HILBERT SCHEMES OF G-ORBITS IN DIMENSION THREE

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ABSTRACT. We study the precise structure of Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ of G-orbits in the space $\mathbf{C}^3$ when the group G is a simple subgroup of $\mathrm{SL}(3,\mathbf{C})$ of either 60 or 168. These are the only possible non-abelian simple subgroups of $\mathrm{SL}(3,\mathbf{C})$.

0. INTRODUCTION

For a given finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$ a somewhat complicated scheme, the Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ of G-orbits has been introduced for the purpose of resolving singularities of the quotient space $\mathbf{C}^3/G$ as well as generalizing McKay correspondence in dimension two [McKay80]. It is defined to be the subscheme of $\mathrm{Hilb}^{|G|}(\mathbf{C}^3)$ parametrizing all the zero dimensional G-invariant subschemes with their structure sheaf isomorphic to the regular representation of the group G as G-modules. This scheme is now known by [BKR99] to be smooth and irreducible and it is a crepant resolution of the quotient space $\mathbf{C}^3/G$. See [Nakamura98] for smoothness and an algorithm of computation in the abelian case. See also [INakajima98].

On the other hand we know all the possibilities of finite subgroups of $\mathrm{SL}(3,\mathbf{C})$ by [Blichfeldt17], [BDM16] and [YY93]. There are exactly 4 infinite series labeled by A(belian), B, C, D, and 8 exceptional cases labeled by E though L. Among them there are only two non-abelian simple subgroups, which are of order either 60 or 168. We denote these subgroups of $\mathrm{SL}(3,\mathbf{C})$ simply by G_{60} and G_{168}. In this article we study $\mathrm{Hilb}^G(\mathbf{C}^3)$ when G is either G_{60} or G_{168}. The structure of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is more or less easily understood over the quotient space $(\mathbf{C}^3/G)\setminus\{0\}$. Since smoothness of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is already known by [BKR99], the remaining problem is to describe the fibre of $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin precisely. However the computations we actually need in the cases $G = G_{60}$ and G_{168} are rather enormous. Therefore we wish to state the result precisely, indicating our ideas of the computations only.

Date: 30 November 1999.

Key words and phrases. Hilbert scheme, G-orbits, Invariant theory, Coinvariant algebra.

Research of the second and third author was supported in part by the Grants-in-aid (No. 08404001, No. 09640063) for Scientific Research, JSPS.

We instead explain in detail the simplest non-abelian case - a trihedral group G_{12} of order 12. In this case we describe the precise structure of $\mathrm{Hilb}^G(\mathbf{C}^3)$. This gives a negative answer to a question posed by [INakamura99, p.227]. In fact, G_{12} has a normal subgroup N of order 4 by definition. The composite Hilbert scheme $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is a crepant resolution of $\mathbf{C}^3/G$ as well as $\mathrm{Hilb}^G(\mathbf{C}^3)$. Both of the fibres over the origin of $\mathbf{C}^3/G$ consist of 3 rational curves, but they have different configurations.

We note that crepant resolutions of $\mathbf{C}^3/G_{60}$ and $\mathbf{C}^3/G_{168}$ have been constructed by [M97] and [Roan96] by using the equations defining the quotient given by [YY93]. It remains to compare the structures of their resolutions and ours.

The authors would like to thank Professor Li Chiang for pointing out an error in the computation of $f_3^3 + \bar{f}_3^3$ in Section 2.

1. Duality of the coinvariant algebra

The result of this section is more or less familiar to the specialists.

1.1. **Complex reflection groups.** Let V be a finite dimensional complex vector space, $V^\vee$ the dual of V and M a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections of V. Here we mean by a complex reflection an automorphism of V of finite order which has exactly one eigenvalue distinct from 1.

Let $S := S(V^\vee)$ be the symmetric algebra of $V^\vee$ over $\mathbf{C}$, $S = \oplus_{k=0}^\infty S_k$ be the homogeneous decomposition of S, $S_+ := \oplus_{k=1}^\infty S_k$. Since M acts on V, hence on $V^\vee$ contravariantly, it naturally acts on S by $g(F)(v) = F(g^{-1}v)$ for $F \in S$ and $v \in V$. Let S^M be the subalgebra of M-invariants of S, $S_+^M := S^M \cap S_+$, $\mathfrak{n}_M := SS_+^M$ and $S_M := S/\mathfrak{n}_M$. We call S_M the coinvariant algebra of M.

Let S^* be the symmetric algebra of V, and we identify it with the algebra of polynomial functions on $V^\vee$. Following [Steinberg64] we define an algebra homomorphism D of S into the endomorphism ring $\mathrm{End}\,(S^*)$ of S^* as follows. Let $v, w \in V^\vee$ and $F \in S^*$. Then we first define

$$(D_vF)(w) = \lim_{t\to 0}\,(F(w+tv) - F(w))/t$$

and extend it as an algebra homomorphism of S into $\mathrm{End}\,(S^*)$, in other words, we extend it to S by the conditions $D_{st} := D_sD_t$, $D_{s+t} = D_s + D_t$ for any $s, t \in S$.

We also define the G-invariant inner product

$$\alpha(s, F) := (D_sF)(0) \qquad (1)$$

for $s \in S$, $F \in S^*$. In fact, since $(D_{\sigma s}\sigma F) = \sigma(D_sF)$ for $\sigma \in M$, we have $\alpha(\sigma s, \sigma F) = \sigma(D_sF)(0) = (D_sF)(0) = \alpha(s, F)$. This inner product extends the inner product between V and $V^\vee$. By the inner product we identify V with $V^\vee$, and S^* with S as well as the subalgebra $(S^*)^M$ of M-invariants in S^* with S^M, for instance in Theorem 1.2 (2) where we apply [Steinberg64, Lemma 3.1]. We use freely the identification in Theorem 1.2, in particular the notation (s, F) and D_sF make sense for $s, F \in S$.

We recall a basic fact from [Bourbaki, Chapitre 5]

Theorem 1.2. *Let V be an n-dimensional vector space and M a finite subgroup of $\mathrm{GL}(V)$ generated by complex reflections of V. Then*

1. *S^M is isomorphic to a polynomial ring of n variables, in other words, there are homogeneous polynomials $P_1, \cdots, P_n \in S$ such that*
$$S^M = \mathbf{C}[P_1, \cdots, P_n],$$
2. *There exists $P \in S$, unique up to constant multiples, such that*
   (2a) *P is skew, that is, $g(P) = \det(g)P$ for any $g \in M$,*
   (2b) *if $P' \in S$ is skew, then P divides P'.*

We note that P is given in two different ways. Let P_i be the set of homogeneous generators of S and x_i a basis of $V^\vee$. First P is given as $P = \mathrm{Jac}(P_1, \cdots, P_n) := \det(\partial P_i/\partial x_j)$.

Let Σ be the set of all reflections in G. For any $g \in \Sigma$ there is an element $e_g \in V^\vee$ unique up to constant multiples such that $g(x) = x + f_g(x)e_g$ $(\forall x \in V^\vee)$ for some $f_g \in V$. Then $e_g = 0$ is a *linear* equation defining the reflection hyperplane in V of g. Then $P = c\prod_{g\in\Sigma} e_g \in S$ for some nonzero constant c. This is the same as [Steinberg64, Theorem 1.4 (a)] though the notation looks slightly different.

The basic degrees (or characteristic degrees) of M are by definition the set of integers $d_i := \deg P_i$ $(1 \le i \le n)$, which is known to be independent of the choice of the generators P_i. It is easy to see $\deg P = \sum_{i=1}^n (d_i - 1) = |\Sigma|$.

The following follows from [Steinberg64] by identifying S with S^*.

Theorem 1.3. *Let $m = \deg P$. Then*

1. *Let $U = \{D_sP; s \in S\}$. Then U is the orthogonal complement in S of $\mathfrak{n}_M$ with respect to α. It is a G-stable finite dimensional subspace of S such that $U \otimes_{\mathbf{C}} S^M \simeq S$ and $S_M \simeq U \simeq \mathbf{C}[M]$ as M-modules,*
2. *$s \in \mathfrak{n}_M$ if and only if $D_sP = 0$,*
3. *$S_k \subset \mathfrak{n}_M$ for $k > \deg P$.*

Proof. We note that (1) follows from [Steinberg64, Theorem 1.2 (c)], while (2) follows from [ibid.,Theorem 1.3 (b)]. (3) follows from (2). □

In particular by Theorem 1.3 (1) the maximum degree of elements in U is attained by $\deg(D_sP) = m$ with $s = 1$.

We define a bilinear form $\beta : S_M \times S_M \to \mathbf{C}$ by
$$\beta(f, g) := (D_{fg}P)(0). \tag{2}$$

Theorem 1.4. *Let S_M be the coinvariant algebra of M, $U_k := U \cap S_k$ and $(S_M)_k :=$ the image of U_k in S_M for $k \le m$. Then $\beta : S_M \times S_M \to \mathbf{C}$ is a nondegenerate bilinear form such that*

1. *$\beta(f, gh) = \beta(fg, h)$,*
2. *$\beta(\sigma f, \sigma g) = \det(\sigma)^{-1}\beta(f, g)$,*

3. $(S_M)_k$ *and* $(S_M)_{m-k}$ *are dual to each other with respect to* β.

