Compactification of the moduli of abelian varieties over $\mathbf{Z}[\zeta_N, 1/N]$

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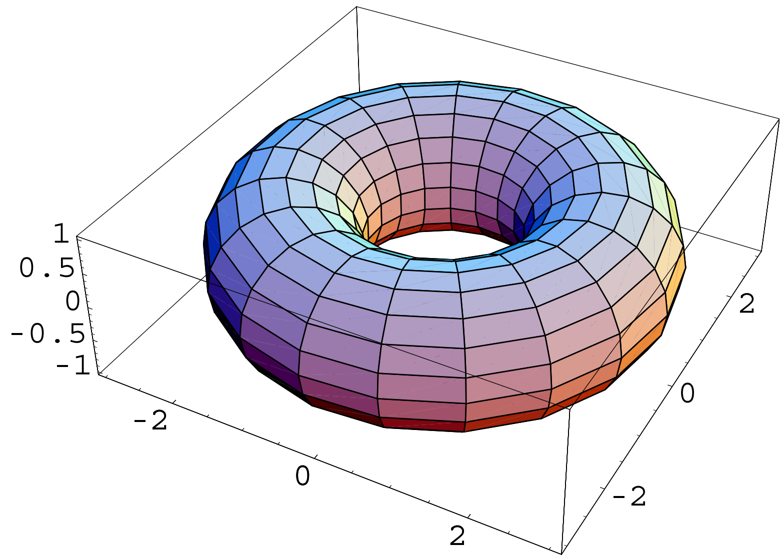
1 Hesse cubic curves

$$C(\mu) : x_0^3 + x_1^3 + x_2^3 - 3\mu\, x_0 x_1 x_2 = 0$$
$$(\mu \in \mathrm{P}^1_{\mathrm{C}})$$

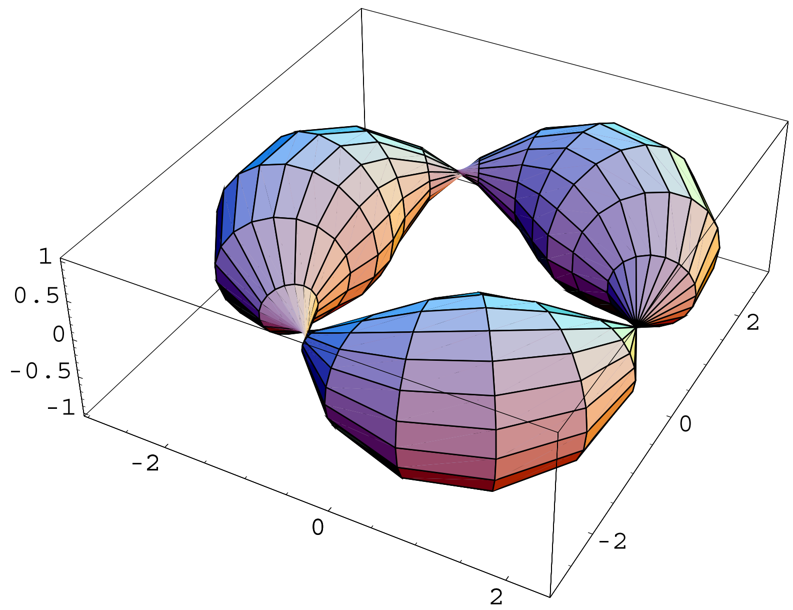


$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu \in \mathrm{C})$$

When $\mu = 1,\ \zeta_3,\ \zeta_3^2,\ \infty$,

it divides into 3 copies of P^1

(two-dimensional spheres).



2 Moduli of cubic curves

Th 1 (Hesse 1849)

(1) Any nonsing. cubic curve is converted into $C(\mu)$ under SL(3).

(2) $C(\mu)$has nine inflection points $[1:-\beta:0]$, $[0:1:-\beta]$,$[-\beta:0:1]$ $(\beta \in \{1, \zeta_3, \zeta_3^2\})$.

(3) $C(\mu)$ and $C(\mu')$ are isomorphic to each other with nine points fixed if and only if $\mu = \mu'$

Th 2 (moduli and compactification)

$A_{1,3}$:= moduli of nonsing. cubic curves

with 9 inflection points

= {nonsing. cubics} /ordered 9 points

$= \mathrm{C} \setminus \{1, \zeta_3, \zeta_3^2\} = \Gamma(3)\backslash \mathrm{H}$

$\overline{A_{1,3}}$:= {slightly general cubic curves}

/ordered nine inflection points

= {Hesse cubic curves} /isom.=identical

= {Hesse cubic curves}

$= A_{1,3} \cup \{C(\infty)\} \cup \{C(1), C(\zeta_3), C(\zeta_3^2)\}$

$= \mathrm{P}^1 = \overline{\Gamma(3)\backslash \mathrm{H}}$

Our goal is

Th 3 (N.'99) (High dim. compactification)

Let K finite symplectic, $\forall$ elm. div. of $K \geq 3$. There exists a fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$ where $N = |K|$. For k : alg.closed, char.k and N:coprime

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{degenerate abelian varieties} \\ \text{with level } G(K)\text{-structure} \\ \text{and a closed SL-orbit} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant degenerate} \\ \text{abelian varieties} \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

$G(K)$: non-abelian Heisenberg group of K

Usually moduli of cubic curves is

moduli of cubics with 9 inflection points

We convert it into G-equivariant theory

with G : Heisenberg group

all cubics in P^2 = all cubic polynom. in x_0, x_1, x_2 $/\mathrm{C}^*$

Take $G(3)$-invariants!

all Hesse cubics $= G(3)$ -inv. cubics

$G = G(3)$:the Heisenberg group of level 3, $(|G| = 27)$

$V = \mathrm{C}x_0 + \mathrm{C}x_1 + \mathrm{C}x_2$: a representation of G

$$\sigma : (x_0, x_1, x_2) \mapsto (x_0, \zeta_3 x_1, \zeta_3^2 x_2)$$

$$\tau : (x_0, x_1, x_2) \mapsto (x_1, x_2, x_0)$$

$x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2 \in S^3V$ are G-invariant

$\Downarrow$ ("Hesse cubic curves" in P^2)

$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0x_1x_2 = 0 \ (\mu \in \mathrm{C})$$

$\Downarrow$

Compactification of moduli of abelian var.

3 Theta functions

Why does $G(3)$ get involved in moduli of cubics ?

