups of M_0^0 and the set of monomial subsemigroups in the polynomial ring $k[\mathbf{A}^r] = k[x_1, \cdots, x_r]$, we will often identify them if no confusion is possible.

Suppose that an ideal I of $k[\mathbf{A}^r]$ is generated by monomials. The monomial generators of I, in an obvious manner, give rise to a subsemigroup $S(I)$ of $k[\mathbf{A}^r]$. Let $S(I)^\sharp$ be the set of monomials in x_i which do not belong to $S(I)$. Then we easily see that $1 \notin S(I)$, $fS(I) \subset S(I)$ for any monomial f. From a certain reason we will explain in Subsection 2.1, we are particularly interested in those I satisfying the condition (ii) in Definition 1.4.

Definition 1.4. For a subsemigroup S of M_0^+ we define a subset $S^\sharp$ of M_0^0 by $S^\sharp := M_0^0 \setminus S$. The subset $S^\sharp$ of M_0^0 is called a *G-graph* if the following conditions are satisfied:

(i) $S + M_0^+ \subset S$, or as is equivalent, if $a+b \in S^\sharp$ for some $a, b \in M_0^0$, then $a \in S^\sharp$ and $b \in S^\sharp$,

(ii) $\mathrm{wt} : S^\sharp \to G^\vee$ is a bijection

where $\mathrm{wt}(\alpha)(g) = \alpha(g)$ $(\alpha \in S(I)^\sharp, g \in G)$.

We denote $S^\sharp$ by $\Gamma(S^\sharp)$ or simply by Γ. We note $0 \in \Gamma(S^\sharp)$ because $S \subset M_0^+$. If $G \subset SL(r,k)$, then $f_1 + f_2 + \cdots + f_r \not\in \Gamma(S^\sharp)$. Therefore $\Gamma(S^\sharp)$ consists of at most r $(r-1)$ -dimensional graphs $\Gamma_k := \Gamma(S^\sharp) \cap (\sum_{i\neq k} \mathbf{R} f_i)$.

Definition 1.5. Let Γ be a G-graph. We define the weight map $\mathrm{wt}_\Gamma : M_0^0 \to \Gamma$ as follows. For any $u \in M_0^0$ there exists a unique element $v \in \Gamma$ such that $\mathrm{wt}(v) \equiv \mathrm{wt}(u)$. We denote v by $\mathrm{wt}_\Gamma(u)$. Then we define cones $\sigma(\Gamma)$ and $\overset{\vee}{\sigma}(\Gamma)$ by

$$\sigma(\Gamma) := \{\alpha \in N_0 \otimes_{\mathbf{Z}} \mathbf{R}; \alpha(v - \mathrm{wt}_\Gamma(v))) \geq 0, \forall v \in M_0^0\},$$
$$\overset{\vee}{\sigma}(\Gamma) := \{v \in M_0 \otimes_{\mathbf{Z}} \mathbf{R}; \alpha(v) \geq 0, \forall \alpha \in \sigma(\Gamma)\},$$

We define $S(\Gamma)$ to be the semigroup of M generated by $v - \mathrm{wt}_\Gamma(v)$ $(\forall v \in M_0^0)$, and $M(\Gamma)$ the sublattice of M generated by $v - \mathrm{wt}_\Gamma(v)$ $(\forall v \in M_0^0)$.

Lemma 1.6. *For any finite abelian subgroup G of $\mathrm{GL}(r,k)$, there exists a G-graph.*

Proof. Assume first that G is a cyclic group of order n. Then let $g = (\cdots, 1/n)$ be a generator of G. Then $\Gamma = \{x f_r; x \in [0, n-1]\}$ is a G-graph. If G is not cyclic, then we prove Lemma by the induction on r. We may assume there exists a $g \in G$ such that $g = (\cdots, 1/\ell)$ is of the maximal order in G where $\ell \geq 2$. Then there exists a subgroup H of $G \cap \mathrm{GL}(r-1,k)$ such that G/H $(\simeq \mathbf{Z}/\ell\mathbf{Z})$ is generated by the image of g. By the induction hypothesis there exists an H-graph $\Gamma' \subset \oplus_{i=1}^{r-1} \mathbf{Z} f_i$. Let $\Gamma := \Gamma' \times ([0, \ell-1] f_r)$. Then it is clear that Γ is a G-graph. □

Lemma 1.7. *Let Γ be a G-graph. Then $S(\Gamma)$ is a finitely generated semigroup and $M(\Gamma) = M$.*

Proof. Though it seems clear, we give a proof. First we note that $M(\Gamma) \subset M$ and $M \cap M_0^0 \subset M(\Gamma)$ because $\mathrm{wt}_\Gamma(v) = 0$ for $v \in M \cap M_0^0$. Since $M \cap M_0^0$ generates M as a group, $M \subset M(\Gamma)$. It follows $M(\Gamma) = M$. There exists a finite subset A of M_0^0 such that M_0^0 is generated by A and $M \cap M_0^0$ as a semigroup. Any $w \in M_0^0$ is written as $w = v + a$ $(\exists a \in A, v \in M \cap M_0^0)$. Therefore

$$w - \mathrm{wt}_\Gamma(w) = v + a - \mathrm{wt}_\Gamma(a)$$

because $\mathrm{wt}_\Gamma v = 0$. Since $M \cap M_0^0$ is finitely generated, $S(\Gamma)$ is finitely generated. The proof shows that $S(\Gamma)$ is generated by A and $M \cap M_0^0$. □

Lemma 1.8. *Let Γ be a G-graph with $\dim\sigma(\Gamma) = r$ and A a finite subset of $M_0^0 \setminus \Gamma$ such that $M_0^0 \setminus \Gamma = A + M_0^0$. Then $S(\Gamma)$ is generated by $v - \mathrm{wt}_\Gamma(v)$ ($v \in A$) as a semigroup.*

Proof. Let $\lambda(v) := v - \mathrm{wt}_\Gamma(v)$ and we denote by $S^\flat(\lambda(A))$, the semigroup generated by $\lambda(A)$. We see $nf_i \in \overset{\vee}{\sigma}(\Gamma) \cap M$ because $n = |G|$ and $\mathrm{wt}_\Gamma(nf_i) = 0$. Hence $\sigma(\Gamma) \subset \Delta$. Since $\dim\sigma(\Gamma) = r$, we can find $\alpha \in \mathrm{Int}(\sigma(\Gamma))$ such that $\alpha(\lambda(v)) > 0$ ($\forall v \in M_0^0 \setminus \Gamma$).

We choose and fix such an $\alpha \in \mathrm{Int}(\sigma(\Gamma))$. Then since $\sigma(\Gamma) \subset \Delta$, we see that $\alpha(v) > 0$ for any $v \in M_0^+$. Now we prove our Lemma by the induction on the value $\alpha(z)$ ($z \in M_0^0 \setminus \Gamma$). If $z \in A$, then $\lambda(z) \in S^\flat(\lambda(A))$. Let $z \in M_0^0$. By the assumption there exist $v \in A$ and $b \in M_0^0$ such that $z = v + b$. We see $\lambda(z) = \lambda(v) + \lambda(\mathrm{wt}_\Gamma(v) + b)$. Since $\alpha(z) = \alpha(v) + \alpha(b) > \alpha(\mathrm{wt}_\Gamma(v) + b)$, we have $\lambda(\mathrm{wt}_\Gamma(v) + b) \in S^\flat(\lambda(A))$ by the induction hypothesis. Hence $\lambda(z) \in S^\flat(\lambda(A))$. This completes the proof. $\square$

The condition $\dim\sigma(\Gamma) = r$ is always satisfied in view of Corollary 2.4.

Definition 1.9. For any G-graph Γ we define

$$\begin{aligned} U_{\sigma(\Gamma)} &:= \mathrm{Spec}\, k[\overset{\vee}{\sigma}(\Gamma) \cap M], \quad V(\Gamma) := \mathrm{Spec}\, k[S(\Gamma)], \\ I(\Gamma) &:= (w^v; v \in M_0^0 \setminus \Gamma) \subset k[\mathbf{A}^r], \\ I^{\mathrm{vers}}(\Gamma) &:= (w^v - s^{v - \mathrm{wt}_\Gamma(v)} w^{\mathrm{wt}_\Gamma(v)}; v \in M_0^0) \\ &\qquad\qquad \subset k[S(\Gamma)] \otimes_k k[\mathbf{A}^r] \subset k[\overset{\vee}{\sigma}(\Gamma) \cap M] \otimes_k k[\mathbf{A}^r] \end{aligned}$$

where w^v resp. s^v is the monomial in $k[\mathbf{A}^r]$ resp. in $k[M]$ corresponding to $v \in M_0^0$ resp. $v \in M$. We note that $U_{\sigma(\Gamma)}$ is the normalization of $V(\Gamma)$. Let $Z(\Gamma)$ (resp. $Z^{\mathrm{vers}}(\Gamma)$) be a G-invariant subscheme of $\mathbf{A}^r$ (resp. a G-invariant subscheme of $\mathbf{A}^r_{V(\Gamma)}$) defined by the ideal $I(\Gamma)$ (resp. $I^{\mathrm{vers}}(\Gamma)$).

2. The G-orbit Hilbert scheme

2.1. **The G-orbit Hilbert scheme.** Let G be a finite subgroup of $\mathrm{GL}(r, k)$ and $n = |G|$. Let $\mathrm{Hilb}^n(\mathbf{A}^3)$ be the Hilbert scheme of n points in $\mathbf{A}^r$ and $S^n(\mathbf{A}^r)$ the n-th symmetric product of $\mathbf{A}^r$. We have a natural morphism π_n from $\mathrm{Hilb}^n(\mathbf{A}^r)$ onto $S^n(\mathbf{A}^r)$. Since π_n is G-equivariant, we have a natural morphism between their G-fixed point sets. We see easily that the quotient variety $\mathbf{A}^r/G$ is one of the irreducible components of the G-fixed point set $S^n(\mathbf{A}^r)^{G-\mathrm{inv}}$ with reduced structure. Then *the G-orbit Hilbert scheme* $\mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^r)$ is by definition the unique irreducible component, endowed with reduced structure, of the G-fixed point set of $\mathrm{Hilb}^n(\mathbf{A}^r)$ which dominates $\mathbf{A}^r/G$ by the map π_n. Let π be the natural morphism from $\mathrm{Hilb}^G(\mathbf{A}^r)$ onto $\mathbf{A}^r/G$. We call a zero-dimensional subscheme (resp. a G-invariant subscheme) Z of $\mathbf{A}^r$ of length n *a cluster* (resp. *a G-cluster*). A G-cluster Z is smoothable if and only if $Z \in \mathrm{Hilb}^G(\mathbf{A}^r)$. Let I be the ideal defining a smoothable cluster (resp. a G-invariant smoothable cluster).