Proof. Nondegeneracy of β follows from Theorem 1.3 (2). (1) is clear from the definition of β. Next we prove (2). In fact, we see

$$\begin{aligned}\beta(\sigma f, \sigma g) &= D_{(\sigma f \sigma g)}P(0)\\ &= \sigma(D_{(fg)}(\sigma^{-1}P))(0)\\ &= \det(\sigma)^{-1}(D_{(fg)}P)(0)\\ &= \det(\sigma)^{-1}\beta(f,g)\end{aligned}$$

because $\sigma(D_sF) = D_{\sigma s}(\sigma F)$ for any $s, F \in S$. See [Steinberg64, p. 392]. □

1.5. **Subgroups of complex reflection groups of index two.** Let V be a finite dimensional complex vector space and G a finite subgroup of $\mathrm{SL}(V)$. Suppose that there is a finite subgroup M of $\mathrm{GL}(V)$ generated by complex reflections of V such that $G = M \cap \mathrm{SL}(V)$ and $[M : G] = 2$. For instance any finite subgroup of $\mathrm{SL}(2, \mathbf{C})$ and the subgroups G_{12}, G_{60} and G_{168} of $\mathrm{SL}(3, \mathbf{C})$ satisfy the conditions as we see later. We will see that the facts observed in [INakamura99, Section 11] are easily derived from [Steinberg64] in a more general situation, though these have been observed already in [GSV83] and [Knörrer85].

Let $S := S(V^\vee)$ be the symmetric algebra of $V^\vee$ over $\mathbf{C}$, $S = \oplus_{k=0}^{\infty} S_k$ be the homogeneous decomposition of S, $S_+ := \oplus_{k=1}^{\infty} S_k$. Let S^G be the subalgebra of G-invariants of S, $S^G_+ := S^G \cap S_+$, $\mathfrak{n}_G := SS^G_+$ and $S_G := S/\mathfrak{n}_G$.

In the rest of this section we compare S^G and S_G with S^M and S_M. First we note $S^M \subset S^G$ and $\mathfrak{n}_M \subset \mathfrak{n}_G$.

Theorem 1.6. *Let* U *be the same as in Theorem 1.4,* $U_k = U \cap S_k$ *and* $(S_G)_k$ *the image of* U_k *in* S_G. *Then the following is true.*

1. $S^G = S^M[P]$, $\mathfrak{n}_G = \mathfrak{n}_M + \mathbf{C}P$.
2. $S_G \simeq \oplus_{k=1}^{m-1} U_k \simeq \mathbf{C}[G] + \mathbf{C}[G]/\mathbf{C}$, $U \cap S^G = \mathbf{C} + \mathbf{C}P$,
3. $(S_G)_k$ *and* $(S_G)_{m-k}$ *are dual to each other if* $1 \le k \le m-1$.

Proof. Since det is a nontrivial character of M/G by our definition of G, we see $1_G^M = 1_M + \det$, and $\mathbf{C}[M]_G = \mathbf{C}[G] + \det_G \otimes \mathbf{C}[G] = \mathbf{C}[G] + \mathbf{C}[G]$. It follows from Theorem 1.2 that $U \cap S^G = \mathbf{C} + \mathbf{C}P$. Therefore we infer the rest of the assertions from Theorem 1.3 and Theorem 1.4. □

We can make Theorem 1.6 (3) more precise as follows.

Theorem 1.7. *Let the notation be the same as in Theorem 1.6. Let* ρ *(resp.* ρ'*) be (an equivalence class of) an irreducible representation of* G *and* $\bar\rho$ *(resp.* $\bar\rho'$*) the complex conjugate of* ρ. *Let* $(S_G)_k[\rho]$ *be the sum of all* G*-submodules of* $(S_G)_k$ *isomorphic to* ρ. *Then*

1. $(S_G)_k[\rho]$ *and* $(S_G)_{m-k}[\bar\rho]$ *are dual with respect to* β,
2. *there is a* G*-submodule of* S_G *isomorphic to* ρ' *in* $S_1((S_G)_k[\rho])$ *if and only if there is a* G*-submodule of* S_G *isomorphic to* $\bar\rho$ *in* $S_1((S_G)_{m-k-1}[\bar\rho'])$.

Proof. Let W be an irreducible G-submodule of $(S_G)_k[\rho]$ isomorphic to ρ. Let W^c be a G-submodule in U_k complementary to W and W^* the orthogonal complement in U_{m-k} to W^c with respect to β. By Theorem 1.6 (3) W^* is dual to W with respect to β. It is clear that $\sigma(W^*) \subset W^*$ for any $\sigma \in G$. For $f \in W$ and $g \in W^*$, we have $\beta(\sigma f, g) = \beta(f, \sigma^{-1} g)$ by Theorem 1.4 (2), whence by $G \subset \mathrm{SL}(V)$

$$\chi_\rho(\sigma) = \mathrm{Tr}\,((\sigma)_W) = \mathrm{Tr}\,((\sigma^{-1})_{W^*}).$$

Hence $\mathrm{Tr}\,((\sigma)_{W^*}) = \overline{\chi_\rho(\sigma)} = \chi_{\bar\rho}(\sigma)$. It follows $W^* \simeq \bar\rho$. This proves (1).

Let W be a G-submodule of $(S_G)_k[\rho]$ and W' a G-submodule of $(S_1 W)[\rho']$ such that $W \simeq \rho$, $W' \simeq \rho'$. Let W^c (resp. $(W')^c$) be a G-submodule of $(S_G)_k$ (resp. $(S_G)_{k+1}$) complementary to W (resp. W'). Let W^* and $(W')^*$ be the orthogonal complement in U_{m-k} and U_{m-k-1} to W^c and $(W')^c$ respectively. Hence by (1) $W^* \simeq \bar\rho$ and $(W')^* \simeq \bar\rho'$. By assumption and by Theorem 1.6 (3) there exist $x \in S_1$, $f \in W$ and $g \in (W')^*$ such that

$$\beta(f, xg) = \beta(xf, g) \neq 0.$$

This implies that $S_1(W')^*$ contains a G-submodule dual to W with respect to β, hence isomorphic to $\bar\rho$ by (1). This completes the proof. □

1.8. **Finite subgroups of SL(2,C).** Let V be a vector space of two dimension. It is well known that for any any finite subgroup of $\mathrm{SL}(V)$ there is a finite complex reflection group M of $\mathrm{GL}(V)$ such that $G = M \cap \mathrm{SL}(V)$ and $[M : G] = 2$. In fact, for G a cyclic group of order n, M is a dihedral group $I(n)$ of order $2n$. For a binary dihedral group G, M is a subgroup of $GL(V)$ generated by G and a permutation matrix $(1, 2)$. For G a binary tetra-, octa-, or icosa-hedral group respectively, M is a complex reflection group with Shephard-Todd number 12, 13 and 22 respectively [ST54, p. 301]. In particular, $M = \mu_4 \cdot G$ for G the binary octa-, or icosa-hedral group where μ_4 is the subgroup of scalar matrices of fourth roots of unity.

2. G_{12}

The purpose of this section is to provide an example which solves negatively the question in [INakamura99, Section 17].

2.1. **The action of G on $V^\vee$.** This Subsection is included just to explain our convention and notation in the subsequent sections.

Let $V = \mathbf{C}^3$ and $V^\vee$ the dual of V. We choose and fix a basis e_i of V once for all. The space $V^\vee$ is spanned by the dual basis $x_1 = x$, $x_2 = y$ and $x_3 = z$ with $x_i(e_j) = \delta_{ij}$. The matrix form of $g \in \mathrm{SL}(V)$ in Subsection 2.2 etc. is that of g with respect to e_i. Hereafter we call this matrix representation ρ and hence $\rho(g) = g$. Then as in Section 1, G acts on $V^\vee$ by the contragredient representation $\rho^\vee$ of ρ and we have $\rho^\vee(g)(v^\vee)(p) = v^\vee(\rho(g^{-1})p)$ for $p \in V$, $v^\vee \in V^\vee$ and $g \in G$. In terms of pull back by the automorphism

$\rho(g^{-1})$ of V, this means that $\rho^\vee(g)(v^\vee) = \rho(g^{-1})^*(v^\vee)$, where $V^\vee$ is regarded as the space of linear functions on V. In particular we have

$$(\rho^\vee(g)(x), \rho^\vee(g)(y), \rho^\vee(g)(z)) = (x, y, z)^t g^{-1}.$$

This is equivalent to $(g^*(x), g^*(y), g^*(z)) = (x, y, z)^t g$ where $g^* = \rho(g)^*$ is the pull back of functions on V by the automorphism g of V.

The action of G on $V^\vee$ via $\rho^\vee$ can be extended to S and hence to S_G, since $\mathfrak{n}$ is G-invariant. We denote this representation of G on S_G by $S_G(\rho^\vee)$. We note also that the action of G on S is the same as the one given in [YY93, p. 38], so we can apply their results for G_{60} and G_{168}.

We use the same notation as above from now on.