$E(\tau)$: an elliptic curve over C

$$E(\tau) = \mathrm{C}/\mathrm{Z} + \mathrm{Z}\tau = \mathrm{C}^*/w \mapsto wq^6 \ (q = e^{2\pi i\tau/6})$$

Def 4 Theta functions ($k = 0, 1, 2$)

$$\theta_k(\tau, z) = \sum_{m\in \mathrm{Z}} q^{(3m+k)^2} w^{3m+k}$$
$$= \sum_{m\in \mathrm{Z}} a(3m + k) w^{3m+k}$$

where $q = e^{2\pi i\tau/6}$, $w = e^{2\pi iz}$,

$a(x) = q^{x^2}$ ($x \in X$), $X = \mathrm{Z}$ and $Y = 3\mathrm{Z}$.

Formula:

$$\theta_k(\tau, z+1) = \theta_k(\tau, z), \quad \theta_k(\tau, z+\tau) = q^{-9} w^{-3} \theta_k(\tau, z)$$

$$\Theta : z \in E(\tau) \ \mapsto \ [\theta_0, \theta_1, \theta_2] \in \mathrm{P}^2_{\mathrm{C}} : \text{ well-def.}$$

$$\theta_k(\tau, z+\tfrac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$
$$\theta_k(\tau, z+\tfrac{\tau}{3}) = q^{-1} w^{-1} \theta_k(\tau, z)$$

Then $z \mapsto z + \frac{1}{3}$ induces (the contragredient repres.)

$$\sigma : [\theta_0, \theta_1, \theta_2] \ \mapsto \ [\theta_0, \zeta_3\theta_1, \zeta_3^2\theta_2]$$

$z \mapsto z + \frac{\tau}{3}$ induces

$$\tau : [\theta_0, \theta_1, \theta_2] \ \mapsto [\theta_1, \theta_2, \theta_0]$$

$$\text{Let } V = \mathbf{C}x_0 + \mathbf{C}x_1 + \mathbf{C}x_2$$

$$\sigma(x_k) = \zeta_k x_k, \quad \tau(x_k) = x_{k+1}$$

$$\sigma\tau\sigma^{-1}\tau^{-1} = (\zeta_3 \cdot \mathrm{id}_V)$$

Def 5 Weil pairing $e_{E(\tau)}(1/3, \tau/3) = \zeta_3$

Def 6 $G(3) :=$ the group generated by σ, τ

the Heisenberg group of level 3, $|G(3)| = 27$.

Formula: $\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z)$

$$\theta_k(\tau, z + \tfrac{\tau}{3}) = q^{-1} w^{-1} \theta_k(\tau, z)$$

$$\Downarrow$$

The cubic curve $\Theta(E(\tau))$ is $G(3)$-invariant.

Since V is $G(3)$-irreducible, by Schur's lemma $G(3)$ determines x_j uniquely up to const. multiple.

x_j is an algebraic theta function

as $G(3)$-modules

$$S^3V = 2 \cdot 1_0 \oplus (1_j)(j = 1, ..., 8) \quad \text{10-dim}$$

$$2 \cdot 1_0 = \{x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2\}$$

$$1_j = \{x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3\},$$

$$1_k = \{x_0^2 x_1 + \zeta_3 x_1^2 x_2 + \zeta_3^2 x_2^2 x_0\} \text{ etc.}$$

$$2 \cdot (1_0) = \{x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2\} \subset S^3V$$

gives the equation of $\Theta(E(\tau))$

$$x_0^3 + x_1^3 + x_2^3 - 3\mu(\tau)x_0x_1x_2 = 0$$

4 Principle for compactifying the moduli

moduli = the set of isomorphism classes

Roughly "moduli"$=X/G$, G: algebraic group

X	the set of geometric objects
G	the group of isomorphisms
x and x' are isom.	their G-orbits are the same
	$O(x) = O(x')$
X_{ps}	stable objects
X_{ss}	semistable objects
Quotient X_{ps}/G	"moduli"
$X_{ss}//G$	"compactification"

A lot of compactifications of the moduli space of abelian varieties are already known.

Satake, Baily-Borel, Mumford etc, Namikawa

What is nice? What is natural?

Wish "to identify isom. classes by invariants"

"moduli": =algebraic moduli

=the space defined by the invariants of isom. classes

Difficult to investigate this

What should be done about it?

”moduli” :=the space defined by

the invariants of isom. classes

It is easier to investigate geometrically.

We limit the geometric objects

to those whose invariants are well defined

Stability and semistability (Mumford:GIT)

To classify the isom. classes by invariants completely

we are led to the space of closed orbits

5 The space of closed orbits

To review again

X	the set of geometric objects
G	the group of isomorphisms
x, x' are isom.	their G-orbits are the same
	$O(x) = O(x')$
X_{ss}	the set of semistable objects
$X_{ss}//G$	”moduli”

Rem stability $\Rightarrow$ closed orbits $\Rightarrow$ semistability

Ex 7 Action on C^2 of $G = \mathrm{C}^*$, $(x, y) \in \mathrm{C}^2$

$$(x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathrm{C}^*)$$

How can we define the quotient space $\mathrm{C}^2 // G$?

Simple answer : the set of G-orbits $(\times)$

Answer : the space defined by the invariant of G $(\bigcirc)$

$t = xy$ is the unique invariant. Hence

$$\mathrm{C}^2 // G = \{t \in \mathrm{C}\}$$

These two spaces disagree with each other.

$$\mathrm{C}^2 // G = \{t \in \mathrm{C}\} \neq \text{the set of } G\text{-orbits}$$