Then we write $I \in \mathrm{Hilb}^n(\mathbf{A}^r)$ resp. $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$ since there seems to be no fear of confusion.

For instance, if $r = 3$ and G is generated by $g = (1/6, 2/6, 3/6)$, then (x^6, y, z), (x^2, y^3, z), and (x^2, xy, y^2, z^2) are examples of ideals generated by monomials in $\mathrm{Hilb}^G(\mathbf{A}^3)$. See Table 2.

By [IN98, Lemma 9.4] if $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$, then the quotient module $k[\mathbf{A}^r]/I$ is isomorphic to the group algebra $k[G]$ (the regular representation of G) as a G-module. In other words, $\dim(k[\mathbf{A}^r]/I)^\rho = 1$ $(\forall \rho \in G^\vee)$ where $(k[\mathbf{A}^r]/I)^\rho$ is the ρ-eigensubspace of $k[\mathbf{A}^r]/I$ for any character $\rho \in G^\vee$. This is just the condition (ii) in Definition 1.4.

Definition 2.2. Let $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbf{A}^r)$ be the normalization of $\mathrm{Hilb}^G(\mathbf{A}^r)$. Since $\mathrm{Hilb}^G_{\mathrm{norm}}(\mathbf{A}^r)$ is a normal torus embedding, it is covered with finitely many toric charts. To each chart there corresponds a cone of $N \otimes \mathbf{Q}$. Since $\mathrm{Hilb}^{|G|}(\mathbf{A}^r)$ is projective over $S^{|G|}(\mathbf{A}^r)$, so is $\mathrm{Hilb}^G(\mathbf{A}^r)$ over $\mathbf{A}^r/G$. As is well known $\mathbf{A}^r/G \simeq \mathrm{Spec}\ k[\overset{\vee}{\Delta} \cap M]$ Therefore there exists a fan $\{\sigma\}$ with support Δ and a covering of $\mathrm{Hilb}^G(\mathbf{A}^r)$ consisting of toric charts U_σ with $\dim \sigma = r$, each of which has a unique zero-dimensional toric stratum corresponding to a G-cluster $Z_\sigma \in \mathrm{Hilb}^G(\mathbf{A}^r)$ defined by a monomial G-ideal I_σ of $k[\mathbf{A}^r]$. There exists a G-graph Γ such that $Z_\sigma = Z(\Gamma)$, $I_\sigma = I(\Gamma)$, equivalently $\sigma = \sigma(\Gamma)$, hence $\dim \sigma(\Gamma) = r$. In particular, $\dim \sigma(\Gamma) = r$ for any smoothable G-cluster $Z(\Gamma)$. However we will see below that $Z(\Gamma)$ is smoothable for any G-graph Γ.

Lemma 2.3. *Let Γ be a G-graph, $Z := Z^{\mathrm{vers}}(\Gamma)$ and $V := V(\Gamma)$ in Definition 1.9. Let Z_s be a closed fibre of Z over $s \in V(k)$. Then*

(i) *Z is V-flat and $Z_s \in \mathrm{Hilb}^G$ for any $s \in V(k)$. In particular, $Z(\Gamma) \in \mathrm{Hilb}^G$.*
(ii) *Γ is a k-basis of $k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma) \otimes k(s)$ for any $s \in V(k)$.*
(iii) *At any point $s \in V(k)$ Z is versal for flat G-equivariant deformations of Z_s with generic support (in the sense defined below in the proof).*

Proof. The scheme Z as well as V admits a torus action $w^v \mapsto \lambda^v w^v$, $s^{v-\mathrm{wt}_\Gamma(v)} \mapsto \lambda^{v-\mathrm{wt}_\Gamma(v)} s^{v-\mathrm{wt}_\Gamma(v)}$ $(\forall v \in M^0_0, \lambda \in \mathbf{G}^r_m)$. For $s \in \mathbf{G}^r_m(k)$, the number of solutions of the system of the equations

$$w^v = s^{v-\mathrm{wt}_\Gamma(v)} w^{\mathrm{wt}_\Gamma(v)} \quad (\forall v \in M^0_0) \tag{1}$$

is equal to $|\Gamma| = |G|$ by Lemma 1.7. Hence Z is flat over $\mathbf{G}^r_m$ $(\subset V)$. Let B be the subscheme of V consisting of all $s \in V$ such that $\dim_k k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma) \otimes k(s) \geq |G| + 1$. The subset B is invariant under $\lambda \in \mathbf{G}^r_m$, whence if $B \neq \emptyset$, then B contains $s = 0$, the unique 0-stratum of V because B is closed. However $\dim_k k[\mathbf{A}^r]/I(\Gamma) = |G|$ because Γ is a G-graph. It follows that $B = \emptyset$, Z is V-flat. In particular, Z_s is smooth for $s \in \mathbf{G}^r_m$ and $Z_s \in \mathrm{Hilb}^G$ for any $s \in V(k)$. This proves (i).

Next we prove (ii). Let $B' = \{s \in V; \Gamma \text{ is not a } k(s)\text{-basis of } k[\mathbf{A}^r]/I^{\mathrm{vers}}(\Gamma)\otimes k(s)\}$. Then B' is a closed $\mathbf{G}_m^r$-invariant subscheme of V. It is easy to see that $0 \notin B'$. This shows $B' = \emptyset$, which proves (ii).

Let R be a complete local domain with $R/m \simeq k$ and K the quotient field, m the maximal ideal of R. Choose and fix $s_0 \in V(k)$.

Let X be an R-flat (embedded) G-equivariant deformation of Z_{s_0} with generic support, in other words, the support of X_K is not contained in $\mathbf{A}_K^r \setminus \mathbf{G}_{m,K}^r$. Let J be a G-invariant ideal of $\Gamma(O_{\mathbf{A}_R^r})$ defining X. Since X is flat and $X_0 = Z_{s_0}$, Γ is a free R-basis of $\Gamma(O_X) \simeq R \otimes k[\mathbf{A}^r]/J$ by (ii). Therefore for any w^v there exist a unique $\gamma \in \Gamma$ and $q_v \in R$ such that $w^v = q_v \cdot w^\gamma \mod J$. In other words, $w^v - q_v w^{\mathrm{wt}_\Gamma(v)} \in J$. Moreover J is generated by the elements of this form because $\Gamma(O_X) \simeq R \otimes k[\mathbf{A}^r]/J$. Hence we see that

$$J = \left(w^v - q_v w^{\mathrm{wt}_\Gamma(v)} \ (\forall v \in M_0^0 \setminus \Gamma)\right) \tag{2}$$

for some $q_v \in R$. Since $X_0 = Z_{s_0}$, we have $q_v(s_0) = s_0^{v - \mathrm{wt}_\Gamma(v)}$. Suppose that we are given $v_j, u_i \in M_0^0$ such that $\sum_j (v_j - \mathrm{wt}_\Gamma(v_j)) = \sum_i (u_i - \mathrm{wt}_\Gamma(u_i))$. Then we prove

$$\prod_j q_{v_j} = \prod_i q_{u_i}. \tag{3}$$

In fact, let $a := \sum_j \mathrm{wt}_\Gamma(v_j)$ and $b := \sum_i \mathrm{wt}_\Gamma(u_i)$. Then

$$w^{a+b} \prod_j q_{v_j} = w^b \prod_j w^{v_j} = w^a \prod_i w^{u_j} = w^{a+b} \prod_i q_{u_j} \mod J.$$

Since X has generic support, the relation (3) follows. This proves (iii). $\square$

Corollary 2.4. *$Z(\Gamma)$ is smoothable and $\dim \sigma(\Gamma) = r$ for any G-graph Γ. The map $I \mapsto \Gamma(S(I)^\sharp)$ is a bijection between the set of $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$ generated by monomials and the set of G-graphs.*

Proof. By Theorem 2.3, $Z(\Gamma)$ is smoothable for any G-graph Γ. Hence $Z(\Gamma)$ is a 0-stratum of a toric chart of $\mathrm{Hilb}^G(\mathbf{A}^r)$ over $\mathbf{A}^r/G$. This implies that $\dim \sigma(\Gamma) = r$. The rest is clear. $\square$

See also [INkjm98] for the case where $G \subset \mathrm{SL}(3, k)$.

Lemma 2.5. *Let Γ be a G-graph and $\sigma(\Gamma)$ the same as before. Let τ be a codimension one face of $\sigma(\Gamma)$. Then there exists $v^* := v_- - v_+ \in M$ indivisible in M by any positive integer ≥ 2 such that $v_- \in M_0^0$, $v_+ = \mathrm{wt}_\Gamma(v_-) \in \Gamma$ and $v_\pm$ has no common factors in M_0^0 (that is, there is no nonzero $u \in M_0^0$ such that $v_\pm - u \in M_0^0$), $\tau = \sigma(\Gamma) \cap (v^*)^\perp$ and $\sigma(\Gamma) \subset \{v_- \geq v_+\}$, v^* is a generator of the lattice $\tau^\perp \cap M$.*

Proof. Since τ is codimension one in $\sigma(\Gamma)$, there exists $v^* := v_- - v_+ \in M$ such that $v_+ = \mathrm{wt}_\Gamma(v_-)$ and $\tau = \sigma(\Gamma) \cap \{v^*\}^\perp$, $\sigma(\Gamma) \subset \{v_- \geq v_+\}$ for some $v_\pm \in M_0$. We note that if $v^* := v_- - v_+ \in M$ such that $v_- \in M_0^0$, $v_+ = \mathrm{wt}_\Gamma(v_-) \in \Gamma$ and if $v_\pm$ has a common factor u, then $v_+ - u \in \Gamma$

because Γ is a G-graph. Similarly $v_- - u \in M_0^0$ and $\mathrm{wt}_\Gamma(v_- - u) = v_+ - u$. Hence without loss of generality we may assume that $v_\pm$ has no factors in common. Moreover we may assume that $v_- - v_+$ is primitive in M, that is, $v_- - v_+$ is divisible in M by no positive integer ≥ 2. In fact, suppose that there exist $e_\pm \in M_0^0$ with no common factors and a positive integer ℓ such that $v_- - v_+ = \ell(e_- - e_+)$ and $e_- - e_+ \in M$. Since M is free, it follows from $v_- + \ell e_+ = v_+ + \ell e_-$ that $v_\pm = \ell e_\pm$. Since Γ is a G-graph, $e_+ \in \Gamma$ whence $e_- \notin \Gamma$ and $\mathrm{wt}_\Gamma(e_-) = e_+$ because $e_- - e_+ \in M$. Moreover $\tau = \sigma \cap \{e_- = e_+\}$, $\sigma \subset \{e_- \geq e_+\}$. Thus we may assume $v_- - v_+$ is divisible in M by no positive integer ≥ 2. This also shows that v^* is a generator of the lattice $\tau^\perp \cap M$. This completes the proof of Lemma. $\square$

Definition 2.6. Let Γ be a G-graph, v^*, v_+ and v_- as in Lemma 2.5. Suppose $v_\pm \neq 0$. A G-igsaw transform Γ' of Γ is then defined to be

$$G_{\mathrm{ig}}(v_+, v_-)(v) = v + c^+_{\max}(v)(v_- - v_+) \in \Gamma', \tag{4}$$

$$\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma) = \{G_{\mathrm{ig}}(v_+, v_-)(v); v \in \Gamma\} \tag{5}$$

where $c^+_{\max}(v) := \max\{c \in \mathbf{Z}; v - cv_+ \in M_0^0\}$.