2.2. **A trihedral group G_{12}.** Let N be an order 4 abelian subgroup of $\mathrm{SL}(V)$ consisting of diagonal matrices with diagonal coefficients ± 1 and $\tau := (\delta_{i,j+1})$. To be more precise

$$N = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \in \mathrm{SL}(V) \right\}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $G = G_{12}$ be the subgroup of $\mathrm{SL}(V)$ of order 12 generated by N and τ. The group G is called *a ternary trihedral group*. It is clear that there is an exact sequence $1 \to N \to G \to \mathbf{Z}/3\mathbf{Z} \to 0$. We note that the group G is a subgroup of index two of a complex reflection group M. In fact, we choose as $\widetilde{N}$ a diagonal subgroup of $\mathrm{GL}(V)$ with diagonal coefficients ± 1 and M the extension of $\widetilde{N}$ by $\{1, \tau, \tau^2\}$. Thus Section 1 can be applied for G and M. The basic degrees of M are equal to $2, 3$ and 4 so that $\deg P = m = (2-1) + (3-1) + (4-1) = 6$ as we have seen in Section 1.

2.3. **Characters of G.** Table 1 is the character table of G, where the first (resp. the second, the third) line gives a representative of (resp. the age of, the number of elements in) each conjugacy class and $\sigma_{12} = \mathrm{diag}(-1, -1, 1)$. For $G = G_{12}$, we have $\rho = \rho^\vee$ and the character of ρ is given by χ_3. For each irreducible character χ_k in Table 1, we choose and fix an irreducible representation ρ_k which affords χ_k.

c.c	1	σ_{12}	τ	τ^2
age	0	1	1	1
$\sharp$	1	3	4	4
χ_{1_1}	1	1	1	1
χ_{1_2}	1	1	ω	ω^2
χ_{1_3}	1	1	ω^2	ω
χ_3	3	-1	0	0

Table 1. Character table of G_{12}

2.4. **The coinvariant algebra of G_{12}.** The ring of all G-invariant polynomials is generated by the following four homogeneous polynomials

$$\begin{aligned} &f_2 = x^2 + y^2 + z^2, \quad f_3 = xyz, \\ &f_4 = x^2y^2 + y^2z^2 + z^2x^2, \\ &f_6 = (x^2 - y^2)(y^2 - z^2)(z^2 - x^2) \end{aligned} \tag{3}$$

where

$$f_6^2 + 4f_4^3 - f_2^2f_4^2 - 18f_2f_4f_3^2 + 4f_2^3f_3^2 + 27f_3^4 = 0. \tag{4}$$

We note that the ring of M-invariant polynomials is generated by f_2, f_3, f_4 and $P = f_6 = (1/4)\,\mathrm{Jac}(f_2, f_3, f_4)$. The variety defined by the equation (4) is an irreducible singular variety $\mathbf{C}^3/G$ with non-isolated singularities.

Let $\mathfrak{n}$ be the ideal generated by these four polynomials and $S_G = S/\mathfrak{n}$ the coinvariant algebra of G. Then S_G is decomposed into irreducible components as in Table 2. We denote the homogeneous component of S_G of degree d by $\bar{S}_d$. We denote the ρ_k-component of $\bar{S}_d$ by $\bar{S}_d[\rho_k]$, which is in view of Table 2 irreducible except for $(d, k) = (3, 3)$ if it is nonzero. We identify S_1 and $\bar{S}_1 = \bar{S}_1[\rho_3]$.

For the notation we define

$$f = x^2 + \omega y^2 + \omega^2 z^2, \quad \bar{f} = x^2 + \omega^2 y^2 + \omega z^2. \tag{5}$$

We remark that

$$\begin{aligned} f\bar{f} &= f_2^2 - 3f_4, \\ f^3 - \bar{f}^3 &= \prod_{i=0}^{2}(f - \omega^i \bar{f}) = 3(\omega^2 - \omega)f_6, \\ f^3 + \bar{f}^3 &= \prod_{i=0}^{2}(f + \omega^i \bar{f}) = 27f_3^2 - 9f_2f_4 + 2f_2^3. \end{aligned}$$

$\bar{S}_d$	1_2	1_3	3	$\dim \bar{S}_d$	$\bar{S}_d[\rho]$
$\bar{S}_1$	0	0	1	3	$\{x, y, z\}$
$\bar{S}_2$	1	1	1	5	$\{f\} + \{\bar{f}\} + \{yz, zx, xy\}$
$\bar{S}_3$	0	0	2	6	$\{xf, \omega^2 yf, \omega zf\} + \{x\bar{f}, \omega y\bar{f}, \omega^2 z\bar{f}\}$
$\bar{S}_4$	1	1	1	5	$\{\bar{f}^2\} + \{f^2\} + \{yzf, \omega^2 zxf, \omega xyf\}$
$\bar{S}_5$	0	0	1	3	$\{x\bar{f}^2, \omega^2 y\bar{f}^2, \omega z\bar{f}^2\}$

TABLE 2. The coinvariant algebra of G_{12}

2.5. **The exceptional locus.** We have a natural morphism from $\mathrm{Hilb}^G(\mathbf{C}^3)$ onto the quotient $\mathbf{C}^3/G$. It is called the Hilbert-Chow morphism π, which is an isomorphism over $(\mathbf{C}^3 \setminus \{\text{Fixed points of } G\})/G$. Now we study the structure of the fibre of π over the origin. We define

$$\begin{aligned} I([a:b]_{1_2}) :&= S\cdot(af+b\bar{f}^2)+\mathfrak{n} \quad (a\neq 0),\\ I([a:b]_{1_3}) :&= S\cdot(a\bar{f}+bf^2)+\mathfrak{n} \quad (a\neq 0),\\ I([a:b]_3) :&= S[G](ax\bar{f}+bxf)+Sf^2+S\bar{f}^2+\mathfrak{n},\\ J :&= S\cdot(xf,yf,zf)+S\cdot\bar{f}^2+\mathfrak{n}=I([0:1]_3),\\ J' :&= S\cdot(x\bar{f},y\bar{f},z\bar{f})+S\cdot f^2+\mathfrak{n}=I([1:0]_3). \end{aligned} \tag{6}$$

Theorem 2.6. *Let $G=G_{12}$. Then the fibre of the Hilbert Chow morphism π over the origin is one of the following*

$$I([a:b]_{1_2})\ (a\neq 0),\ I([a:b]_{1_3})\ (a\neq 0),\ I([a:b]_3).$$

Proof. Let $S=\mathbf{C}[x,y,z]$ and $\mathfrak{m}$ the maximal ideal of S defining the origin. Let I be any ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin. Then $\mathfrak{n}\subset I\subset\mathfrak{m}$ because $S/I\simeq\mathbf{C}[G]$. Let $\bar{I}:=I/\mathfrak{n}$. Since $S/\mathfrak{n}+S(f+b\bar{f}^2)\simeq\mathbf{C}[G]$ by Table 2, we have $I=\mathfrak{n}+S(f+b\bar{f}^2)$ if $f+b\bar{f}^2\in I$. Similarly if $\bar{f}+bf^2\in I$, then $I=\mathfrak{n}+S(\bar{f}+bf^2)$. Suppose $f+b\bar{f}^2\notin I$ and $\bar{f}+bf^2\notin I$ for any $b\in\mathbf{C}$. Then $f^2\in I$ and $\bar{f}^2\in I$, $\bar{S}_5[\rho_3]\subset\bar{I}$. If $\bar{S}_2[\rho_3]\subset\bar{I}$, then $\bar{S}_4[\rho_3]\subset I$ and $\bar{S}_3[\rho_3]\cap I\neq\{0\}$, which is absurd. Hence $\bar{S}_2[\rho_3]\not\subset\bar{I}$. Similarly we see that if $I\not\subset\mathfrak{m}^3+\mathfrak{n}$, then we have a contradiction. Hence $I\subset\mathfrak{m}^3+\mathfrak{n}$ and $\bar{I}\cap\bar{S}_3[\rho_3]\neq\{0\}$. It follows that $I=I([a:b]_3)$ for some $[a:b]\in\mathbf{P}^1$. This completes the proof. □

Corollary 2.7. *The fibre $\pi^{-1}(0)$ is a chain of three smooth rational curves intersecting transversally.*

Proof. We see that in $\mathrm{Hilb}^G(\mathbf{C}^3)$

$$\lim_{a\to 0} I([a:b]_{1_2})=J,\quad \lim_{a\to 0} I([a:b]_{1_3})=J'.$$

Let $C_{1_2}:=\{I([a:b]_{1_2}),J;a\neq 0\}\simeq\mathbf{P}^1$, $C_{1_3}:=\{I([a:b]_{1_3}),J';a\neq 0\}\simeq\mathbf{P}^1$ and $C_3:=\{I([a:b]_3);[a:b]\in\mathbf{P}^1\}\simeq\mathbf{P}^1$. Then $C_{1_2}\cap C_{1_3}=\emptyset$, $C_{1_2}\cap C_3=\{J\}$ and $C_{1_3}\cap C_3=\{J'\}$. The intersection of C_k is transversal. In fact, the tangent space of $\mathrm{Hilb}^G(\mathbf{C}^3)$ is the direct sum of $\mathrm{Hom}(I[\rho_i],S/I[\rho_i])$ for ρ_i distinct, hence for instance at J, C_{1_2} and C_3 are transversal because so are $\mathrm{Hom}(I[\rho_{1_2}],S/I[\rho_{1_2}])$ ($\simeq\mathbf{C}$) and $\mathrm{Hom}(I[\rho_3],S/I[\rho_3])$ ($\simeq\mathbf{C}$). This completes the proof. See also the proof of Corollary 4.6. □

2.8. $\mathbf{Hilb}^{G/N}(\mathbf{Hilb}^N(\mathbf{C}^3))$**.** By [Nakamura98] $\mathrm{Hilb}^N(\mathbf{C}^3)$ is a crepant resolution of $\mathbf{C}^3/N$, on which G/N acts naturally. Hence $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is a crepant resolution of $\mathrm{Hilb}^N(\mathbf{C}^3)/(G/N)$, hence it is a crepant resolution of $\mathbf{C}^3/G$. Let $\phi:\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))\to\mathbf{C}^3/G$ be the natural morphism, and $\psi:\mathrm{Hilb}^N(\mathbf{C}^3)\to\mathbf{C}^3/N$ the Hilbert-Chow morphism.