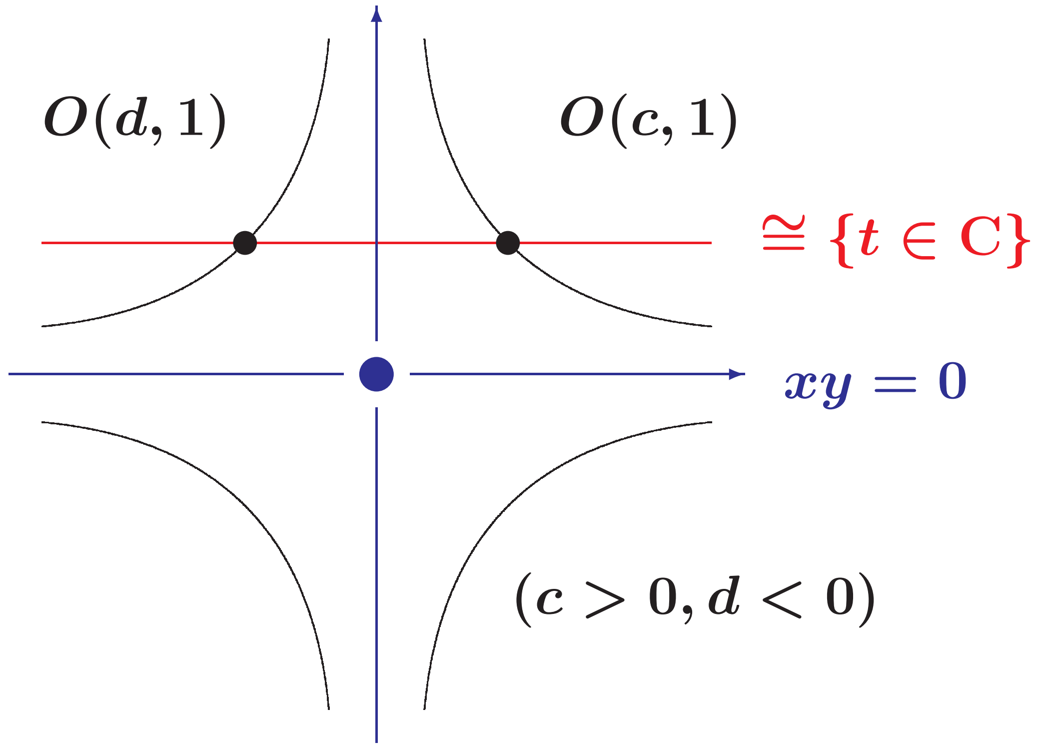
$\mathbb{C}^2//G = \{t \in \mathbb{C}\} \neq$ the set of G-orbits

Reason $\{xy = t\}$ ($t \neq 0$: constant) is a G-orbit

But, $\{xy = 0\}$ is the union of 3 G-orbits

$\mathbb{C}^* \times \{0\}, \{0\} \times \mathbb{C}^*, \{(0,0)\}$.

We cannot distingush them by t.



$\{(0,0)\}$ is the only closed orbit of $\{xy = 0\}$.

Th 8 The space $C^2//G$ defined by G-invariants $=$ the set of closed G-orbits in C^2, $(G = C^*)$.

More generally

Th 9 (Mumford,Seshadri)

Let G : a reductive group, (e.g. $G = C^*$)

Let X_{ss} : the set of all semistable points. Then

$$X_{ss}//G := \text{the space defined by } G\text{-invariants}$$
$$= \text{the set of closed orbits.}$$

Here closed means closed in X_{ss}.

We limit the objects to those
with closed orbits

$\Downarrow$

Abelian varieties and PSQASes

PSQASes : the degenerate abelian varieties which have closed orbits

$\Downarrow$

Can compactify the moduli
of abelian varieties with these.

6 GIT-stability and stable critical points

Definition of GIT-stability has nothing to do with stable critical points, But it has to do with them

V : vector space, G : reductive group

K : maximal compact of G, $\|\cdot\|$: K-invariant metric

$$p_v(g) := \|g \cdot v\| \text{ for } \quad v \in V$$

Th 10 (Kempf-Ness 1979) The following are equiv.

(1) v has a closed G-orbit

(2) p_v attains a minimum on the orbit $O(v)$

(3) p_v attains a critical point on $O(v)$

7 Stable curves of Deligne-Mumford

Def 11 C is called a stable curve of genus g if

(1) A conn. proj. curve with finite autom. group,

(2) Sing. of C are like $xy = 0$,

(3) $\dim H^1(O_C) = g$.

Th 12 (Deligne-Mumford)

Let M_g : moduli of nonsing. curves of genus g,

$\overline{M_g}$: moduli of stable curves. Then

$\overline{M_g}$ is compact,

M_g is Zariski open in $\overline{M_g}$.

Th 13 **The following are equivalent**

(1) C **is a stable curve.**

(2) Hilbert point of $\Phi_{|mK|}(C)$ **is GIT-stable.**

(3) Chow point of $\Phi_{|mK|}(C)$ **is GIT-stable.**

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(1)$\Leftrightarrow$(2) Gieseker 1982

(1)$\Leftrightarrow$(3) Mumford 1977

8 Stability of cubic curves

cubic curves	stability
smooth elliptic	closed orbits, stable
3 lines, no triple point	closed orbits
a line+a conic, not tangent	semistable
irreducible, a node	semistable
the others	not semistable

Th 14 The following are equivalent:

(1) it has a SL(3)-closed orbit.

(2) smooth elliptic or a circle of 3 lines (3-gon).

(3) Hesse cubic curves, that is, $G(3)$-invariant.

$G(3)$-invariance leads to the moduli

Let $V = \{x_0, x_1, x_2\}$, as $G(3)$-modules

$S^3V = 2 \cdot 1_0 \oplus (1_j)(j = 1, ..., 8)$ 10-dim

$2 \cdot 1_0 = \{x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2\}$

$1_j = \{x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3\}$,

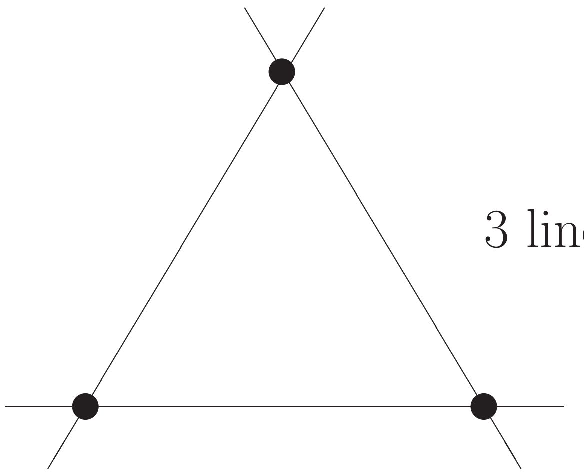
$1_k = \{x_0^2 x_1 + \zeta_3 x_1^2 x_2 + \zeta_3^2 x_2^2 x_0\}$ etc.