The effect by $G_{\mathrm{ig}}(v_+, v_-)$ is to chop elements off one extreme edge of the G-graph Γ, and glue them back at the opposite extreme to preserve the condition Definition 1.4 (ii). See Section 5 and Figures 2 and 3. We will see in Lemma 2.8 that Γ' is also a G-graph adjacent to Γ in the sense that $\sigma(\Gamma)$ and $\sigma(\Gamma')$ are adjacent, *i.e.* have a codimension one face in common.

2.7. **One parameter family of clusters.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be respectively the maximal ideal of $k[\mathbf{A}^r]$ at the origin, and the ideal of $k[\mathbf{A}^r]$ generated by all G-invariant monomials vanishing at the origin.

Let $I \in \mathrm{Hilb}^G(\mathbf{A}^r)$ and $\{I(q); q \in \mathbf{A}^1\}$ an $\mathbf{A}^1$-flat G-equivariant deformation of I. Suppose $I(q) \subset \mathfrak{m}$. Then $\mathfrak{n} \subset I(q)$ because $k[\mathbf{A}^r]/I(q) \simeq k[G]$, hence the G-invariant part of $k[\mathbf{A}^r]/I(q)$ is spanned by constant functions. Then $\dim_k k[\mathbf{A}^r]/I(q)$ is independent of q so that it determines a natural morphism $\phi : \mathbf{A}^1 = \mathbf{P}^1 \setminus \{\infty\} \to \mathrm{Gr}(V, n)$ into the Grassmann variety $\mathrm{Gr}(V, n)$ of codimension n subspaces of V where $V := k[\mathbf{A}^r]/\mathfrak{n}$, $n = |G|$. Since $\mathrm{Gr}(V, n)$ is projective, ϕ extends to $\mathbf{P}^1$. We denote by the same ϕ the extension of ϕ to $\mathbf{P}^1$. Since $\phi(\infty) \in \mathrm{Gr}(V, n)$, it determines a unique ideal $I_{\phi(\infty)}$ of $k[\mathbf{A}^r]$ such that $\phi(\infty) = I_{\phi(\infty)}/\mathfrak{n} \in \mathrm{Gr}(V, n)$.

Lemma 2.8. *Let Γ be a G-graph, τ and $v^* := v_- - v_+$ the same as in Lemma 2.5. Suppose $v_\pm \neq 0$. Define a deformation $\{I(\Gamma, v^*)(q)\}_{q \in \mathbf{A}^1}$ of $I(\Gamma)$ in the v^*-direction by*

$$I(\Gamma, v^*)(q) = (w^{v_-} - qw^{v_+}, w^v \ (v \in M_0^0 \setminus \Gamma, v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp)). \tag{6}$$

Then $I(\Gamma, v^*)(\infty) := (\lim_{q\to\infty} I(\Gamma, v^*)(q)/\mathfrak{n}) + \mathfrak{n} = I(G_{\mathrm{ig}}(v_+, v_-)(\Gamma))$. In particular, $G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$ is a G-graph.

Proof. First we prove $I(\Gamma, v^*)(0) = I(\Gamma)$. Let $\sigma = \sigma(\Gamma)$ and $J(\Gamma) := I(\Gamma, v^*)(0)$. Let s^v $(v \in M)$ be the torus coordinate of the torus embedding $U_\sigma := \operatorname{Spec} k[\overset{\vee}{\sigma} \cap M]$. Let $q = s^{v_- - v_+}$. It is clear that $J(\Gamma) \subset I(\Gamma)$. Conversely let $v \in M_0^0 \setminus \Gamma$. If $v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp$, then $w^v \in J(\Gamma)$. Assume $v - \mathrm{wt}_\Gamma(v) \in \tau^\perp$. It follows that there exists $n \geq 1$ such that $v - \mathrm{wt}_\Gamma(v) = n(v_- - v_+)$. Hence $v - nv_- \in M_0^0$ and $\mathrm{wt}_\Gamma(v) - nv_+ \in M_0^0$. Hence $w^v = w^{nv_-}w^{v-nv_-} \in w^{nv_-}k[\mathbf{A}^r] \subset J(\Gamma)$. Therefore $I(\Gamma) = J(\Gamma)$.

Let $\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Then we see

$$M_0^0 \setminus \Gamma' = \{v \notin \Gamma, v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp\} \cup \{v \in \Gamma; v - v_+ \in M_0^0\}. \tag{7}$$

In fact, if $v \notin \Gamma$, then $v \notin \Gamma'$ if and only if $v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp$. If $v \in \Gamma$, then $v \in \Gamma'$ if and only if $v - v_+ \notin M_0^0$. In other words, if $v \in \Gamma$, then $v \notin \Gamma'$ if and only if $v - v_+ \in M_0^0$. This proves (7).

We can easily prove that $I(\Gamma, v^*)(q)$ is coflat over $\mathbf{A}^1_{v^*} \simeq \operatorname{Spec} k[q]$ and that Γ is a k-basis of $k[\mathbf{A}^3]/I(\Gamma, v^*)(q)$. Since $I(\Gamma, v^*)(q) \subset \mathfrak{m}$, this implies that $\mathfrak{n} \subset I(\Gamma, v^*)(q)$. Therefore we can define $I(\Gamma, v^*)(\infty) := \lim_{q\to\infty} I(\Gamma, v^*)(q)$ as in Subsection 2.7. Hence $\mathfrak{n} \subset I(\Gamma, v^*)(\infty)$. Since $I(\Gamma, v^*)(\infty)$ is $\mathbf{G}_m^r$-invariant, there is a G-graph Γ'' such that $I(\Gamma, v^*)(\infty) = I(\Gamma'')$. We prove $\Gamma' = \Gamma''$. Since $|\Gamma'| = |\Gamma''| = |G|$, it suffices to prove $\Gamma'' \subset \Gamma'$. If $v \notin \Gamma$ and $v - \mathrm{wt}_\Gamma(v) \notin \tau^\perp$, then $w^v \in I(\Gamma, v^*)(q)$ for any q, hence $w^v \in I(\Gamma, v^*)(\infty)$. If $v \in \Gamma$ and $v - v_+ \in M_0^0$, then $w^v = w^{v-v_+}w^{v_+} \in I(\Gamma, v^*)(\infty)$. Hence by (7) we see $(M_0^0 \setminus \Gamma') \subset (M_0^0 \setminus \Gamma'')$. Hence $\Gamma'' \subset \Gamma'$. It follows that $I(\Gamma, v^*)(\infty) = I(G_{\mathrm{ig}}(v_+, v_-)(\Gamma))$. This completes the proof. □

Corollary 2.9. *Let Γ be a G-graph and $\Gamma' := G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Let $v^* := v_- - v_+$, $v_+ \in \Gamma$, $v_- \in \Gamma'$ and $h := s^{v_- - v_+}$ the same as before. Then $k[S(\Gamma)][1/h] \simeq k[S(\Gamma')][h]$.*

Proof. For any $v \in M_0^0$, there exists a unique integer $n \geq 0$ such that $\mathrm{wt}_{\Gamma'}(v) = \mathrm{wt}_\Gamma(v) + n(v_- - v_+)$, that is, $v - \mathrm{wt}_\Gamma(v) = v - \mathrm{wt}_{\Gamma'}(v) + n(v_- - v_+)$. In particular, $v_+ - \mathrm{wt}_{\Gamma'}(v_+) = v_+ - v_-$. Hence we have

$$w^{v-\mathrm{wt}_\Gamma(v)} = w^{v-\mathrm{wt}_{\Gamma'}(v)}w^{n(v_- - v_+)} \tag{8}$$

$$w^{v_- - v_+} = (w^{v_+ - v_-})^{-1}. \tag{9}$$

This proves the isomorphism. □

Remark 2.10. Thus by Corollary 2.9 we can glue together all $V(\Gamma)$ ($\Gamma \in \mathrm{Graph}(G)$) to obtain an irreducible variety $W(G) := T_M(\mathrm{Graph}(G))$. We see also that there is a zero-dimensional universal subscheme Z^{vers} over $W(G)$ such that $Z^{\mathrm{vers}} \times_{W(G)} V(\Gamma) \simeq Z^{\mathrm{vers}}(\Gamma)$. The torus embedding $T_M(\mathrm{Fan}(G))$ is the normalization of $W(G)$. It is not known whether $V(\Gamma)$ as well as Hilb^G is normal.

Theorem 2.11. *Let G be a finite abelian subgroup of $\mathrm{GL}(r, k)$ whose order is prime to the characteristic of k. Let $\mathrm{Graph}(G)$ be the set of all G-graphs and $\mathrm{Fan}(G)$ the set of all $\sigma(\Gamma)$ with $\Gamma \in \mathrm{Graph}(G)$. Then*

(i) $\mathrm{Fan}(G)$ *is a finite fan with its support* Δ,
(ii) $\mathrm{Fan}(G)$ *is obtained from a G-graph by G-igsaw tranformations,*
(iii) $T_M(\mathrm{Graph}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^r)$ *and it is projective over* $\mathbf{A}^r/G$,
(iv) $T_M(\mathrm{Fan}(G))$ *is the normalization of* $\mathrm{Hilb}^G(\mathbf{A}^r)$*, projective over* $\mathbf{A}^r/G$.