By [Nakamura98] the fan which describes the torus embedding $\mathrm{Hilb}^N(\mathbf{C}^3)$ is given by a decomposition of the triangle $\langle e_1, e_2, e_3\rangle$ in $\mathbf{R}^3$ by junior elements $\sigma_{12}, \sigma_{23}, \sigma_{31}$ of N:

$$\begin{aligned}\Delta_0 &= \langle \sigma_{12}, \sigma_{23}, \sigma_{31}\rangle,\\ \Delta_1 &= \langle \sigma_{23}, e_2, e_3\rangle,\\ \Delta_2 &= \langle \sigma_{31}, e_3, e_1\rangle,\\ \Delta_3 &= \langle \sigma_{12}, e_1, e_2\rangle.\end{aligned} \tag{7}$$

Let x, y, z be the standard coordinate of $\mathbf{C}^3$. The one-dimensional strata $\langle \sigma_{12}, \sigma_{23}\rangle$, $\langle \sigma_{23}, \sigma_{31}\rangle$, $\langle \sigma_{31}, \sigma_{12}\rangle$ corresponds to torus orbit rational curves C_1, C_2, C_3. The fibre of ψ over the origin is the union of C_i. The curves C_i meet at a unique point where they intersect as three axes in the affine space $\mathbf{C}^3$. We also note that the fixed locus of the action of N consists of three coordinate axes $\ell_x : x = 0$, $\ell_y : y = 0$ and $\ell_z : z = 0$ where the action of N reduces to a cyclic group of order two, say A_1. This implies that the structure of $\mathrm{Hilb}^N(\mathbf{C}^3)$ over the coordinate axis is a $\mathbf{P}^1$-bundle. Let $D_i := \psi^{-1}(\ell_i)$ $(i = x, y, z)$. The group $G/N \simeq \mathbf{Z}/3\mathbf{Z}$ permutes D_x, D_y and D_z cyclically.

We write the chart of $\mathrm{Hilb}^N(\mathbf{C}^3)$ corresponding to Δ_0 as

$$U_0 := \mathrm{Spec}\ \mathbf{C}[p, q, r]$$

where $p = yz/x, q = zx/y, r = xy/z$. The action of τ turns out to be $\tau^*(p) = q, \tau^*(q) = r, \tau^*(r) = p$. It follows that the fixed point locus of the (induced) action of τ on $\mathrm{Hilb}^N(\mathbf{C}^3)$ is $\ell : p = q = r$, along which the action of τ is A_2. Therefore the structure of $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ over ℓ is a union of two $\mathbf{P}^1$-bundles E_1, E_2 meeting transversally along a section over ℓ. In particular the fibre of $(p, q, r) = (0, 0, 0)$ is the union of two rational curves m_1, m_2. G/N permutes the curves $n_i := D_i \cap \psi^{-1}(0)$ $(i = x, y, z)$, whence it yields a unique rational curve n on $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$. It follows that the fibre of ϕ over the origin is the union of three rational curves m_1, m_2 and n. Taking it into account that the geometry about m_i and n_j is G/N-symmetric, the three rational curves meet at a unique point. By a calculation we see that they meet as three coordinate axes of $\mathbf{C}^3$ at the intersection. We also see that the normal bundle of n in $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ is $O_n(-1)^{\oplus 2}$. We also see that the exceptional divisors of ϕ are E_1,E_2 and D where D is the image of D_i mod G/N. The divisor D is a $\mathbf{P}^1$-bundle over $\ell \setminus \{0\}$ with $D_0 = \phi^{-1}(0) \cap D = \phi^{-1}(0)$. $\mathrm{Hilb}^G(\mathbf{C}^3)$ is obtained from $\mathrm{Hilb}^{G/N}(\mathrm{Hilb}^N(\mathbf{C}^3))$ by a flop with center n.

3. G_{60}

3.1. **Characters of G_{60}.** Let $G = G_{60}$. The group G is isomorphic to the alternating group of degree 5 and is the normal subgroup of index 2 of the Coxeter group H_3. We note $H_3 = G \times \{\pm 1\}$.

Let $V = \mathbf{C}^3$, and $V^\vee$ the dual of V. The space $V^\vee$ is spanned by the dual basis x, y and z as before. By [YY93, p.72] G is realized as a subgroup of $\mathrm{SL}(V)$ generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^4 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}, \ \upsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \ \tau = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}$$

where $\epsilon = \exp(2\pi i/5)$, $s = \epsilon^2 + \epsilon^3$ and $t = \epsilon + \epsilon^4$.

c. c.	1	$\sigma\tau$	υ	σ	σ^2
age	0	1	1	1	1
$\sharp$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_{3_1}	3	0	-1	$-s$	$-t$
χ_{3_2}	3	0	-1	$-t$	$-s$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

TABLE 3. Character table of G_{60}

Table 3 is the character table of G. The first (resp. the second, the third) line of Table 3 gives a representative of (resp. the age of, the number of elements in) each conjugacy class. For each irreducible character χ_k in Table 3, we choose and fix an irreducible representation ρ_k which affords χ_k. We note that $\bar{\rho}_k \simeq \rho_k$ for any k and $\rho = \rho_{3_1} \simeq \rho^\vee$. We also note that $\rho_{3_1}(g)$ has an eigenvalue one for g belonging to any junior conjugacy class, whence by [IR96], there is no compact irreducible divisor in the fibre of the Hilbert-Chow morphism over the origin. See Corollary 3.6.

3.2. **The coinvariant algebra of G_{60}.** The ring of invariant polynomials of G is generated by the four polynomials h_i $(i = 1, 2, 3, 4)$ of degrees 2, 6, 10 and 15 respectively ([YY93, pp. 72-74])

$$\begin{gathered} h_1 = x^2 + yz, \ h_2 = 8x^4yz - 2x^2y^2z^2 - x(y^5 + z^5) + y^3z^3, \\ h_3 = (1/25)(-256h_1^5 + \mathrm{BH}(h_2, h_1) + 480h_1^2h_2), \\ h_4 = (1/10)\,\mathrm{Jac}(h_2, h_1, h_3), \end{gathered} \tag{8}$$

where $\mathrm{BH}(h_2, h_1)$ is the bordered Hessian of h_2, and h_1. See [YY93, p. 71].

Let $\mathfrak{n}$ be the ideal generated by these four polynomials and $S_G = S/\mathfrak{n}$ the coinvariant algebra of G. We can apply Section 1 to H_3 and G. Let $\bar{S}_d$ be the homogeneous component of S_G of degree d. Then $\bar{S}_d$ is decomposed into irreducible components as in Table 4.

We will denote the ρ_k component of $\bar{S}_d$ by $\bar{S}_d[\rho_k]$, which is irreducible in view of Table 4 if it is nonzero. We identify S_1 and $\bar{S}_1$ and we decompose

$\bar{S}_d$	1	3_1	3_2	4	5	$\dim \bar{S}_d$
$\bar{S}_0$	1	0	0	0	0	1
$\bar{S}_1$	0	1	0	0	0	3
$\bar{S}_2$	0	0	0	0	1	5
$\bar{S}_3$	0	0	1	1	0	7
$\bar{S}_4$	0	0	0	1	1	9
$\bar{S}_5$	0	1	1	0	1	11
$\bar{S}_6$	0	1	0	1	1	12
$\bar{S}_7$	0	0	1	1	1	12
$\bar{S}_8$	0	0	1	1	1	12
$\bar{S}_9$	0	1	0	1	1	12
$\bar{S}_{10}$	0	1	1	0	1	11
$\bar{S}_{11}$	0	0	0	1	1	9
$\bar{S}_{12}$	0	0	1	1	0	7
$\bar{S}_{13}$	0	0	0	0	1	5
$\bar{S}_{14}$	0	1	0	0	0	3

TABLE 4. The coinvariant algebra of G_{60}

$S_1\bar{S}_d[\rho_k]$ for each d and k. The result is summarized in Diagram G_{60} in Subsection 4.7. Omitting the details we just mention how we calculated Diagram G_{60}.

1. Choose a monomial basis $x^iy^jz^k$ of $\bar{S}_d$ for each d.
2. Calculate the projections of each basis element to each component $\bar{S}_d[\rho_k]$, and thus we get a basis for every $\bar{S}_d[\rho_k]$.
3. Multiply the basis elements of $\bar{S}_d[\rho_k]$ by x, y, z and calculate again their projections to the components $\bar{S}_{d+1}[\rho_{k'}]$ for all k'.