$2 \cdot (1_0) = \{x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2\} \subset S^3V$

gives the equations

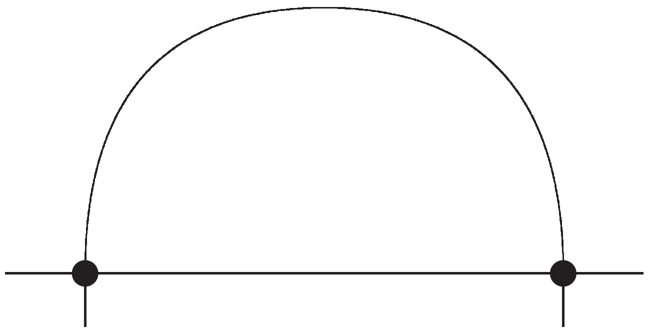
$x_0^3 + x_1^3 + x_2^3 - 3\mu(\tau)x_0x_1x_2 = 0$

semistable cubic curves



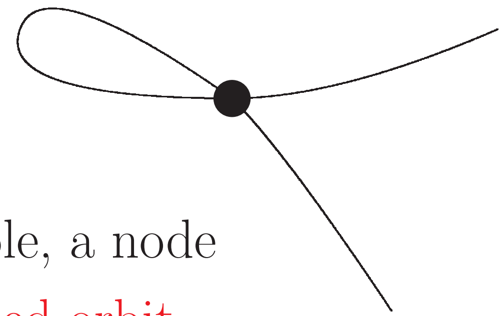
3 lines with no triple pts

closed orbit



a line+a conic, not tangent

nonclosed orbit



irreducible, a node

nonclosed orbit

9 Stability-higher-dim.

Th 15 (N, 1999) k is alg. closed, $K = H \oplus H^\vee$ ($H^\vee$:dual of H), H: a finite abelian group, any elm. div≥ 3; $|H|$ and char.k coprime; $V = k[H]$: the gp ring of H; Assume X ($\subset$ P(V)) is a limit of abelian var. with K-torsions. The following are equiv:

(1) X has a closed SL(V)-orbit.

(2) X is invariant under $G(K)$: Heisenbg gp of K, .

(3) X is one of PSQASes.

What is a PSQAS ?

It is a generalization of Tate curves

Tate curve : $\mathbb{C}^*/w \mapsto qw$

Hesse cubics : $\mathbb{G}_m(K)/w \mapsto q^3 w$ **or**

Hesse cubics : $\mathbb{G}_m(K)/w^n \mapsto q^{3mn} w^n (m \in \mathbb{Z})$

The general case : B positive definite

$$(\mathbb{G}_m^g(K))/w^x \mapsto q^{B(x,y)} w^x (y \in \mathbb{Z}^g)$$

(Hesse) 3-gon and PSQASes

are natural limits as $q \to 0$

Th 16 For cubic curves the following are equiv.:

(1) it has a closed SL(3)-orbit.

(2) it is equiv. to a Hesse cubic, *i.e.*, $G(3)$-invariant.

(3) it is smooth elliptic or a circle of 3 lines (3-gon).

Th 17 (N.'99) Let X be a degenerate abelian variety (including the case when X is an abelian variety)

The following are equiv. under natural assump.:

(1) it has a closed SL(V)-orbit.

(2) X is invariant under $G(K)$.

(3) it is one of the above PSQASes.

10 Moduli over $\mathbf{Z}[\zeta_N, 1/N]$

Th 18 (The theorem of Hesse) (a new version)

The projective moduli $SQ_{1,3} \simeq \mathbf{P}^1$ over $\mathbf{Z}[\zeta_3, 1/3]$

(1) The univ. cubic $\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$

$(\mu_0, \mu_1) \in SQ_{1,3} = \mathbf{P}^1$

(2) when k is alg. closed and char. $k \neq 3$

$$SQ_{1,3}(k) = \left\{ \begin{array}{l} \text{closed orbit cubic curves} \\ \text{with level 3-structure} \end{array} \right\} / \text{isom.}$$

$$= \left\{ \begin{array}{l} \text{Hesse cubics} \\ \text{with level 3-structure} \end{array} \right\}$$

Th 19 (N.'99) (High dim. version)

Let K finite symplectic, $\forall$ elm. div. of $K \geq 3$. There exists a fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$ where $N = |K|$. For k : alg.closed, char.k and N:coprime

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{degenerate abelian varieties} \\ \text{with level } G(K)\text{-structure} \\ \text{and a closed SL-orbit} \end{array} \right\} / \text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant degenerate} \\ \text{abelian varieties} \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant PSQASes} \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

A very similar complete moduli was constructed by

Alexeev

$\overline{A_{g,1}}$ **over Z**

11 Faltings-Chai degeneration data

R : a discrete valuation ring R,

m the max. ideal of R, $\quad k(0) = R/m$

$k(\eta)$: the fraction field of R

Let (G, L) an abelian scheme over R,

(G_η, L_η) : abelian variety over $k(\eta)$

$({}^tG, {}^tL)$: the (connected) Neron model of $({}^tG_\eta, {}^tL_\eta)$

Suppose G_0 is a split torus over $k(0)$,

May then suppose that

$({}^tG_0, {}^tL_0)$ is a split torus over $k(0)$

Then we have a Faltings Chai degeneration data

Let $X = \mathrm{Hom}(G_0, \mathrm{G}_m), \quad Y = \mathrm{Hom}({}^tG_0, \mathrm{G}_m)$

G_m : a 1-dim. torus

$$\Downarrow$$

$X \simeq \mathrm{Z}^g$, $Y \simeq \mathrm{Z}^g$

Y : a sublattice of X of finite index.

BECAUSE $\exists$ a natural surjective morphism $G \to {}^t G$,

$\exists$ a surjective morphism $G_0 \to {}^t G_0$,

$\exists$ $\mathrm{Hom}({}^tG_0, \mathrm{G}_m) \to \mathrm{Hom}(G_0, \mathrm{G}_m)$,

Hence $\exists$ an injective homom. $Y \to X$ $\square$

Rem

$$K = X/Y \oplus (X/Y)^{\vee},$$

$G(K)$: Heisenberg group (suitably defined)

$H^0(G, L)$: a finite R-module

an "irreducible" $G(K)$ -module

$\Rightarrow$ an ess. unique basis

θ_k of $H^0(G, L)$

Let G_{for} : form. compl. of G

$G_{\text{for}} \simeq (\mathbb{G}_{m,R})_{\text{for}}$

Theta functions θ_k $(k \in X/Y)$ (nat. basis of $H^0(G, L)$) are expanded as

$$\theta_k = \sum_{y \in Y} a(x+y) w^{x+y}$$

Rem Theta θ_k $(k \in X/Y)$ are expanded as

$$\theta_k = \sum_{y \in Y} a(x+y) w^{x+y}$$

These $a(x)$ satisfy the conditions:

(1) $a(0) = 1$, $a(x) \in k(\eta)^\times \quad (\forall x \in X)$,

(2) $b(x, y) := a(x+y) a(x)^{-1} a(y)^{-1}$

is bilinear $(x, y \in X)$

(3) $B(x, y) := \mathrm{val}_q(a(x+y) a(x)^{-1} a(y)^{-1})$

is positive definite $(x, y \in X)$, e.g. $B = E_8$

Def 20 $a(x)$ are called

a Faltings-Chai degeneration data of (G, L)

Rem In the complex case

$$a(x) = e^{2\pi\sqrt{-1}(x,Tx)},$$
$$b(x,y) = e^{2\pi\sqrt{-1}\cdot 2(x,Ty)}$$

where T : symm. and

$$b(x,y) = a(x+y)a(x)^{-1}a(y)^{-1}$$

Theta functions ($k \in X/Y$)

$$\theta_k(\tau, z) = \sum_{y\in Y} a(y+k)w^{y+k}$$
$$= \sum_{m\in \mathbf{Z}} q^{(3m+k)^2} w^{3m+k} \quad \text{(Hesse cubics)}$$

Def 21

$$\widetilde{R} := R[a(x)w^x\vartheta, x \in X]$$

an action of Y on $\widetilde{R}$ by

$$S_y(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta$$

$\mathrm{Proj}(\widetilde{R})$: locally of finite type over R

$\mathcal{X}$: the formal completion of $\mathrm{Proj}(\widetilde{R})$

$\mathcal{X}/Y$: the top. quot. of $\mathcal{X}$ by Y

$O_{\mathcal{X}}(1)$ descends to $\mathcal{X}/Y$: ample

Grothendieck (EGA) guarantees

$\exists$ a projective R-scheme $(Z, O_Z(1))$

s.t. the formal completion Z_{for}

$$Z_{\text{for}} \simeq \mathcal{X}/Y$$

$$(Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$$

(the stable reduction theorem)

This algebraizes the quotient $\mathrm{G}_m(K)/Y$,

which generalizes the Tate curve.

Ex 22 $g=1,\ X=\mathbf{Z},\ Y=3\mathbf{Z}.$

$$a(x)=q^{x^2},\ (x\in X)$$

$$\mathcal{X}=\mathrm{Proj}(\widetilde{R})$$

The scheme $\mathcal{X}$ is covered with affine

$$V_n=\mathrm{Spec}R[a(x)w^x/a(n)w^n,x\in X]$$

$$V_n\simeq\mathrm{Spec}R[x_n,y_n]/(x_ny_n-q^2)$$

$$(n\in\mathbf{Z})$$

$$x_n=q^{2n+1}w,\ y_n=q^{-2n+1}w^{-1}.$$

$$(V_n)_0=\{(x_n,y_n)\in k(0)^2;x_ny_n=0\}$$

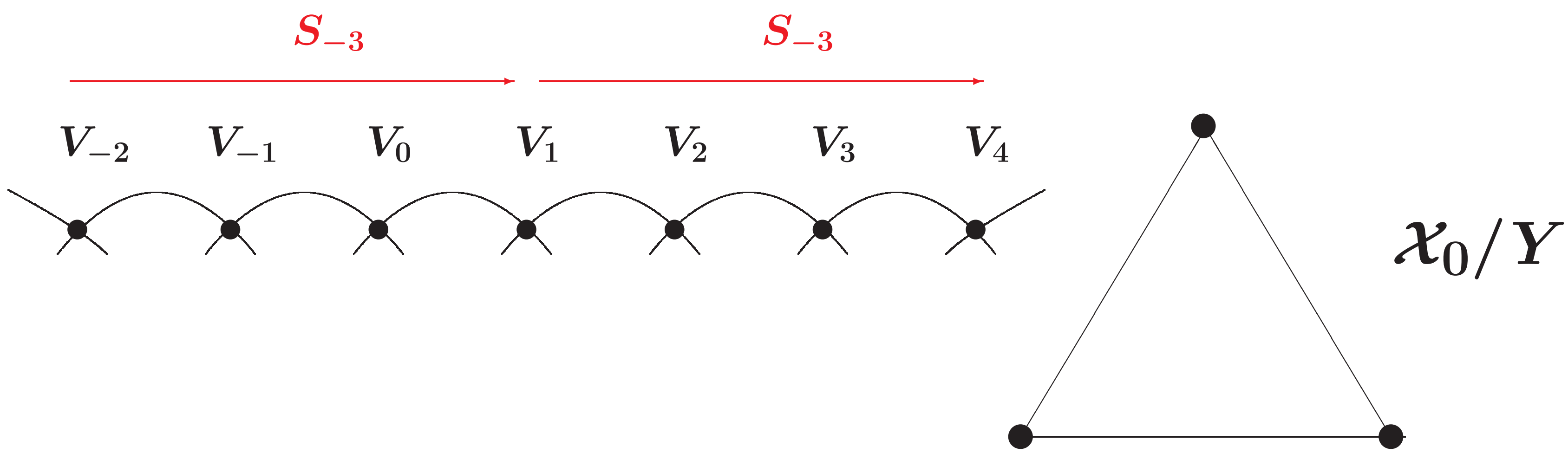
$\mathcal{X}_0$: a chain of infinitely many $\mathbf{P}^1_{k(0)}$

$$Y \text{ acts on } \mathcal{X}_0 \text{ as } V_n \overset{S_{-3}}{\to} V_{n+3},$$

$$(x_n, y_n) \overset{S_{-3}}{\mapsto} (x_{n+3}, y_{n+3}) = (x_n, y_n)$$

$$\mathcal{X}_0/Y : \text{ a cycle of 3 } \mathrm{P}^1_{k(0)}$$

$$(\mathcal{X}/Y)^{\mathrm{alg}}_\eta : \text{ a Hesse cubic over } k(\eta),$$



12 Limits of theta functions

$E(\tau)$ is embedded in P^2 by theta θ_k :

$$\theta_k(q, w) = \sum_{m \in \mathrm{Z}} q^{(3m+k)^2} w^{3m+k}$$

$q = e^{2\pi i \tau/6}$, $w = e^{2\pi i z}$.