Proof. First we notice that $\mathbf{A}^r/G \simeq \mathrm{Spec}\ k[\overset{\vee}{\Delta} \cap M]$, the latter of which we denote by $T_M(\Delta)$ where $\overset{\vee}{\Delta}$ is the dual cone of Δ in $M_{\mathbf{R}} := M \otimes \mathbf{R}$.

Now we prove (iii) and (iv). In fact, let $W(G) = T_M(\mathrm{Graph}(G))$ and X the normalization of it. By the universality of $\mathrm{Hilb}(\mathbf{A}^r)$ the subscheme Z^{vers} gives rise to a morphism $h : W(G) \to \mathrm{Hilb}^G := \mathrm{Hilb}^G(\mathbf{A}^r)$. We prove that h is an isomorphism. Let H be the normalization of Hilb^G. Since Hilb^G hence H admits a T_M-action, H is a normal torus embedding. Hence we have a sequence of morphisms of normal torus embeddings

$$X \xrightarrow{\tilde{h}} H \to \mathbf{A}^r/G$$

where $\tilde{h}$ is the morphism induced from $h : X \to \mathrm{Hilb}^G$. If $\tilde{h}$ is not an isomorphism, some toric 1-strata will be contracted by $\tilde{h}$, hence by h because X is finite over $W(G)$. This contradicts Lemma 2.3 (iii). Hence $X \simeq H$, whence $W(G)$ is finite and generically bijective over Hilb^G. Since $W(G)$ is locally isomorphic to Hilb^G by Lemma 2.3 (iii), $W(G) \simeq \mathrm{Hilb}^G$. Hence $W(G)$ is projective over $\mathbf{A}^r/G$. The normalization $T_M(\mathrm{Fan}(G))$ of $W(G)$ is also projective over $\mathbf{A}^r/G$ because it is finite over $W(G)$.

Next we prove (i) and (ii). Since $T_M(\mathrm{Fan}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^r)$ is projective over $\mathbf{A}^r/G \simeq T_M(\Delta)$, the support of $\mathrm{Fan}(G)$ is exactly Δ. Let $B := nM_0^+ + M_0^0$. Since $w^v \in I(\Gamma)$ $(\forall v \in B)$ for any G-graph Γ, Γ is contained in the fixed finite subset $M_0^0 \setminus B$ of M_0^0. Therefore there are only finitely many G-graphs. Hence $\mathrm{Fan}(G)$ is a finite fan with its support Δ. For any G-graph Γ, the subscheme $Z(\Gamma)$ is smoothable by Lemma 2.3. Hence by Lemmas 2.5 and 2.8 and the connectivity of Δ any G-graph is obtained from a given G-graph by G-igsaw transformations. □

3. G-graphs in dimension three

In what follows we study $\mathrm{Hilb}^G(\mathbf{A}^3)$ when $G \subset \mathrm{SL}(3, k)$.

3.1. **G-admissible fans.** Let $n = |G|$ and $g = (a/n, b/n, c/n) \in G$ such that $0 \leq a, b, c \leq n-1$. Then the sum $(a+b+c)/n$ is equal to either 1 or 2 if g is not the identity of G. If it is equal to 1 (resp. 2), then we call g *junior* (resp. *senior*). Let $\mathrm{Jun}(G)$ be the set of all junior elements in G, and $\mathrm{Jun}^*(G) := \{e_1, e_2, e_3\} \cup \mathrm{Jun}(G)$.

We notice $\mathbf{A}^3/G \simeq T_M(\Delta)$, a torus embedding. We subdivide Δ into simplicies with apices in $\mathrm{Jun}^*(G)$ in an arbitrary manner to get a fan F_{AN}. F_{AN} is called *G-admissible* if any 3-dimensional cone $\tau \in \mathrm{F}_{\mathrm{AN}}$ is generated by three vertices $v_1, v_2, v_3 \in \mathrm{Jun}^*(G)$ such that $\tau \cap N = \{v_1, v_2, v_3\}$. By [Roan94], the torus embedding $T_M(\mathrm{F}_{\mathrm{AN}})$ is a crepant smooth resolution of

$\mathbf{A}^3/G$ if F_{AN} is G-admissible. Any crepant smooth resolution of $\mathbf{A}^3/G$ is obtained this way.

The purpose of this section is to prove that given an abelian subgroup G of $\mathrm{SL}(3,k)$ the fan $\mathrm{Fan}(G)$ is G-admissible in the above sense and that $\mathrm{Hilb}^G \simeq T_M(\mathrm{Fan}(G))$.

Lemma 3.2. *Assume $G \subset \mathrm{SL}(3,k)$. Let Γ be a G-graph and $p_k = \max\{u \in \mathbf{Z}; uf_k \in \Gamma\}$ $(k \in \mathbf{Z}/3\mathbf{Z})$. Then there exist unique non-negative integers a_{k+1}, b_{k-1} such that*

$$(p_k+1)f_k \equiv a_{k+1}f_{k+1} + b_{k-1}f_{k-1} \in \Gamma.$$

Proof. We prove that $(p_1+1)f_1 \equiv \nu f_2 + \mu f_3 \in \Gamma$ for some ν, μ. In fact, by the definition of p_1 we have $(p_1+1)f_1 \not\in \Gamma$. Then by Definition 1.4 there exists a unique $\lambda f_1 + \nu f_2 + \mu f_3 \in \Gamma$ such that $(p_1+1)f_1 \equiv \lambda f_1 + \nu f_2 + \mu f_3 \in \Gamma$. In particular, $(p_1+1)f_1 \neq \lambda f_1 + \nu f_2 + \mu f_3$. If $\lambda \geq 1$, then $(\lambda-1)f_1 + \nu f_2 + \mu f_3 \in \Gamma$ by Definition 1.4, which contradicts Definition 1.4 because $p_1 f_1 \equiv (\lambda-1)f_1 + \nu f_2 + \mu f_3 \in \Gamma$ and $p_1 f_1 \in \Gamma$. Therefore $\lambda = 0$. □

Lemma 3.3. (Unique Valley Lemma) *Suppose $r \geq 0$, $s \geq 0$. Assume*

$$rf_{k+1} + sf_{k-1} \in \Gamma, \quad (r+1)f_{k+1} + sf_{k-1} \in \Gamma,$$
$$rf_{k+1} + (s+1)f_{k-1} \in \Gamma, \quad (r+1)f_{k+1} + (s+1)f_{k-1} \not\in \Gamma.$$

Then $r = a_{k+1}$, $s = b_{k-1}$ and $(p_k+1)f_k \equiv rf_{k+1} + sf_{k-1}$.

Proof. Let $k=1$ for simplicity. By Definition 1.4 there exists $af_1 + bf_2 + cf_3 \in \Gamma$ such that $(r+1)f_2 + (s+1)f_3 \equiv af_1 + bf_2 + cf_3$. Since $(r+1)f_2 + (s+1)f_3 \not\in \Gamma$, we have $(r+1)f_2 + (s+1)f_3 \neq af_1 + bf_2 + cf_3$. If $b \geq 1$, then $rf_2 + (s+1)f_3 \equiv af_1 + (b-1)f_2 + cf_3 \in \Gamma$, which contradicts Definition 1.4 because $rf_2 + (s+1)f_3 \in \Gamma$. Hence $b = 0$. Similarly $c = 0$. It follows that $(r+1)f_2 + (s+1)f_3 \equiv af_1 \in \Gamma$. Since $f_1 + f_2 + f_3 \equiv 0$, we have $rf_2 + sf_3 \equiv (a+1)f_1$, where $rf_2 + sf_3 \in \Gamma$. It follows from Definition 1.4 that $(a+1)f_1 \not\in \Gamma$. Therefore $a = p_1$. □

Definition 3.4. We call $\mathfrak{v}_k := a_{k+1}f_{k+1} + b_{k-1}f_{k-1}$ $(k \in \mathbf{Z}/3\mathbf{Z})$ *a virtual valley* of Γ if the condition of Lemma 3.2 is satisfied. We also call $\mathfrak{s}_k := p_k f_k$ *a summit* of Γ. The vector $rf_{k+1} + sf_{k-1}$ of Γ is called *a valley* of Γ if the conditions in Lemma 3.3 are satisfied. By definition Γ has always three virtual valleys, but may not have valleys.

Lemma 3.3 is true even if $r = 0$ or if $s = 0$. By Lemma 3.3 Γ has at most a unique valley on $\Gamma_k := \Gamma \cap (\mathbf{R}f_i + \mathbf{R}f_j)$ $(i, j \neq k)$. It gives a geometric way of computing a_k and b_k.

3.5. **Classification of G-graphs.** By Lemma 3.3 any G-graph Γ is one of the following:

(i) a linear graph,
(ii) a planar graph with no valleys,
(iii) a planar graph with a valley,

(iv) three planar graphs with no valleys,
(v) a planar graph with a valley and two planar graph with no valleys,
(vi) two planar graphs, each with a valley and a planar graph with no valley,
(vii) three planar graphs, each with a valley.

Definition 3.6. Let Γ be a G-graph. Let $\mathfrak{v}_i := a_{i+1}f_{i+1} + b_{i-1}f_{i-1}$ be the virtual valleys of Γ ($i \in \mathbf{Z}/3\mathbf{Z}$). Then we define

$$\begin{aligned} v_i &:= (p_i + 1)f_i - \mathfrak{v}_i, \\ w_i &:= f_1 + f_2 + f_3 - v_i, \\ v_c = w_c &:= f_1 + f_2 + f_3. \end{aligned}$$

A G-graph Γ is called *v-crepant* (resp. *w-crepant*) if $v_1 + v_2 + v_3 = f_1 + f_2 + f_3$ (resp. $w_1 + w_2 + w_3 = f_1 + f_2 + f_3$).

We call Γ *crepant* if Γ is either v-crepant or w-crepant. We remark that no G-graph Γ is v-crepant and w-crepant.