In view of Theorem 1.6 (3), we only need to calculate $S_1\bar{S}_d$ up to $d = 7$ to complete Diagram G_{60}.

Lemma 3.3. *Let I be a G-invariant ideal of S containing $\mathfrak{n}$ with $S/I \simeq \mathbf{C}[G]$ as G-modules. Then $I_b \subset I \subset I_t$, where I_b and I_t are the ideals of S containing $\mathfrak{n}$ such that*

$$\begin{aligned} I_b/\mathfrak{n} &= \sum_{d=10}^{14} \bar{S}_d + \bar{S}_9[\rho_4] + \bar{S}_9[\rho_5], \\ I_t/\mathfrak{n} &= \sum_{d=7}^{14} \bar{S}_d + \bar{S}_6[\rho_{3_1}]. \end{aligned} \tag{9}$$

Proof. Let I be an ideal satisfying the conditions in the lemma and put $\bar{I} = I/\mathfrak{n}$. Let $\rho = \rho_{3_1}$. We first consider the ρ component $\bar{I}[\rho]$ of $\bar{I}$. Since S/I is isomorphic to $\mathbf{C}[G]$ as a G-module, $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$ and $\bar{I}[\rho] \simeq \rho^{\oplus 3}$ in view of Theorem 1.6. Take an irreducible (not necessarily homogeneous) submodule W of $\bar{I}[\rho]$ and let d_0 be the largest number such that $W \subset \bar{S}_{\geq d_0} := \sum_{d=d_0}^{14} \bar{S}_d$. Then by Diagram G_{60}, we see that $S_{14-d_0}W = \bar{S}_{14}$ and thus $\bar{S}_{14} \subset \bar{I}$. We may assume that $W \neq \bar{S}_{14}$, since $\bar{I}[\rho] \simeq \rho^{\oplus 3}$. Then again by Diagram G_{60} we see that $pr_{[3_1]}S_{10-d_0}W = \bar{S}_{10}[\rho] \mod \bar{S}_{14}$, where $pr_{[3_1]}$ is the projection onto the ρ_{3_1}-component, and thus $\bar{S}_{10}[\rho] + \bar{S}_{14}[\rho] \subset \bar{I}$.

Next we consider the ρ_{3_2} component of $\bar{I}$. Again by Diagram G_{60}, we see $\bar{S}_{12}[\rho_{3_2}] \subset \bar{I}$. By $\bar{I}[\rho_{3_2}] \simeq \rho_{3_2}^{\oplus 3}$, we can choose an irreducible submodule W of $\bar{I}[\rho_{3_2}]$ such that $W \neq \bar{S}_{12}[\rho_{3_2}]$. Then $pr_{[3_2]}S_{10-d_0}W = \bar{S}_{10}[\rho_{3_2}] \mod \bar{S}_{12}$ by a similar argument and by a similar definition of d_0. Hence $\bar{S}_{10}[\rho_{3_2}] + \bar{S}_{12}[\rho_{3_2}] \subset \bar{I}$.

Similar arguments can be applied to ρ_4 and ρ_5 to conclude $I_b \subset I$.

Next we prove $I \subset I_t$. Suppose that I is not contained in I_t. Then there is an irreducible G-module W of $\bar{I}$ which is not contained in $\bar{I}_t := I_t/\mathfrak{n}$. For instance assume $W \simeq \rho_{3_1}$. Since W is not contained in $\bar{I}_t[\rho_{3_1}] = \oplus_{d\geq 6}\bar{S}_d[\rho_{3_1}]$, by Diagram G_{60} we see $\bar{S}_9[\rho_{3_1}] \subset \bar{I}$, whence $\rho_{3_1}^{\oplus 4} \subset \bar{I}$, which is a contradiction. In the other cases we can proceed in the same way to derive a contradiction. This completes the proof. □

3.4. **The eigenvectors $v_d(k)$.** Let $1 \leq d \leq 14$, and $k \in \{3_1, 4, 5\}$ (or resp. $k = 3_2$). Suppose $\bar{S}_d[\rho_k] \neq 0$. Then we define $v_d[k]$ to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue ϵ (or resp. ϵ^2) in $\bar{S}_d[\rho_k]$. The eigenvector $v_d[k]$ is unique up to constant multiples.

Now we define

$$\begin{aligned}
I([a:b]_{3_1}) &= S[G](av_6[3_1] + bv_9[3_1]) + \mathfrak{n},\ (a \neq 0)\\
I([a:b]_k) &= S[G](av_7[k] + bv_8[k]) + \mathfrak{n},\ (a \neq 0, k = 3_2, 4)\\
I([a:b]_5) &= S[G](av_7[5] + bv_8[5]) + \mathfrak{n},\ (ab \neq 0) \qquad (10)\\
I_1 &= S[G]v_7[5] + S[G]v_9[3_1] + \mathfrak{n},\\
I_0 &= S[G]v_8[3_2] + S[G]v_8[4] + S[G]v_8[5] + \mathfrak{n}
\end{aligned}$$

where $S[G] = S \otimes_{\mathbf{C}} \mathbf{C}[G]$. It is easy to check that all $I([a:b]_k)$ $(k = 3_1, 3_2, 4, 5)$ and I_i $(i = 0, 1)$ belong to $\mathrm{Hilb}^G(\mathbf{C}^3)$.

Theorem 3.5. *Let $G = G_{60}$. Then the fiber of the Hilbert-Chow morphism π over the origin consists of the following ideals:*

$$I([a:b]_{3_1})\ (a \neq 0),\ I([a:b]_{3_2})\ (a \neq 0),$$
$$I([a:b]_4)\ (a \neq 0),\ I([a:b]_5)\ (ab \neq 0),\ I_1,\ I_0.$$

Proof. Let I be an ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin and $\bar{I} = I/\mathfrak{n}$. Then since $\bar{I}_b := I_b/\mathfrak{n} \subset \bar{I} \subset \bar{I}_t$, by Lemma 3.3, $\bar{I}$ contains $W := \mathbf{C}[G](av_6[3_1] + bv_9[3_1])$ for some $[a:b] \in \mathbf{P}_1$. If $a \neq 0$, then $SW + \mathfrak{n}$ is G-stable and $S/(SW + \mathfrak{n})$ is isomorphic to $\mathbf{C}[G]$. It follows that $I = SW + \mathfrak{n} = I([a:b]_{3_1})$.

Next we assume $a = 0$. Hence $v_9[3_1] \in I$ and $\bar{I}[\rho_{3_1}] = \oplus_{d\geq 9}\bar{S}_d[\rho_{3_1}] \simeq \rho_{3_1}^{\oplus 3}$.

By Lemma 3.3 $(\bar{S}_{10}+\bar{S}_{12})[\rho_{3_2}] \subset \bar{I}$. Hence $\bar{I}$ contains $W := \mathbf{C}[G](av_7[3_2]+bv_8[3_2])$ for some $[a:b] \in \mathbf{P}_1$. If $a \neq 0$, we see that $I = SW+\mathfrak{n} = I([a:b]_{3_2})$. Now we assume $a = 0$. Then $v_8[3_2] \in \bar{I}$ and $\bar{I}[\rho_{3_2}] = \oplus_{d\geq 8}\bar{S}_d[\rho_{3_2}] \simeq \rho_{3_2}^{\oplus 3}$. Hence by Diagram G_{60} we see $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 9$. Hence there exists $[a:b] \in \mathbf{P}^1$ such that $av_7[4] + bv_8[4] \in \bar{I}$. If $a \neq 0$, then $I = I([a:b]_4)$. If $a = 0$, then $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 8$ while $\bar{S}_d[\rho_5] \subset \bar{I}$ for $d \geq 9$. Hence there is $[a:b] \in \mathbf{P}^1$ such that $av_7[5] + bv_8[5] \in \bar{I}$. If moreover $ab \neq 0$, then $I = I([a:b]_5)$. If $b = 0$, then $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 8$ and $\bar{S}_d[\rho_4] \subset \bar{I}$ for $d \geq 9$ and $k = 3_1, 3_2, 5$. Hence $I = I_1$. If $a = 0$, then $\bar{S}_d[\rho_k] \subset \bar{I}$ for $d \geq 8$ and $k = 4, 5$. Thus we see $I = I_0 = \oplus_{d\geq 8}\bar{S}_d + \mathfrak{n}$. This completes the proof. □

Corollary 3.6. *The fibre $\pi^{-1}(0)$ is a connected curve consisting of four smooth rational curves. Three of the four meet at a point as three coordinate axes of $\mathbf{C}^3$, while two of the four intersect transversally at another point.*

Proof. Let $C_{3_1} = \{I_1, I([a:b]_{3_1}), a \neq 0\}$, $C_k = \{I_0, I([a:b]_k), a \neq 0\}$ $(k = 3_2, 4)$ and $C_5 = \{I_0, I_1, I([a:b]_5), ab \neq 0\}$. Then C_k is a smooth rational curve. In fact, we easily see

$$\lim_{a\to 0} I([a:b]_{3_1}) = \lim_{b\to 0} I([a:b]_5) = I_1,$$
$$\lim_{a\to 0} I([a:b]_k)) = I_0 \quad (k = 3_2, 4, 5).$$

This completes the proof. See also the proof of Corollary 4.6. □

4. G_{168}

4.1. **Characters of G_{168}.** Let $G = G_{168}$. The group G is isomorphic to the simple group $\mathrm{PSL}(2,7)$. Let $V = \mathbf{C}^3$ and $V^\vee$ the dual of V. The space $V^\vee$ is spanned by x, y and z as before. The group G is realized as a subgroup of $\mathrm{SL}(V)$ generated by the elements given below :

$$\sigma = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^4 \end{pmatrix}, \ \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\upsilon = \frac{-1}{\sqrt{-7}} \begin{pmatrix} \beta^4-\beta^3 & \beta^2-\beta^5 & \beta-\beta^6 \\ \beta^2-\beta^5 & \beta-\beta^6 & \beta^4-\beta^3 \\ \beta-\beta^6 & \beta^4-\beta^3 & \beta^2-\beta^5 \end{pmatrix},$$

where $\beta = \exp(2\pi i/7)$. See [YY93, p. 74]. Let $M = G \times \{\pm 1\}$. Then $[M:G] = 2$ and M is a complex reflection group with Shephard-Todd number 24 [ST54, p. 301]. Hence we can apply Section 1 to G.