$$\theta_0^3 + \theta_1^3 + \theta_2^3 = 3\mu(q)\theta_0\theta_1\theta_2$$

$X = \mathrm{Z}$ and $Y = 3\mathrm{Z}$

$$\theta_k = \sum_{y \in Y} a(y+k) w^{y+k}$$

Let $R = \mathrm{C}[[q]]$, $I = qR$, $w = q^{-1}u$

$u \in R \setminus I$, $\overline{u} = u \mod I$

$$\theta_k = \sum_{y \in Y} a(y+k) w^{y+k}$$

Wish to compute the limits

$$\lim_{q\to 0}[\theta_0, \theta_1, \theta_2]$$

$$\theta_0(q,w) = \sum_{m\in Z} q^{9m^2} w^{3m} = 1 + q^9 w^3 + q^9 w^{-3} + \cdots$$

$$\theta_1(q,w) = \sum_{m\in Z} q^{(3m+1)^2} w^{3m+1} = qw + q^4 w^{-2} + \cdots$$

$$\theta_2(q,w) = \sum_{m\in Z} q^{(3m+2)^2} w^{3m+2} = qw^{-1} + q^4 w^2 + \cdots)$$

$$\lim_{q\to 0}[\theta_0, \theta_1, \theta_2] = [1,0,0]$$

This also leads to

$\lim_{\tau\to\infty} E(\tau) = [1,0,0]$ 0-dim. ????? **Wrong!**

Correct computation

$$\theta_0(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{9m^2-3m} u^{3m}$$

$$= 1 + q^6u^3 + q^{12}u^{-3} + \cdots$$

$$\theta_1(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{(3m+1)^2-3m-1} u^{3m+1}$$

$$= u + q^6u^{-2} + q^{12}u^4 + \cdots$$

$$\theta_2(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{(3m+2)^2-3m-2} u^{3m+2}$$

$$= q^2 \cdot (u^2 + u^{-1} + q^{18}u^5 + q^{18}u^{-4} + \cdots)$$

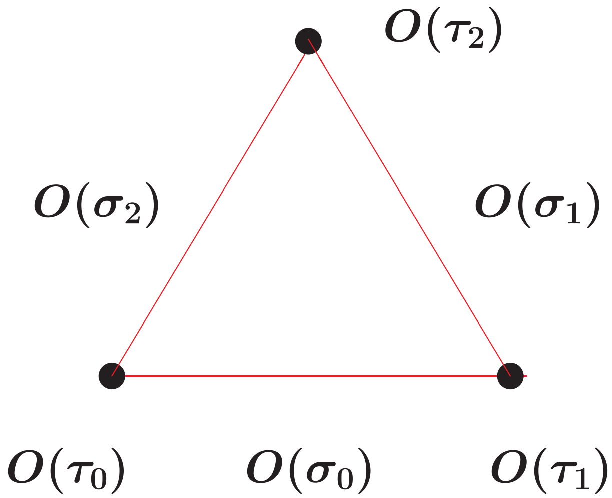
$$\lim_{q\to 0} [\theta_k(q, q^{-1}u)]_{k=0,1,2} = [1, \overline{u}, 0] \in \mathbb{P}^2$$

In $\mathbf{P}^2$

$$\lim_{q\to 0}\,[\theta_k(q, q^{-1}u)]_{k=0,1,2} = [1, \overline{u}, 0]$$

$$\lim_{q\to 0}\,[\theta_k(q, q^{-3}u)]_{k=0,1,2} = [0, 1, \overline{u}]$$

$$\lim_{q\to 0}\,[\theta_k(q, q^{-5}u)]_{k=0,1,2} = [\overline{u}, 0, 1]$$

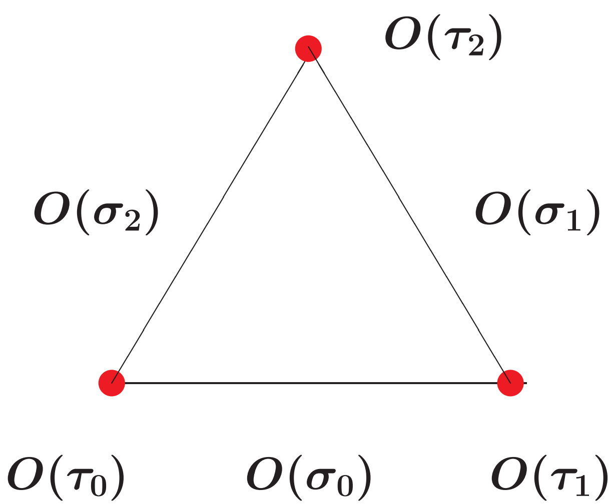


$w = q^{-2\lambda}u$ and $u \in R \setminus I$.

$\lim_{q\to 0}\,[\theta_k(q, q^{-2\lambda}u)] =$

$$\begin{cases} [1,0,0] & (\text{if } -1/2 < \lambda < 1/2), \\ [0,1,0] & (\text{if } 1/2 < \lambda < 3/2), \\ [0,0,1] & (\text{if } 3/2 < \lambda < 5/2). \end{cases}$$

We get a 3-gon



Limits of thetas are described by combinatorics.

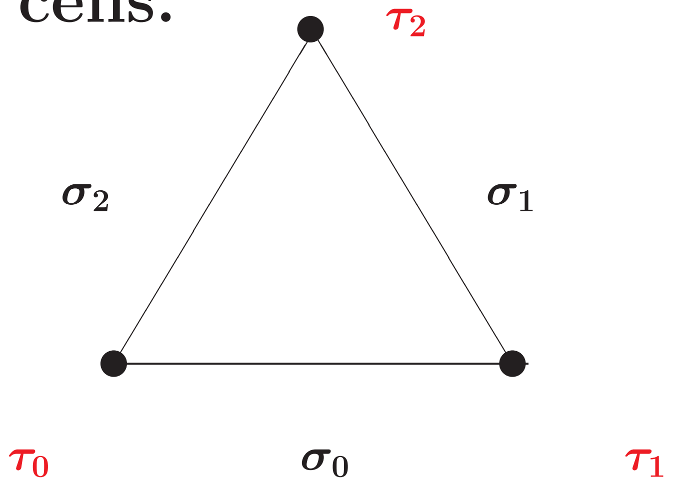
Def 23 For $\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}$ fixed

$$F_\lambda(x) = x^2 - 2\lambda x \quad (x \in X = \mathbb{Z})$$

Define $D(\lambda)$ (a Delaunay cell) by the conv. closure of all $a \in X$ s.t. $F_\lambda(a) = \min\{F_\lambda(x); x \in X\}$

e.g. σ_i, τ_j are Delaunay cells.