Lemma 3.7. *Let Γ be a G-graph. Then $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$. If Γ is v-crepant (resp. w-crepant), then $S(\Gamma) = \mathbf{Z}_+v_1 + \mathbf{Z}_+v_2 + \mathbf{Z}_+v_3$ (resp. $S(\Gamma) = \mathbf{Z}_+w_1 + \mathbf{Z}_+w_2 + \mathbf{Z}_+w_3$). In particular, $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$.*

Proof. By Lemma 1.8 and Subsection 3.5, $S(\Gamma)$ is generated by v_i, w_i and v_c ($i = 1, 2, 3$). If Γ is v-crepant, then $v_c = v_1 + v_2 + v_3$ and $w_i = v_c - v_i$. Hence $S(\Gamma) = \mathbf{Z}_+v_1 + \mathbf{Z}_+v_2 + \mathbf{Z}_+v_3$. Since $M = M(\Gamma)$, we have $S(\Gamma) = \overset{\vee}{\sigma}(\Gamma) \cap M$. The rest is similar. □

Notation 3.8. Let Γ be a G-graph. For $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ we denote Γ by $\Gamma = \Gamma(a, b)$ (resp. $\Gamma^*(a, b)$) if Γ is v-crepant (resp. w-crepant). For the same a and b we also define

$$\begin{aligned} \|a\| = a_1 + a_2 + a_3, \quad a * b = a_1b_2 + a_2b_3 + a_3b_1, \\ d(a, b) = 1 + \|a\| + \|b\| + a * a + a * b + b * b. \end{aligned}$$

Lemma 3.9. *Let Γ be a G-graph. With the same notation as above,*

(i) *If $\Gamma = \Gamma(a, b)$, then $w_i = v_{i-1} + v_{i+1}$, $p_i = a_i + b_i$ and $|\Gamma| = d(a, b)$,*
(ii) *If $\Gamma = \Gamma^*(a, b)$, then $v_i = w_{i-1} + w_{i+1}$, $p_i = a_i + b_i + 1$ and $|\Gamma| = 3 + \|a\| + \|b\| + d(a, b)$.*

Proof. Clear. □

3.10. **Crepant G-graphs.** Any v-crepant Γ in the first six classes is viewed as one of the special cases of crepant graphs in (vii). In fact, if at least one of a_i or b_j equals zero, (vii) is reduced to the cases (i)-(vi). For instance, $\Gamma_k := \Gamma \cap \{x_k = 0\}$ has no valley if and only if $a_{k-1}b_{k+1} = 0$, while the case (i) is obtained by setting $a = (a_1, 0, 0)$ and $b = (b_1, 0, 0)$ with $a_1 > 0$, $b_1 > 0$, in which case $\Gamma = \Gamma(a, b) = \{if_1; i \in [0, a_1 + b_1]\}$. It is v-crepant but is not w-crepant. Since various proofs and computations are carried out more or less in the same manner as (vii), we mainly discuss the case (vii) in what follows. In the case (vii) Γ is given by

$$\Gamma = \left\{ \begin{matrix} if_k + jf_{k+1}; & (i,j) \in [0, p_k] \times [0, b_{k+1}] \ \text{ or} \\ (k \in \mathbf{Z}/3\mathbf{Z}) & (i,j) \in [0, a_k] \times [b_{k+1}+1, p_{k+1}] \end{matrix} \right\}$$

where $p_k = a_k + b_k$.

Let $\Gamma = \Gamma^*(a, b)$ be a w-crepant G-graph. Then $p_i = a_i + b_i + 1 \geq 1$ by Lemma 3.9. Therefore (i)-(iii) is impossible.

Let us consider the case (iv). Then by Lemma 3.9 we have

$$|\Gamma| = 1 + \|p\| + p * p = 4 + 2\|p\| + a * a + a * b + b * a + b * b.$$

Hence $3 + \|a\| + \|b\| + b * a = 0$, which is impossible. Similarly we derive a contradiction in the cases (v) and (vi). Thus there are no w-crepant G-graphs in (i)-(vi). However there do exist infinitely many w-crepant G-graphs in (vii). If Γ is w-crepant with $a_k = b_k = 0$, then $\Gamma = \{0, f_1, f_2, f_3\}$, and G is an abelian group $\simeq \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ of order 4 consisting of diagonal matrices with entries ± 1 and determinant one. See also Corollary 2.4.

The following Lemmas follow from Definition 2.6 readily.

Lemma 3.11. *Let $\Gamma = \Gamma(a, b)$ be a v-crepant G-graph, and $v_1 = v_- - v_+$, $v_- = (p_1 + 1)f_1$ and $v_+ = a_2 f_2 + b_3 f_3$. Then the G-igsaw transformation $G_{\mathrm{ig}}(v_-, v_+)(\Gamma)$ of Γ is a crepant G-graph adjacent to Γ and it is given by*

$$G_{\mathrm{ig}}(v_+, v_-)(\Gamma) = \begin{cases} \Gamma^*(a - f_2, b - f_3) & \text{if } \ a_2 \geq 1, b_3 \geq 1 \\ \Gamma(a', b') & \text{if } \ a_2 \geq 1, b_3 = 0 \\ \Gamma(a'', b'') & \text{if } \ a_2 = 0, b_3 \geq 1 \end{cases} \tag{10}$$

where

$$\begin{aligned} a' &= (([p_2/a_2] - 1)(p_1 + 1) + a_1, (1 + [p_2/a_2])a_2 - p_2 - 1, 0), \\ b' &= (p_1 + 1, p_2 - [p_2/a_2]a_2, a_3), \\ a'' &= (p_1 + 1, b_2, p_3 - [p_3/b_3]b_3), \\ b'' &= (([p_3/b_3] - 1)(p_1 + 1) + b_1, 0, (1 + [p_3/b_3])a_3 - p_3 - 1). \end{aligned}$$

Proof. Let v_i be as before and $\Gamma' = G_{\mathrm{ig}}(v_+, v_-)(\Gamma)$. Then by Lemma 2.8, Γ' is a G-graph. Therefore we have $v'_i := v_i(\Gamma')$ and $w'_i := w_i(\Gamma')$ by Definition 3.6. The G-graph Γ' is given as above. In fact, suppose $a_2 \geq 1, b_3 \geq 1$. Then we see that $p_1(\Gamma') = p_1(\Gamma) + 1$, $a_2(\Gamma') = a_2(\Gamma) - 1$ and $b_3(\Gamma') = b_3(\Gamma) - 1$ and for the other invariants $a_i(\Gamma') = a_i(\Gamma)$ $(i = 1, 3)$, $b_j(\Gamma') = b_j(\Gamma)$ $(j = 1, 2)$ and $p_k(\Gamma') = p_k(\Gamma)$ $(k = 2, 3)$. Therefore $p_2(\Gamma') = a_2(\Gamma') + b_2(\Gamma')$, $\Gamma' = \Gamma^*(a - f_2, b - f_3)$ and $w_i(\Gamma') = (1 - \delta_{i1})(f_1 + f_2 + f_3) - v_i$. In particular, $w'_1 = -v_1$, $w'_1 + w'_2 + w'_3 = f_1 + f_2 + f_3$. Hence Γ' is w-crepant.

If $a_2 \geq 1, b_3 = 0$, then $v'_1 = (k + 1)v_1 + v_2$, $v'_2 = -v_1$, $v'_3 = v_3 - (k - 1)v_1$ where $k = [p_2/a_2]$. Hence $v'_1 + v'_2 + v'_3 = f_1 + f_2 + f_3$. This proves that Γ' is v-crepant. The rest is similar. $\square$

Lemma 3.12. *Let* $\Gamma = \Gamma^*(a,b)$ *be a* w*-crepant* G*-graph and* $w_1 = w_- - w_+$, $w_- = (a_2+1)f_2 + (b_3+1)f_3$ *and* $w_+ = p_1 f_1$. *Then* $G_{\mathrm{ig}}(w_+, w_-)(\Gamma)) = \Gamma(a+f_2, b+f_3)$ *and it is* v*-crepant and adjacent to* Γ.

Proof. This case is the converse to the first case in Lemma 3.11. □

Lemma 3.13. *Any* G*-graph is crepant if* $G \subset \mathrm{SL}(3,k)$.

Proof. Suppose that there is at least a crepant G-graph Γ. Then any G-graph adjacent to Γ is obtained by a G-igsaw transformation from Γ by Theorem 2.11 and it is crepant by Lemmas 3.11 and 3.12. Thus we can find adjacent crepant G-graphs successively so that we can eventually cover the whole cone Δ with $\sigma(\Gamma)$ for crepant G-graphs Γ. It remains to find a crepant G-graph.

If G is cyclic, then let $g = (r/n, s/n, 1/n)$ be a generator with $1 \le r \le s \le n-1$ and $1+r+s = n$. Let Γ be a linear graph in Subsection 3.5 (i) with $\Gamma = \{xf_3; x \in [0, n-1]\}$. Then it is easy to check that Γ is a crepant G-graph. We see $\Gamma = \Gamma(a,b)$ where $a_i = b_i = 0$ $(i = 1,2)$, $a_3 = s$ and $b_3 = r$.

Next consider an abelian group G with two generators. Let m be the maximal order of elements in G. Then without loss of generality we may assume that G is a group of order ℓm generated by $g := (r/m, s/m, 1/m)$ and $f := (-1/\ell, 1/\ell, 0)$ for some $2 \le \ell \le m$. Let $\Gamma = \{pf_2 + qf_3; p \in [0, \ell-1], q \in [0, m-1]\}$, a planar G-graph in Subsection 3.5 (ii) with no valleys. Let $a_1 = b_1 = b_2 = 0$, $a_2 = \ell - 1$, a_3 the unique integer such that $0 \le a_3 \le m-1$, $a_3 = \ell s \mod m$, and $b_3 := m - 1 - a_3$ (≥ 0). Let $a = (a_i)$, $b = (b_i)$. Then it turns out that $\Gamma = \Gamma(a,b)$ and Γ is v-crepant. This proves the existence of crepant G-graphs. This completes the proof of Lemma. □

Table 1. G-graphs

type	v/w	conditions	valleys
(i)	v	$a_k = a_{k+1} = b_k = b_{k+1} = 0$	none
(ii)	v	$a_k = b_k = a_{k-1}b_{k+1} = 0$	none
(iii)	v	$a_k = b_k = 0$	$\mathfrak{v}_k$
(iv)	v	$a_2b_1 = a_3b_2 = a_1b_3 = 0$	none
(v)	v	$a_kb_{k-1} = a_{k+1}b_k = 0$	$\mathfrak{v}_k$
(vi)	v	$a_{k+1}b_k = 0$	$\mathfrak{v}_k, \mathfrak{v}_{k+1}$
(vii)	v	$a_k \ge 1$, $b_k \ge 1$	$\mathfrak{v}_k$ $(k = 1,2,3)$
(vii)	w	$a_k \ge 0$, $b_k \ge 0$	$\mathfrak{v}_k$ $(k = 1,2,3)$

3.14. **Generators of** $\sigma(\Gamma)$**.** By Definition 1.5 $\sigma(\Gamma)$ is given explicitly. Let

$$\mathfrak{a}(a,b) = \begin{pmatrix} F(a_2,a_3,b_2,b_3) \\ F(a_3,a_1,b_3,b_1) \\ F(a_1,a_2,b_1,b_2) \end{pmatrix} \tag{11}$$

where $a = (a_1,a_2,a_3)$, $b = (b_1,b_2,b_3)$ and $F(x,y,u,v) = (xy + xv + uv)/|\Gamma|$.