Table 5 is the character table of G, where $a = \beta+\beta^2+\beta^4$. We choose β as a primitive 7-th root of unity to define a conjugacy class to be junior or senior in Table 5. For each irreducible character χ_k in Table 5, we choose and fix an irreducible representation ρ_k which affords χ_k, in particular $\rho_{3_1} = \rho^\vee$ and $\rho_{3_2} = \rho$. We note $\bar{\rho}_{3_1} \simeq \rho_{3_2}$. There is a unique junior conjugacy class with all eigenvalues of ρ_{3_1} distinct from one. Hence by [IR96], there is a unique

c. c.	1	υ	τ	$\sigma\upsilon$	σ	σ^3
age	0	1	1	1	1	2
$\sharp$	1	21	56	42	24	24
χ_1	1	1	1	1	1	1
χ_{3_1}	3	-1	0	1	$-1-a$	a
χ_{3_2}	3	-1	0	1	a	$-1-a$
χ_6	6	2	0	0	-1	-1
χ_7	7	-1	1	-1	0	0
χ_8	8	0	-1	0	1	1

TABLE 5. Character table of G_{168}

compact irreducible divisor in the fibre of the Hilbert-Chow morphism over the origin. See Corollary 4.6.

4.2. **The coinvariant algebra of G.** By [YY93] the ring of G-invariant polynomials is generated by the four polynomials h_i $(i = 1, 2, 3, 4)$ of degrees 4, 6, 14 and 21 respectively:

$$\begin{gathered} h_1 = xy^3 + yz^3 + zx^3, \quad h_2 = 5x^2y^2z^2 - x^5y - y^5z - z^5x, \\ h_3 = (1/9)\,\mathrm{BH}(h_1, h_2),\ \ h_4 = (1/4)\,\mathrm{Jac}(h_1, h_2, h_3). \end{gathered} \tag{11}$$

Let S_G be the coinvariant algebra of G, and $\bar{S}_d$ the homogeneous component of S_G of degree d. $\bar{S}_d$ is decomposed into irreducible components $\bar{S}_d[\rho_k]$ as in Table 6. In view of Table 6, $\bar{S}_d[\rho_k]$ is irreducible except when it is zero, or $(d, k) = (7, 8)$, $(14, 8)$. We give the irreducible decomposition of $S_1\bar{S}_d[\rho_k]$ for each (d, k) in Diagram G_{168} in Subsection 4.7.

For $(d, k) = (7, 8)$ or $(14, 8)$, the decomposition of $S_1\bar{S}_d[\rho_k]$ is given more precisely as follows. $W_{7,\ell} := pr_{[8]}S_1\bar{S}_6[\rho_\ell]$ is irreducible and isomorphic to ρ_8 for $\ell = 6, 7, 8$ and they are however all distinct. Moreover

$$\begin{aligned} S_1W_{7,6} &= \bar{S}_8[\rho_6] + \bar{S}_8[\rho_7] + \bar{S}_8[\rho_8], \\ S_1W_{7,\ell} &= \bar{S}_8, \quad \text{for} \quad \ell = 7, 8. \end{aligned}$$

Similarly $W_{14,\ell} := pr_{[8]}S_1\bar{S}_{13}[\rho_\ell]$ is irreducible and isomorphic to ρ_8 for $\ell = 3_2, 6, 7, 8$ and they are all distinct. Moreover

$$\begin{aligned} S_1W_{14,3_2} &= \bar{S}_{15}[\rho_7] + \bar{S}_{15}[\rho_8], \\ S_1W_{14,\ell} &= \bar{S}_{15}, \quad \text{for} \quad \ell = 6, 7, 8. \end{aligned}$$

In a manner similar to the case of G_{60}, we can prove the following by chasing Diagram G_{168}.

Lemma 4.3. *Let I be a G-invariant ideal of S containing $\mathfrak{n}$ with $S/I \simeq$ $\mathbf{C}[G]$ as G-modules. Then $I_b \subset I \subset I_t$, where I_b and I_t are ideals of S*

$\bar{S}_d$	1	3_1	3_2	6	7	8	$\dim \bar{S}_d$
$\bar{S}_0$	1	0	0	0	0	0	1
$\bar{S}_1$	0	1	0	0	0	0	3
$\bar{S}_2$	0	0	0	1	0	0	6
$\bar{S}_3$	0	0	1	0	1	0	10
$\bar{S}_4$	0	0	0	1	0	1	14
$\bar{S}_5$	0	0	1	0	1	1	18
$\bar{S}_6$	0	0	0	1	1	1	21
$\bar{S}_7$	0	0	0	0	1	2	23
$\bar{S}_8$	0	1	0	1	1	1	24
$\bar{S}_9$	0	1	0	1	1	1	24
$\bar{S}_{10}$	0	0	1	1	1	1	24
$\bar{S}_{11}$	0	1	0	1	1	1	24
$\bar{S}_{12}$	0	0	1	1	1	1	24
$\bar{S}_{13}$	0	0	1	1	1	1	24
$\bar{S}_{14}$	0	0	0	0	1	2	23
$\bar{S}_{15}$	0	0	0	1	1	1	21
$\bar{S}_{16}$	0	1	0	0	1	1	18
$\bar{S}_{17}$	0	0	0	1	0	1	14
$\bar{S}_{18}$	0	1	0	0	1	0	10
$\bar{S}_{19}$	0	0	0	1	0	0	6
$\bar{S}_{20}$	0	0	1	0	0	0	3

TABLE 6. The coinvariant algebra of G_{168}

containing $\mathfrak{n}$ *such that*

$$\begin{aligned} I_b/\mathfrak{n} &= \bar{S}_{\geq 14} + \sum_{k=6,7,8} \bar{S}_{13}[\rho_k] + \sum_{k=7,8} \bar{S}_{12}[\rho_k], \\ I_t/\mathfrak{n} &= \bar{S}_{\geq 10} + \bar{S}_9[\rho_6] + \bar{S}_9[\rho_{3_1}] + \bar{S}_8[\rho_{3_1}]. \end{aligned} \tag{12}$$

4.4. **The eigenvectors $v_d(k)$.** Let $1 \leq d \leq 20$ and $k = 3_1, 3_2, 6, 7, 8$. Suppose $(d,k) \neq (7,8), (14,8)$ and $\bar{S}_d[\rho_k] \neq 0$. Then we define $v_d[k]$ to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue β (resp. β^3) in $\bar{S}_d[\rho_k]$ for $k \neq 3_2$ (resp. $k = 3_2$). The eigenvector $v_d[k]$ is unique up to constant multiples. Also we define $v'_d[6]$ (resp. $v''_d[7]$) to be an eigenvector of $S_G(\rho^\vee)(\sigma^{-1})$ with eigenvalue β^2 (resp. β^0) in $\bar{S}_d[\rho_6]$ (resp. $\bar{S}_d[\rho_7]$).