Delaunay decomposition



13 PSQAS and its shape

PSQAS is a geometric limit of theta functions, a generalization of 3-gons.

"Limits of theta functions, and PSQASes are described by the Delaunay decomposition."

Almost the same as PSQAS was already introduced by Namikawa and Nakamura (1975)

Delaunay decomposition was considered in the study of quadratic forms at the beginning of the last century (Voronoi 1908).

Assume that $X = \mathbf{Z}^g$, and let B a positive symmetric integral bilinear form on $X \times X$.

$\|x\| = \sqrt{B(x,x)}$: a distance of $X \otimes \mathbf{R}$ (fixed)

Def 24 D is a Delaunay cell if for some $\alpha \in X \otimes \mathbf{R}$ D is a convex closure of a lattice (X point) which is closest to α

It depends on $\alpha \in X \otimes \mathbf{R}$, we discribe it as $D = D(\alpha)$ If $\alpha \in X$, $D = \{\alpha\}$. All the Delaunay cells constitute a polyhedral decomp. of $X \otimes_{\mathbf{Z}} \mathbf{R}$ the Delaunay decomposition ass. to B

Each PSQAS,

and its decomposition into torus orbits

(its stratification), is described

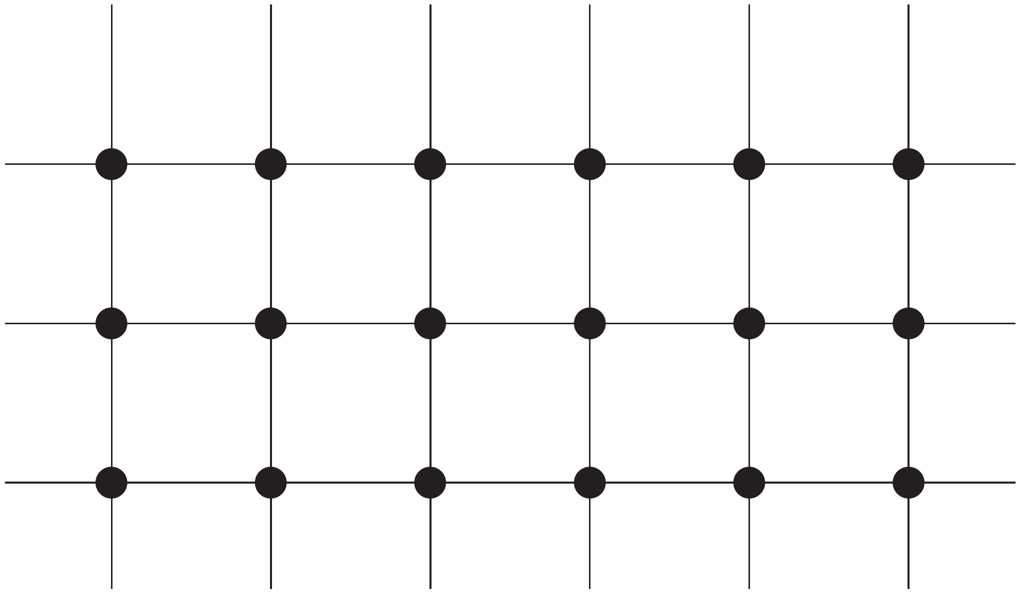
by a Delaunay decomposition

Each positive B defines a Delaunay decomposition,

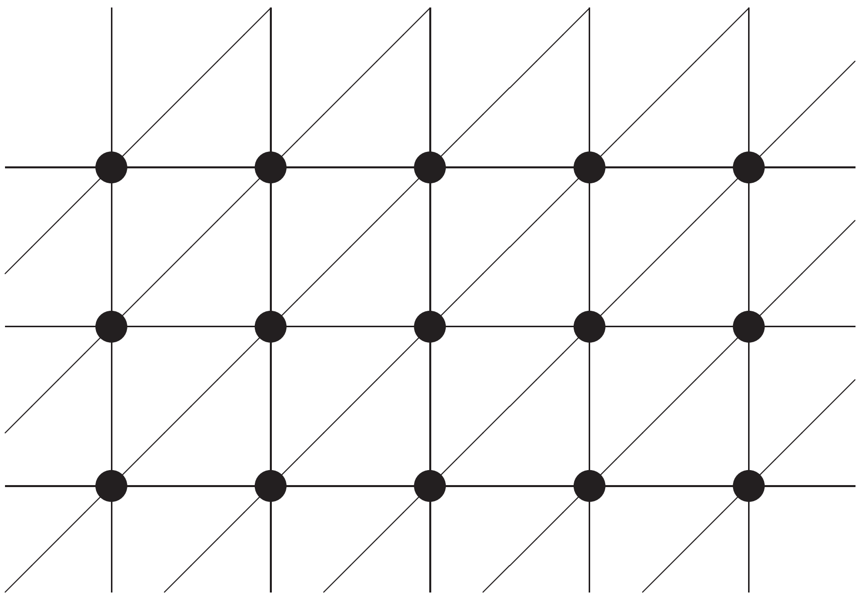
Different B can correspond to the same Delaunay

decomp. and the same PSQAS.

Ex 25 This decomp. (mod Y) is a PSQAS, a union of $\mathrm{P}^1 \times \mathrm{P}^1$ for $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.



For $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, the decomp. below (mod Y) is a PSQAS. It is a union of P^2, each triangle denotes a P^2, 6 P^2 intersects at a point, while each line segment is a P^1.



15 Cohomology groups

Th 26 (Sugawara and N. 2006) Let (Q_0, L_0) be a PSQAS. Then $H^q(Q_0, L_0^n) = 0$ for any $q, n > 0$.

Cor 27 $H^0(Q_0, L_0)$ is irred. $G(K)$-module of wt. one.

Th 28 (Q_0, L_0) is GIT-stable in the sense that the $\mathrm{SL}(V \otimes k)$-orbit of any of the Hilbert points of (Q_0, L_0) is closed in the semistable locus.

To construct the moduli $SQ_{g,K}$ a weaker form of this theorem was sufficient.

Cor 29 (a valuative criterion for separatedness of the moduli) Let R be a complete DVR. For two proper flat families of PSQASes with $G(K)$-actions $(Q, L, G(K))$ and $(Q', L', G(K))$ over R, assume $\exists\, G(K)$-isomorphism

$$\phi_\eta : (Q, L, G(K)) \to (Q', L', G(K))$$

over $k(\eta)$:the fraction field of R.
Then ϕ_η extends to a $G(K)$-isomorphism over R.