It turns out that $\sigma(\Gamma)$ is generated by three junior elements of G via the isomorphism $G \simeq N/N_0$. Let $\Gamma = \Gamma(a,b)$ and

$$A = A(\Gamma) = \begin{pmatrix} a_1+b_1+1 & -a_2 & -b_3 \\ -b_1 & a_2+b_2+1 & -a_3 \\ -a_1 & -b_2 & a_3+b_3+1 \end{pmatrix}. \tag{12}$$

Then we have

$$\begin{aligned} A^{-1} &:= (\mathfrak{a}_1(\Gamma), \mathfrak{a}_2(\Gamma), \mathfrak{a}_3(\Gamma)) \\ &:= (\mathfrak{a}(a+f_2, b+f_3), \mathfrak{a}(a+f_3, b+f_1), \mathfrak{a}(a+f_1, b+f_2)). \end{aligned} \tag{13}$$

The column vectors $\mathfrak{a}_k(\Gamma)$ are junior elements of G via the isomorphism $G \simeq N/N_0$ because $(1,1,1)A = (1,1,1)$ by the v-crepancy of Γ. Next take $\Gamma = \Gamma^*(a,b)$ and set

$$A^* = A^*(\Gamma) = \begin{pmatrix} -a_1-b_1-1 & a_2+1 & b_3+1 \\ b_1+1 & -a_2-b_2-1 & a_3+1 \\ a_1+1 & b_2+1 & -a_3-b_3-1 \end{pmatrix} \tag{14}$$

and $(A^*)^{-1} = (\mathfrak{a}_1^*(\Gamma), \mathfrak{a}_2^*(\Gamma), \mathfrak{a}_3^*(\Gamma))$. The vectors $\mathfrak{a}_k^*(\Gamma)$ are junior elements of G via the isomorphism $G \simeq N/N_0$ where

$$\begin{aligned} \mathfrak{a}_1^*(\Gamma) &= \mathfrak{a}(a+f_3+f_1, b+f_1+f_2), \\ \mathfrak{a}_2^*(\Gamma) &= \mathfrak{a}(a+f_1+f_2, b+f_2+f_3), \\ \mathfrak{a}_3^*(\Gamma) &= \mathfrak{a}(a+f_2+f_3, b+f_3+f_1). \end{aligned} \tag{15}$$

Thus $\sigma(\Gamma(a,b))$ is generated by $\mathfrak{a}_k(\Gamma(a,b))$ $(k \in \mathbf{Z}/3\mathbf{Z})$, while $\sigma(\Gamma^*(a,b))$ is generated by $\mathfrak{a}_k^*(\Gamma^*(a,b))$ $(k \in \mathbf{Z}/3\mathbf{Z})$.

4. A crepant smooth resolution

Lemma 4.1. *Let* $\mathrm{Graph}(G)$ *be the set of all G-graphs. Let* $\mathrm{Fan}(G)$ *be the set of all* $\sigma(\Gamma)$ *such that* $\Gamma \in \mathrm{Graph}(G)$*. Then* $\mathrm{Fan}(G)$ *is a finite fan with* Δ *its support, which consists of exactly* $|G|$ *3-dimensional G-cones and their faces. The associated torus embedding* $T_M(\mathrm{Fan}(G))$ *is a crepant smooth resolution of* $T_M(\Delta) \simeq \mathbf{A}^3/G$*.*

Proof. By Lemma 3.7 $T_M(\mathrm{Fan}(G))$ is smooth. It remains to prove crepancy. Let $\Gamma \in \mathrm{Graph}(G)$ and let u_i be a v-basis or a w-basis of M. Then $X := T_M(\mathrm{Fan}(G))$ has a globally well-defined rational 3-form

$$\omega_{\mathrm{rat}} := \prod_{i=1}^{3} dw^{u_i}/w^{u_i} = |G| \prod_{i=1}^{3} dw^{f_i}/w^{f_i}. \tag{16}$$

By the crepancy of $\Gamma \in \mathrm{Graph}(G)$, $w^{f_1+f_2+f_3}\omega_{\mathrm{rat}}$ is a nonvanishing regular 3-form on X because char. k is prime to $n = |\Gamma|$, whence the dualizing sheaf of X is trivial. This proves Lemma. $\square$

Theorem 4.2. $T_M(\mathrm{Fan}(G)) \simeq \mathrm{Hilb}^G(\mathbf{A}^3)$.

Proof. By Theorem 2.11 and Lemma 3.7 we see $\mathrm{Hilb}^G \simeq T_M(\mathrm{Graph}(G)) \simeq T_M(\mathrm{Fan}(G))$. □

Remark 4.3. The crepancy of G-graphs is interpreted almost as smoothness of versal generic-support-deformation spaces of $Z(\Gamma)$ when $G \subset \mathrm{SL}(3,k)$.

With the notation in Subsection 3.1 $\mathbf{A}^3 \simeq \mathrm{Spec}\ k[\overset{\vee}{\Delta} \cap M_0] \simeq \mathrm{Spec}\ k[x,y,z]$ where x,y and z are the torus coordinates corresponding to the characters of M_0 f_1, f_2 and f_3 respectively. We also denote $k[x,y,z]$ by $k[\mathbf{A}^3]$.

Let Γ be a G-graph and $Z(\Gamma)$ the associated G-cluster.

The generic-support G-equivariant deformation theory of the cluster $Z(\Gamma)$ is more or less the same for any crepant G-graph Γ. The cases (i)-(vi) are obtained by specialization of the case $(\mathrm{vii})_v$. Now suppose that $\Gamma := \Gamma(a,b)$ is a v-crepant G-graph in the class (vii). The ideal $I(\Gamma)$ defining $Z(\Gamma)$ is given by

$$I(\Gamma) := \begin{pmatrix} x^{p_1+1}, y^{p_2+1}, z^{p_3+1}, xyz \\ y^{a_2+1}z^{b_3+1}, z^{a_3+1}x^{b_1+1}, x^{a_1+1}y^{b_2+1} \end{pmatrix}.$$

Let $A^v := k[s_1,s_2,s_3]$ and Z^v a G-invariant subscheme of $\mathbf{A}^3_{A^v}$ defined by the ideal

$$I^v(\Gamma)(s) := \begin{pmatrix} x^{p_1+1} - s_1 y^{a_2} z^{b_3}, y^{p_2+1} - s_2 z^{a_3} x^{b_1}, z^{p_3+1} - s_3 x^{a_1} y^{b_2} \\ y^{a_2+1}z^{b_3+1} - s_2 s_3 x^{p_1}, z^{a_3+1}x^{b_1+1} - s_3 s_1 y^{p_2} \\ x^{a_1+1}y^{b_2+1} - s_1 s_2 z^{p_3}, xyz - s_1 s_2 s_3 \end{pmatrix}.$$

This is the versal deformation given in Lemma 2.3. Similarly for a w-crepant G-graph $\Gamma := \Gamma^*(a,b)$ the versal deformation of $Z^w := Z(\Gamma)$ is given by

$$I^w(\Gamma)(t) := \begin{pmatrix} x^{p_1+1} - t_2 t_3 y^{a_2} z^{b_3}, y^{p_2+1} - t_3 t_1 z^{a_3} x^{b_1}, z^{p_3+1} - t_1 t_2 x^{a_1} y^{b_2} \\ y^{a_2+1}z^{b_3+1} - t_1 x^{p_1}, z^{a_3+1}x^{b_1+1} - t_2 y^{p_2} \\ x^{a_1+1}y^{b_2+1} - t_3 z^{p_3}, xyz - t_1 t_2 t_3 \end{pmatrix}.$$

Thus in each case the distinguished term $xyz - s_1 s_2 s_3$ or $xyz - t_1 t_2 t_3$ is present in the ideal defining a versal deformation, which implies crepancy of $\mathrm{Hilb}^G(\mathbf{A}^3)$.

By Lemma 3.7 $U_{\sigma(\Gamma)} := \mathrm{Spec}\ k[\overset{\vee}{\sigma}(\Gamma) \cap M]$ turns out to be

$$U_{\sigma(\Gamma(a,b))} := \mathrm{Spec}\ k[w^{v_1}, w^{v_2}, w^{v_3}] \simeq \mathrm{Spec}\ k[s_1,s_2,s_3],$$
$$U_{\sigma(\Gamma^*(a,b))} := \mathrm{Spec}\ k[w^{w_1}, w^{w_2}, w^{w_3}] \simeq \mathrm{Spec}\ k[t_1,t_2,t_3].$$

See Definition 3.6 for the notation.

4.4. **The exceptional set.** Let $\pi : \mathrm{Hilb}^G \to \mathbf{A}^3/G$ be the natural morphism and $E(G)$ the execeptional set of π. Any irreducible component of $E(G)$ is by Theorem 4.2 a smooth torus embedding corresponding to a junior element $g \in G$, which we denote by $E(g)$. Hence $E(G) = \cup_{g \in \mathrm{Jun}(G)} E(g)$. Let $\mathrm{Graph}(G,g)$ be the set of $\Gamma \in \mathrm{Graph}(G)$ with $g \in \sigma(\Gamma)$, and $U_g(\sigma(\Gamma)) := \mathrm{Spec}\ k[\overset{\vee}{\sigma}(\Gamma) \cap g^{\perp} \cap M]$. Then $E(g)$ is covered with $U_g(\sigma(\Gamma))$ $(\Gamma \in \mathrm{Graph}(G,g))$.