Here is a list of some of these polynomials used later. Notice that these polynomials are in $\bar{S}$, i.e. taken modulo $\mathfrak{n}$.

$$\begin{aligned}
v_8[3_1] &= 3x^8 - 29xz^7 - 112y^5z^3 - 149xy^7,\\
v_9[3_1] &= 48y^7z^2 - 6xy^2z^6 + z^9 - 198x^2y^4z^3,\\
v_{11}[3_1] &= -198x^2yz^8 - 990xy^6z^4 + y^{11} - 748y^4z^7,\\
v_{10}[3_2] &= -264y^8z^2 + x^{10} - 116xy^3z^6 + 161yz^9,\\
v_{12}[3_2] &= 3y^{12} - 11693xy^7z^4 - 344xz^{11} - 9988y^5z^7,\\
v_{13}[3_2] &= 2z^{13} + 3315xy^9z^3 + 559xy^2z^{10} + 3198y^7z^6,\\
v'_9[6] &= x^9 + 40x^2z^7 + 360xy^5z^3 + 298y^3z^6,\\
v'_{10}[6] &= 4y^5z^5 + 5xy^7z^2,\\
v'_{11}[6] &= 538xy^9z - 68xy^2z^8 + 338y^7z^4 + 3z^{11},\\
v'_{12}[6] &= 49xy^4z^7 + 10y^9z^3 - 6y^2z^{10},\\
v''_{10}[7] &= 19xy^8z + 4xyz^8 + 14y^6z^4,\\
v''_{11}[7] &= 34xy^3z^7 + 7y^8z^3 - 5yz^{10}.
\end{aligned} \tag{13}$$

Now we define G-invariant ideals of S by

$$\begin{aligned}
I([a:b:c]_{3_1}) &:= S[G](av_8[3_1] + bv_9[3_1] + cv_{11}[3_1]) + \mathfrak{n},\\
I([a:b]_6) &:= S[G](av'_{10}[6] + bv'_{11}[6]) + S[G]v_{11}[3_1] + \mathfrak{n},\\
I([a:b]_7) &:= S[G](av''_{10}[7] + bv''_{11}[7]) + S[G]v_{11}[3_1] + \mathfrak{n},\\
I([a:b]_8) &:= S[G](av_{10}[8] + bv_{11}[8]) + \mathfrak{n},\\
I_0 &:= \sum_{k=3_1,6,7,8} S[G]v_{11}[k] + \mathfrak{n}
\end{aligned} \tag{14}$$

where $S[G] = S \otimes_{\mathbf{C}} \mathbf{C}[G]$, $[a:b:c]_{3_1} \neq [0:0:1]$ and $[a:b]_k \neq [0:1]$. It is not difficult to check by using Diagram G_{168} that these ideals belong to $\mathrm{Hilb}^G(\mathbf{C}^3)$. Then we have

Theorem 4.5. *Let $G = G_{168}$. Then the fiber of the Hilbert-Chow morphism π over the origin consists of the following ideals:*

$$\begin{gathered}
I([a:b:c]_{3_1}) \quad (a,b) \neq (0,0),\\
I_0,\ I([a:b]_k) \quad a \neq 0,\ k = 6,7,8.
\end{gathered}$$

Proof. Let I be an ideal in $\mathrm{Hilb}^G(\mathbf{C}^3)$ over the origin and $\bar{I} = I/\mathfrak{n}$. In view of Lemma 4.3 $\bar{I}$ contains $v := av_8[3_1]+bv_9[3_1]+cv_{11}[3_1]$ for some $[a:b:c] \in \mathbf{P}_2$. Let $W = \mathbf{C}[G]v$ and $W^* = \sum_{\ell=1}^4 S_\ell W$. We have $[SW : \rho_{3_2}] \leq 3$, because $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$. Since $\bar{S}_{20}[\rho_{3_1}] \subset SW$, the multiplicity $[W^* : \rho_{3_2}]$ is at most two. To calculate the multiplicity, we consider the eigenspace of $S_G(\rho^\vee)(\sigma^{-1})$ in $pr_{[3_2]}W^*$ with eigenvalue β^3. It is easy to see that this space is spanned by the images by $pr_{[3_2]}$ of the four vectors yv, x^2v, xy^2zv and z^4v. By calculation it turns out that

$$\begin{aligned} pr_{[3_2]}(yv) &= -28bv_{10}[3_2] + (28/3)cv_{12}[3_2],\\ pr_{[3_2]}(x^2v) &= 42av_{10}[3_2] + 7cv_{13}[3_2],\\ pr_{[3_2]}(xy^2zv) &= -2av_{12}[3_2] - bv_{13}[3_2],\\ pr_{[3_2]}(z^4v) &= -3pr_{[3_2]}(xy^2zv). \end{aligned}$$

We easily see

$$pr_{[3_2]}(3ayv + 2bx^2v + 14cxy^2zv) = 0,$$

whence the β^3-eigenspace of $S_G(\rho^\vee)(\sigma^{-1})$ in $pr_{[3_2]}W^*$ is exactly 2-dimensional. Since W^* is a G-module, $pr_{[3_2]}$ is an isomorphism on $W^*[\rho_{3_2}]$. Hence $[SW : \rho_{3_2}] = 3$ for any $[a : b : c] \in \mathbf{P}_2$. Similarly it can be verified that $pr_{[6]}S_{\leq 3}W$ is generated by

$$\begin{aligned} (1/84)pr_{[6]}(xv) &= 3av'_9[6] + 49bv'_{10}[6] + 49cv'_{12}[6],\\ (-1/4)pr_{[6]}(y^2zv) &= av'_{11}[6] + 49bv'_{12}[6] \end{aligned}$$

as an $S[G]$-module. Hence $[SW : \rho_6] = 6$ if $(a, b) \neq (0, 0)$. Also we see $[SW : \rho_k] = k$ for $k = 7$ and 8 if $(a, b) \neq (0, 0)$. Thus if $(a, b) \neq (0, 0)$, then $SW \simeq \mathbf{C}[G]/\mathbf{C}$ and $I = SW + \mathfrak{n} = I([a : b : c]_{3_1})$.

Now we assume $(a, b) = (0, 0)$. Then $\bar{I}[\rho_{3_1}] = \oplus_{k\geq 11}\bar{S}_k[\rho_{3_1}]$. It follows $\bar{I}[\rho_{3_2}] = \oplus_{k\geq 12}\bar{S}_k[\rho_{3_2}]$, and $\bar{S}_k[\rho_6] \subset \bar{I}[\rho_6]$ for $k \geq 12$. Since $\bar{I} \simeq \mathbf{C}[G]/\mathbf{C}$, there exists $av'_{10}[6] + bv'_{11}[6] \in \bar{I}$ for some $[a : b] \in \mathbf{P}^1$ by Lemma 4.3. If $a \neq 0$, then we see $I = I([a : b]_6)$. If $a = 0$, then $\bar{I}[\rho_6] = \oplus_{k\geq 11}\bar{S}_k[\rho_6]$ and $\bar{S}_k[\rho_7] \subset \bar{I}[\rho_7]$ for $k \geq 12$.

It follows from Lemma 4.3 that there exists $av''_{10}[7] + bv''_{11}[7] \in \bar{I}$ for some $[a : b] \in \mathbf{P}^1$. If $a \neq 0$, then we see $I = I([a : b]_7)$. If $a = 0$, then $\bar{I}[\rho_7] = \oplus_{k\geq 11}\bar{S}_k[\rho_7]$ and $\bar{S}_k[\rho_8] \subset \bar{I}[\rho_8]$ for $k \geq 12$. Therefore there exists $av_{10}[8] + bv_{11}[8] \in \bar{I}$ for some $[a : b] \in \mathbf{P}^1$ by Lemma 4.3. If $a \neq 0$, then we see $I = I([a : b]_8)$. If $a = 0$, we see $I = I_0$. This completes the proof. □

Corollary 4.6. *The fibre $\pi^{-1}(0)$ is the union of a smooth rational curve and a doubly blown-up projective plane with infinitely near centers, both intersecting transversally at a unique point.*

Proof. Let $C_8 := \{I_0, I([a : b]_8)\ (a \neq 0)\}$. Then $C_8 \simeq \mathbf{P}^1$ because $I_0 = \lim_{a\to 0} I([a : b]_8)$. To be more precise we construct a G-invariant zero-dimensional subscheme Z of $C_8 \times \mathbf{C}^3$ flat over C_8 such that the fibre of Z over $I \in C_8$ is Spec S/I. For the purpose we define a G-invariant ideal $\mathcal{I}$ of $O_{C_8} \otimes_{\mathbf{C}} S$ by

$$\mathcal{I} = O_{C_8}(-1)I([a : b]_8) + O_{C_8}(\sum_{k=3_1,6,7} S[G]v_{11}[k] + \mathfrak{n}).$$

We note that $\mathcal{I}_{[a:b]\times\mathbf{C}^3} = I([a : b]_8)$ if $a \neq 0$, while $\mathcal{I}_{[0:1]\times\mathbf{C}^3} = I_0$. Since $\dim S/\mathcal{I}_{[a:b]}$ is constant $(= 168)$ on C_8, the subscheme Z of $C_8 \times \mathbf{C}^3$ defined by $\mathcal{I}$ is flat over C_8.

Next let T be a doubly blown-up projective plane with infinitely near centers and we construct a G-invariant zero-dimensional subscheme Z of $T \times \mathbf{C}^3$ flat over T such that

(i) any fibre of Z over T is one of the subschemes Spec S/I defined by the ideals I among $I([a:b:c]_{3_1})$, $I([a:b]_k)$ $(k=6,7)$ and I_0,

(ii) the natural morphism ϕ of T into $\mathrm{Hilb}^G(\mathbf{C}^3)$ is a closed immersion, *in other words*,

(ii′) ϕ is an injection and for any $t \in T$ the Kodaira-Spencer map of ϕ is a monomorphism of the tangent space $T_t(T)$ of T at t into the subspace $\mathrm{Hom}_S(I/\mathfrak{n}, S/I)$ of $\mathrm{Hom}_S(I, S/I)$ where I is the unique ideal of S such that the fibre of Z over t is Spec S/I.

In fact, (ii′) is easy to check by the construction of Z below.

For the purpose let $U = \mathbf{P}^2 \setminus \{p_0\}$ and we first define a G-invariant ideal $\mathcal{I}_U$ on $U \times \mathbf{C}^3$ by

$$\mathcal{I}_U := O_U(-1)I([a:b:c]_{3_1}) + O_U\mathfrak{n}$$

where $O_U(-1) = O_{\mathbf{P}^2}(-1)_{|U}$ and $[a:b:c] \in \mathbf{P}^2$. Then we extend it to T so that $(O_T \otimes_{\mathbf{C}} S)/\mathcal{I}$ may be O_T-flat.

This is done as follows. Let $p_0 := [0:0:1] \in \mathbf{P}^2$ and let ℓ_0 be the (-1)-curve on $Q_{p_0}(\mathbf{P}^2)$. Let p_1 be a point of ℓ_0 and $T = Q_{p_1}Q_{p_0}(\mathbf{P}^2)$. Since T is a torus embedding, we may assume that we can choose an open covering $\{U, U_1, U_2, U_3\}$ of T such that $U = \mathbf{P}^2 \setminus \{p_0\}$, $U_k = \mathrm{Spec}\ \mathbf{C}[s_k, t_k]$ where

$$p = a/c,\ q = b/c,$$
$$s_1 = p,\ t_1 = q/p,\ s_2 = p/q,\ t_2 = q^2/p,\ s_3 = p/q^2,\ t_3 = q.$$

Let $v = s_1v_8[3_1] + s_1t_1v_9[3_1] + v_{11}[3_1]$. We define on U_1

$$\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n} + O_T \otimes S[G](3v''_{10}[7] + 7t_1v''_{11}[7]).$$

We see $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_1) \times \mathbf{C}^3$ because if $s_1 \neq 0$, then

$$(-1/56s_1)pr_{[7]}(yzv) = 3v''_{10}[7] + 7t_1v''_{11}[7].$$

We note that $\mathcal{I}_{\tau \times \mathbf{C}^3} = I([3:7t_1]_7)$ where $\tau := (s_1, t_1) = (0, t_1) \in U_1$. It is evident that $\mathcal{I} = O_T \otimes S[G]I([s_1, s_1t_1, 1]_{3_1})$ if $(s_1, t_1) \in U_1$ and $s_1 \neq 0$.

Let $v = s_2^2t_2v_8[3_1] + s_2t_2v_9[3_1] + v_{11}[3_1]$. Next we define on U_2

$$\begin{aligned}\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n} + O_T \otimes S[G](3s_2t_2v'_9[6] + 49t_2v'_{10}[6] - v'_{11}[6])\\ + O_T \otimes S[G](3s_2v''_{10}[7] + 7v''_{11}[7]) + O_T \otimes S[G]v_{11}[8].\end{aligned}$$

We see $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_2) \times \mathbf{C}^3$ because if $s_2t_2 \neq 0$, then

$$\begin{aligned}-(1/56s_2t_2)pr_{[7]}(yzv) &= 3s_2v''_{10}[7] + 7v''_{11}[7],\\ (1/84)pr_{[6]}(xv) &= 3s_2^2t_2v'_9[6] + 49s_2t_2v'_{10}[6] + 49v'_{12}[6],\\ -(1/4s_2t_2)pr_{[6]}(y^2zv) &= s_2v'_{11}[6] + 49v'_{12}[6],\end{aligned}$$

whence

$$(1/84s_2)pr_{[6]}(xv) + (1/4s_2^2t_2)pr_{[6]}(y^2zv) \\ = 3s_2t_2v'_9[6] + 49t_2v'_{10}[6] - v'_{11}[6].$$

We note that $\mathcal{I}_{\tau\times\mathbf{C}^3} = I_0$ if $\tau := (s_2, t_2) = (0, 0)$ while $\mathcal{I}_{\tau\times\mathbf{C}^3} = I([3s_2 : 7]_7)$ (resp. $I([49t_2 : (-1)]_6)$ if $\tau = (s_2, 0), s_2 \neq 0$ resp. if $\tau = (0, t_2)$, $t_2 \neq 0$.

Finally let $v = s_3t_3^2v_8[3_1] + t_3v_9[3_1] + v_{11}[3_1]$. We define on U_3

$$\mathcal{I} = O_T \otimes S[G]v + O_T\mathfrak{n} \\ + O_T \otimes S[G](3s_3t_3v'_9[6] + 49v'_{10}[6] - s_3v'_{11}[6]).$$

It is easy to see that $\mathcal{I} = \mathcal{I}_U$ on $(U \cap U_3) \times \mathbf{C}^3$. We note that $\mathcal{I}_{\tau\times\mathbf{C}^3} = I([49 : (-s_3)]_6)$ for $\tau = (s_3, 0) \in U_3$.

Let Z be a subscheme of $T \times \mathbf{C}^3$ defined by $\mathcal{I}$. Then Z is T-flat because $(O_T \otimes S)/\mathcal{I}$ is locally O_T-free of rank 168. This completes the proof. □

4.7. **Diagrams.** Diagram G_{60} and Diagram G_{168} express the decomposition of $S_1\bar{S}_i[j]$. The rows are indexed by degrees and the columns by irreducible representations. Each vertex corresponds to $\bar{S}_i[j]$ and we join $\bar{S}_i[j]$ and $\bar{S}_{i+1}[k]$ when $\bar{S}_{i+1}[k]$ appears in $S_1\bar{S}_i[j]$. Two double circles in Diagram G_{168} mean that the multiplicities of ρ_8 in $\bar{S}_7$ and $\bar{S}_{14}$ are equal to two. We note that Diagram G_{60} is symmetric with center at degrees 7 and 8 because $\bar{\rho}_k \simeq \rho_k$ for any irreducible representation ρ_k. However Diagram G_{168} loses apparent symmetry with center at degrees 10 and 11 because $\bar{\rho}_{3_1} \simeq \rho_{3_2}$.

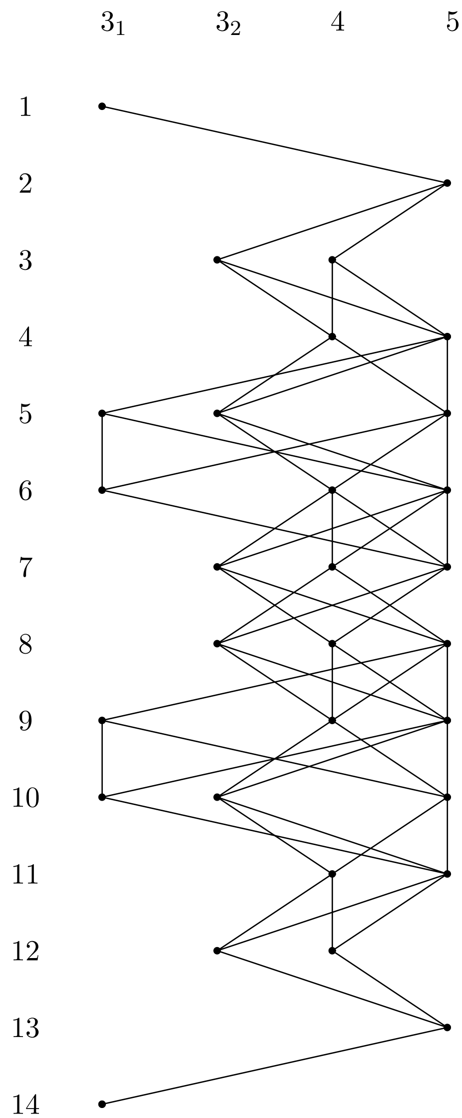

Diagram G_{60}

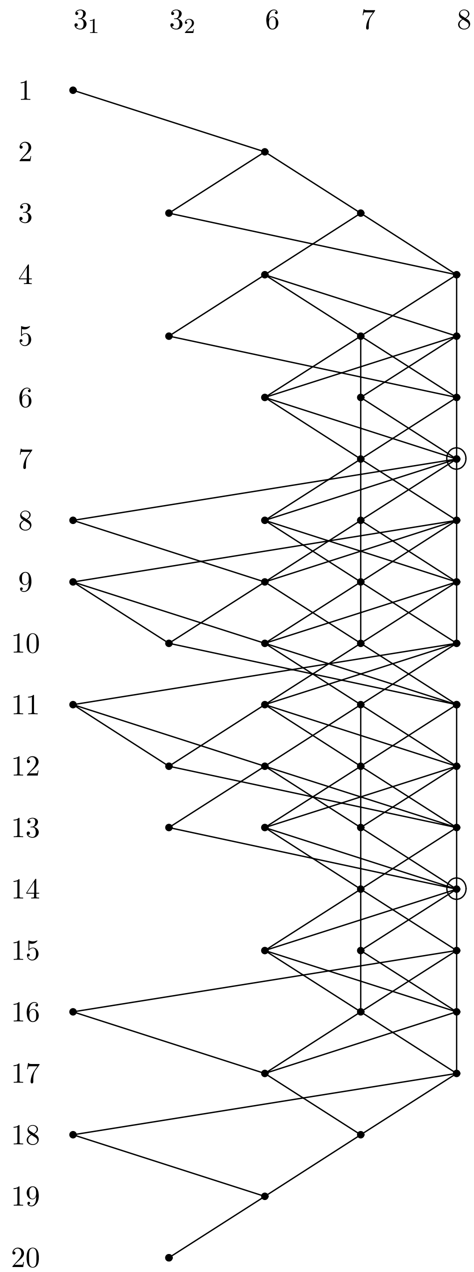

Diagram G_{168}

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