Definition 4.5. Let $g_i \in \mathrm{Jun}(G)$ $(i = 1, 2, 3)$ be distinct elements. Then $E(g_1) \cap E(g_2)$ is a smooth rational curve if and only if $\mathrm{Graph}(G, g_1) \cap \mathrm{Graph}(G, g_2) \neq \emptyset$ if and only if $\sigma(g_1, g_2) := \mathbf{R}_+ g_1 + \mathbf{R}_+ g_2 \in \mathrm{Sk}^2 \mathrm{Fan}(G)$. Similarly $\cap_{i=1,2,3} E(g_i)$ is a point if and only if $\cap_{i=1,2,3}\ \mathrm{Graph}(G, g_i) \neq \emptyset$ if and only if $\sigma(g_1, g_2, g_3) := \sum_{i=1,2,3} \mathbf{R}_+ g_i \in \mathrm{Sk}^3 \mathrm{Fan}(G)$. In this case $|\cap_{i=1,2,3}\ \mathrm{Graph}(G, g_i)| = 1$. Thus $E(G)$ is stratified into 0,1,2-strata labeled by $\mathrm{Sk}^p \mathrm{Fan}(G)$ $(p = 3, 2, 1)$ respectively. Let $\tau \in \mathrm{Sk}^p \mathrm{Fan}(G)$ and $E^0(\tau)$ a $(3-p)$-stratum of $E(G)$ corresponding to τ and $E(\tau)$ the closure of $E^0(\tau)$. This means that $E^0(\sigma) \cap E^0(\tau) = \emptyset$ for $\sigma \neq \tau$ and $E(\sigma) \subset E(\tau)$ if $\tau \prec \sigma$ $(\tau, \sigma \in \mathrm{Fan}(G))$, where $E^0(\tau) = \mathrm{Spec}\, k[\tau^\perp \cap M] \simeq \mathbf{G}_m^{3-p}$.

Definition 4.6. For a subset A of M_0 and $\tau \in \mathrm{Fan}(G)$ we define

$$\mathrm{Graph}(G, \tau) := \cap_{g \in \tau \cap \mathrm{Jun}(G)} \mathrm{Graph}(G, g),$$
$$\Gamma(\tau) := \cap_{\Gamma \in \mathrm{Graph}(G,\tau)} \Gamma, \quad \Gamma(\tau, A) := \cap_{A \subset \Gamma \in \mathrm{Graph}(G,\tau)} \Gamma.$$

Now one easily sees

Lemma 4.7. *Let $\Gamma = \Gamma(a - f_2, b - f_3)$ with $a_i, b_j \geq 1$ and $g := \mathfrak{a}_1(\Gamma) = \mathfrak{a}(a, b)$ with the notation in Subsection 3.14. Then*

$$\mathrm{Graph}(G, g) = \begin{Bmatrix} \Gamma^*(a - f_i - f_{i+1}, b - f_{i+1} - f_{i-1}) \\ \Gamma(a - f_{i+1}, b - f_{i+2})\ (i \in \mathbf{Z}/3\mathbf{Z}) \end{Bmatrix}.$$

Corollary 4.8. *Let the G-graph Γ and the junior element g be as above. Then $E(g)$ is a regular hexagon – $\mathbf{P}^2$ blown up at three distinct points.*

Remark 4.9. As in [IN98] there is a certain (rather complicated) correspondence between the set of irreducible exceptional subvarieties and the set of minimal G-submodules generating $I \in \mathrm{Hilb}^G$. This would be a kind of phenomenon generalizing [McKay80] from the view point of representations of G. However it seems difficult to understand this correspondence only from irreducible decompositions of tensor products. We omit it. A generalization of McKay correspondence in dimension three analogous to [GSV83] is given in [INkjm98].

5. Examples in dimension 3

5.1. **The case $G = G_{1,2,3}$.** We will explain the theory in the previous sections by the examples in smaller values. Let G be a cyclic group of order 6, $g = (1, 2, 3)/6$ a generator of G. We denote $G = G_{1,2,3}$ when necessary. Figure 1 plots all the junior elements for $G = G_{1,2,3}$. Any $\Gamma \in \mathrm{Graph}(G)$ is one of the following table (Table 2). Def_Γ is the set of toric parameters for deforming the G-cluster $Z(\Gamma)$.

First we start with the ideal $I(\Gamma_1) = (x^6, y, z)$. It is clear $\dim k[\mathbf{A}^3]/I(\Gamma_1) = 6$, and that $k[\mathbf{A}^3]/I(\Gamma_1)$ is spanned by Γ_1. We can deform $Z(\Gamma_1)$ into smooth G-subschemes by $(x^6 - s, y, z)$, whence $Z(\Gamma_1) \in \mathrm{Hilb}^G$. The versal deformation of $Z(\Gamma_1)$ is given by the ideal

$$(x^6 - s, y - tx^2, z - ux^3, x^4y - st, x^3z - us, xyz - stu)$$

where x^2 and x^3 are the unique elements of Γ_1 such that $\mathrm{wt}(y) = \mathrm{wt}(x^2)$, $\mathrm{wt}(z) = \mathrm{wt}(x^3)$. Def_{Γ_1} is now understood as 3 free parameters s, t, u hence may be understood as x^6, yx^{-2} and zx^{-3}. In this case there are two directions yx^{-2} and zx^{-3} of deformations which give rise to monomial G-ideals (*i.e.* G-ideals generated by monomials). This is written in Figure 2 as $b-2a$ and $c-3a$. The graph Γ_2 is obtained from Γ_1 by replacing $2a$ in any vector of Γ_1 by b repeatedly. In general, in the process of taking the limit of clusters in the direction $v_- - v_+$ the second G-graph is obtained from the first G-graph by replacing v_+ by v_- repeatedly.

For instance $v_+ = a + b$ and $v_- = c$ in the process from Γ_2 to Γ_4. Hence $a + b \in \Gamma_2$ (resp. $a + 2b$) is transformed into c (resp. $b + c$) so that Γ_2 is changed into Γ_4 under $(G_{1,2,3})_{\mathrm{ig}}(a + b, c)$. Figure 2 lists the possible solutions to the jigsaw puzzle (the G-igsaw puzzle !) between G-graphs when $G = G_{1,2,3}$.

TABLE 2. $G_{1,2,3}$-graphs/cones

No.	Γ	$I(\Gamma)$
1	$\{1, x, x^2, x^3, x^4, x^5\}$	(x^6, y, z)
2	$\{1, x, y, xy, y^2, xy^2\}$	(x^2, y^3, z)
3	$\{1, y, z, yz, y^2, y^2z\}$	(x, y^3, z^2)
4	$\{1, x, y, z, yz, y^2\}$	$(xy, y^2z, zx, x^2, y^3, z^2)$
5	$\{1, x, y, z, yz, zx\}$	(x^2, xy, y^2, z^2)
6	$\{1, x, x^2, z, xz, x^2z\}$	(x^3, y, z^2)
No.	**Def_Γ**	**$\sigma(\Gamma)$**
1	$x^6, y/x^2, z/x^3$	(e_2, e_3, g)
2	$x^2/y, y^3, z/xy$	(e_3, g, g^3)
3	$x/y^2z, y^3, z^2$	(e_1, g^3, g^4)
4	$xy/z, y^2z/x, zx/y^2$	(g, g^3, g^4)
5	$x^2/y, y^2/zx, z^2$	(g, g^2, g^4)
6	$x^3/z, y/x^2, z^2$	(e_2, g, g^2)

5.2. **Jigsaw puzzles.** For a given G-graph the possible solutions to the jigsaw puzzle are clear at first sight if the graph is complicated enough, as we see in the case of $G_{1,5,31}$ below. See Figure 3. Let $\Gamma = \Gamma(g^{10}, g^{11}, g^{17})$. Then the G-graph Γ is v-crepant. There are exactly three jigsaw puzzle solutions only in the v_k-directions where v_k is among $4a-2b-c$, $5c-2a-b$ and $4b-a-3c$. For instance, the adjacent G-graph of Γ in the $(5c-2a-b)$-direction is obtained by moving the hook at $2a+b$ consisting of 4 small black dots, each surrounded by a large white circle, into the position $5c$ as in the lower-left of the Figure 3. On the other hand there are exactly three jigsaw puzzle solutions only in the w_k-directions for any w-crepant Γ, for instance, $\Gamma = \Gamma(g^4, g^{10}, g^{11})$. See also Table 3 for another example of G-graphs/cones.

Table 3. $G_{1,2,4}$-graphs/cones

No.	Γ	$I(\Gamma)$
0	$\{1, x, y, z, xy, yz, zx\}$	(x^2, y^2, z^2, xyz)
1	$\{1, y, z, z^2, z^3, yz, yz^2\}$	(x, y^2, z^4, yz^3)
2	$\{1, z, x, x^2, x^3, zx, zx^2\}$	(y, z^2, x^4, zx^3)
3	$\{1, x, y, y^2, y^3, xy, xy^2\}$	(z, x^2, y^4, zx^3)
4	$\{1, z, z^2, z^3, z^4, z^5, z^6\}$	(x, y, z^7)
5	$\{1, y, y^2, y^3, y^4, y^5, y^6\}$	(x, y^7, z)
6	$\{1, x, x^2, x^3, x^4, x^5, x^6\}$	(x^7, y, z)
No.	Def_Γ	$\sigma(\Gamma)$
0	$x^2/y, y^2/z, z^2/x$	(g, g^2, g^4)
1	$x/z^2, z^4/y, y^2/z$	(e_1, g^2, g^4)
2	$y/x^2, x^4/z, z^2/x$	(e_2, g, g^2)
3	$z/y^2, y^4/x, x^2/y$	(e_3, g, g^4)
4	$z^7, x/z^2, y/z^4$	(e_1, e_2, g^2)
5	$y^7, z/y^2, x/y^4$	(e_3, e_1, g^4)
6	$x^7, y/x^2, z/x^4$	(e_2, e_3, g)

6. Miscellaneous remarks

6.1. **A singular example in dimension four.** There is an example of singular $\mathrm{Hilb}^G(\mathbf{A}^4)$. Let G be an abelian subgroup of $\mathrm{SL}(4, k)$ consisting of all diagonal matrices with entries ± 1, which we denote by G_{2^3}. We will prove that $\mathrm{Hilb}^G(\mathbf{A}^4)$ is singular. The group G has order 8 with 6 junior elements and 12 monomial G-ideals. Among them there are essentially three different types of G-ideals whose G-graphs are Γ_1, Γ_2 and Γ_3 in the Table 4. Let $U_k := \operatorname{Spec} k[\overset{\vee}{\sigma}(\Gamma_k) \cap M]$. We see that U_k $(k = 1, 2)$ is smooth, while U_3 is singular. Hence $\mathrm{Hilb}^G(\mathbf{A}^4)$ is singular. The versal generic-support-G-equivariant deformation of $Z(\Gamma_k)$ $(k = 1, 2, 3)$ is given by the ideal

$$I(\Gamma_1)(s) := \begin{pmatrix} x - s_1 yzw, y^2 - s_2, z^2 - s_3, w^2 - s_4 \\ xyzw - s_1 s_2 s_3 s_4 \end{pmatrix},$$

$$I(\Gamma_2)(t) := \begin{pmatrix} x^2 - t_1 t_2 t_3 t_4, y^2 - t_1 t_4, z^2 - t_2 t_4, w^2 - t_3 t_4 \\ xy - t_1 zw, xz - t_2 yw, xw - t_3 yz \\ yzw - t_4 x, xyzw - t_1 t_2 t_3 t_4^2 \end{pmatrix},$$

$$I(\Gamma_3)(u) := \begin{pmatrix} x^2 - u_1, y^2 - u_2, z^2 - u_3, w^2 - u_4 \\ yz - u_5 xw, yw - u_6 xz, zw - u_7 xy \\ xyzw - u_1 u_2 u_7 \end{pmatrix}$$

where

$$u_2 u_7 = u_3 u_6 = u_4 u_5,\ u_2 u_3 = u_1 u_4 u_5^2,$$
$$u_2 u_4 = u_1 u_3 u_6^2,\ u_3 u_4 = u_1 u_2 u_7^2.$$

Table 4. G_{2^3}-graphs/cones

No.	Γ	$I(\Gamma)$
1	$\{1, y, z, w, yz, yw, zw, yzw\}$	(x, y^2, z^2, w^2)
2	$\{1, x, y, z, w, yz, yw, zw\}$	$(xy, xz, xw, x^2, y^2, z^2, w^2, yzw)$
3	$\{1, x, y, z, w, xy, xz, xw\}$	$(yz, yw, zw, x^2, y^2, z^2, w^2)$

6.2. **The case of no crepant resolutions.** There is an example of a smooth $\mathrm{Hilb}^G(\mathbf{A}^4)$ which is not a crepant resolution of $\mathbf{A}^4/G$. For instance let G be the subgroup of $\mathrm{SL}(4,k)$ of order two generated by minus the identity. We easily see that $\mathrm{Hilb}^G(\mathbf{A}^4)$ is smooth, while it is well known that $\mathbf{A}^4/G$ has no crepant resolutions because it is terminal.

6.3. **Abelian subgroups in GL.** There is also an example of singular Hilb^G for a finite abelian subgroup G in $\mathrm{GL}(3,k)$ by Reid. Very recently R. Kidoh [Kidoh98] gave a complete description of $\mathrm{Hilb}^G(\mathbf{A}^2)$ for an abelian finite subgroup G of $\mathrm{GL}(2,k)$ by using two kinds of continued fractions.

6.4. **Quot-schemes.** There are infinitely many algebro-geometric relatives of Hilbert schemes such as Grothedieck's Quot-schemes of coherent sheaves.

Problem 6.5. Suppose $G \subset \mathrm{SL}(3,k)$. Then for which G-invariant coherent sheaf F on $\mathbf{A}^3$ is $\mathrm{Quot}^G(F)$ a projective crepant resolution of $\mathbf{A}^3/G$? Is any projective crepant resolution of $\mathbf{A}^3/G$ isomorphic to $\mathrm{Quot}^G(F)$ for some G-invariant coherent sheaf F on $\mathbf{A}^3$?

A similar problem was communicated to us by Nakajima in the quiver variety formulation.

References

[Kidoh98] R. Kidoh, *Hilbert schemes and cyclic quotient surface singularities*, Preprint (1998).

[GSV83] G. Gonzalez-Sprinberg, J. Verdier, *Construction géométrique de la correspondence de McKay*, Ann. scient. Éc. Norm. Sup. **16** (1983) 409–449.

[INkjm98] Y. Ito, H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, Preprint (1998), to appear in Topology.

[IN96] Y. Ito, I. Nakamura, *McKay correspondence and Hilbert schemes*, Proc. Japan Acad. **72** (1996), 135–138.

[IN98] Y. Ito, I. Nakamura, *Hilbert schemes and simple singularities*, New trends in algebraic geometry (Proc. of the July 1996 Warwick European alg. geom. conf.), Cambridge university press (1999) 151-233.

[IR94] Y. Ito, M. Reid, *The McKay correspondence for finite subgroups of* $\mathrm{SL}(3,\mathbf{C})$, in Higher Dimensional Complex Varieties (Trento, June 1994), M. Andreatta and others Eds., de Gruyter, 1996, 221–240.

[McKay80] J. McKay, *Graphs, singularities, and finite group*, Proc. Symp. Pure Math., AMS **37** (1980), 183–186.

[Roan94] S-S.Roan, On $c_1 = 0$ resolution of quotient singularities, Intern. J. Math. **5** (1994), 523–536.

[Reid97] M. Reid, *McKay correspondence*, Preprint (1997).

FIGURE 1. FAN($G_{1,2,3}$)

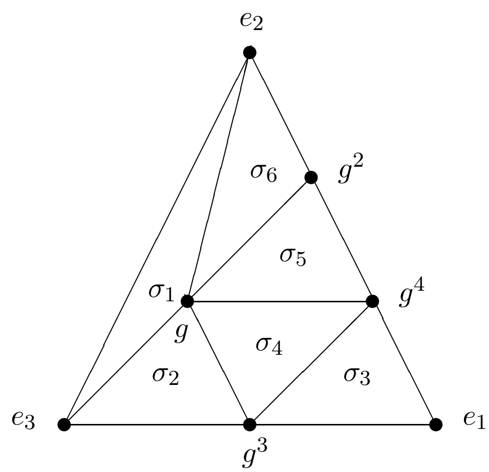


FIGURE 2. JIGSAW PUZZLE OF $G_{1,2,3}$-GRAPHS

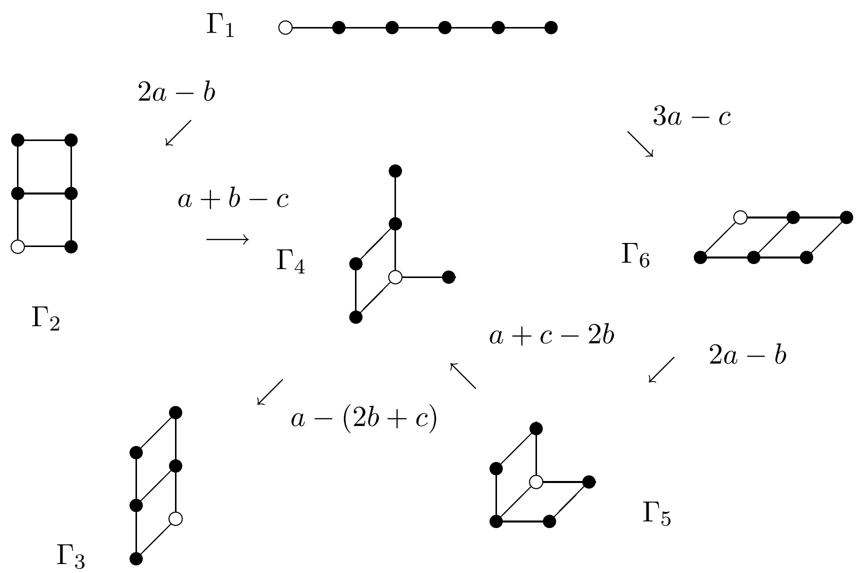
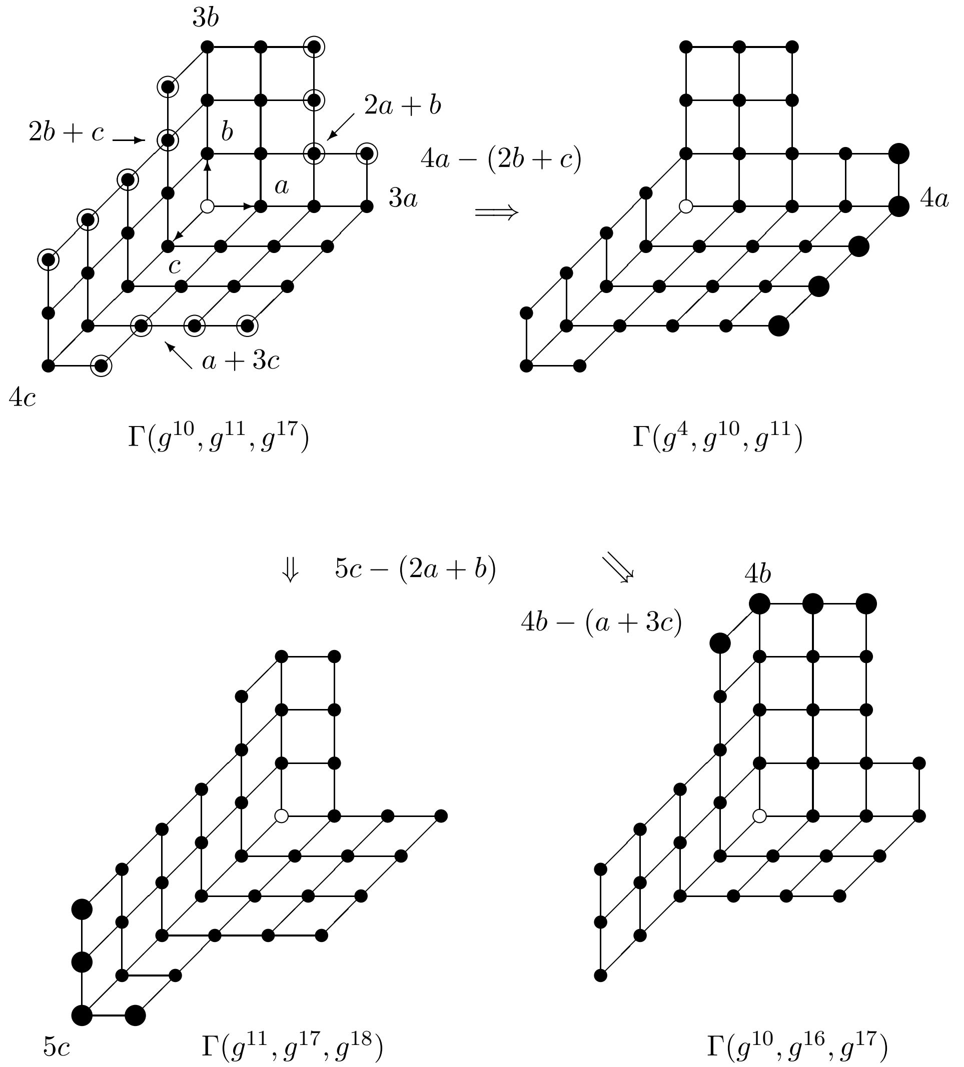


Figure 3. JIGSAW PUZZLE OF $G_{1,5,31}$-GRAPHS



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