Rem We note that the isomorphism class V of irreducible $G(K)$-modules of weight one is unique.

The proof of Corollary 29 goes roughly as follows.

Note that L and L' are very ample.

Hence the isom. ϕ_η is an element of $\mathrm{GL}(V \otimes k(\eta))$ which commutes with $G(K)$-action.

Since V is irreducible over $k(\eta)$,

by the lemma of Schur, ϕ_η is a scalar matrix,

which reduces to the identity of $\mathrm{P}(V \otimes k(\eta))$,

hence extends to the identity of $\mathrm{P}(V \otimes R)$.

16 Degeneration associated with E_8

Assume B is unimodular and even positive definite. Then (Q_0, L_0) is nonreduced anywhere, but GIT-stable.

Th 30 Let $B = E_8$. Assume $X = Y$ for simplicity. Then $Q_0 = (V'_1 + \cdots + V'_{135}) + (V''_1 + \cdots + V''_{1920})$, each V'_j (each V''_k) isom. resp. along which generically

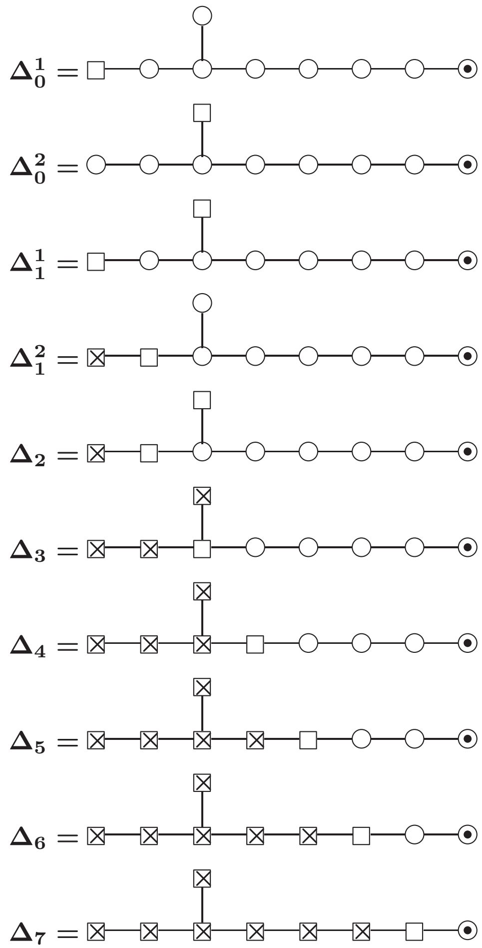
$$Q_0 : x^2 = 0 \text{ along } V'_j, \quad Q_0 : y^3 = 0 \text{ along } V''_k,$$
$$(L_0)^8_{V'_j} = 2^7 = 128, \quad (L_0)^8_{V''_k} = 1. \qquad \square$$

Hence (Q_η, L_η) is principally polar. with

$$L_0^8 = 135 \cdot 2 \cdot 128 + 1920 \cdot 3 = 40320 = 8! = L_\eta^8.$$

17 The Wythoff-Coxeter construction

The Delaunay decomposition of E_8 is described by decorated diagrams:



18 Voronoi cells $V(0)$

Def 31 for a Delaunay cell D :

$$V(D) := \{\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}; D = D(\lambda)\}$$

We call it a Voronoi cell.

$$\{V(D); D : \text{a Delaunay cell}\}$$

is a (Voronoi) decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$

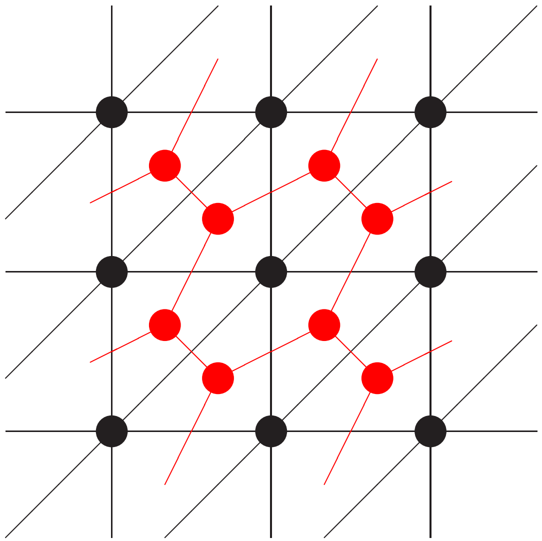
$$\begin{aligned}\overline{V(0)} &= \{\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R};\ \|\lambda\| \leqq \|\lambda - q\|, (\forall q \in X)\} \\ &= \{\lambda; \text{the nearest lattice pt. to } \lambda \text{ is the origin}\}\end{aligned}$$

Once we know $V(0)$, then we see Delaunay decomp.

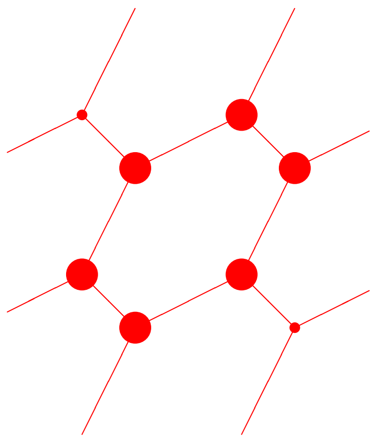
For $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

the red decomp. is Voronoi,

the black decomp. is Delaunay.

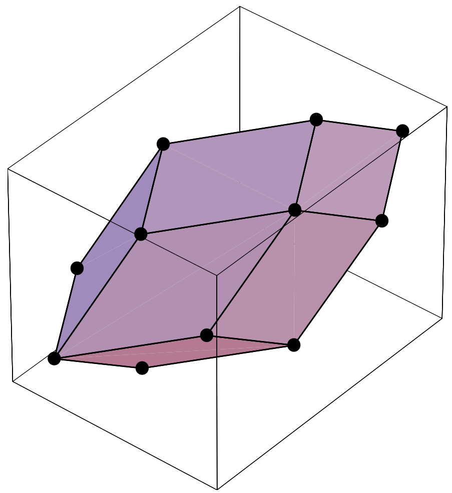


The following is a 2-dim Voronoi cell $V(0)$ (a Red Hexagon) for $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$



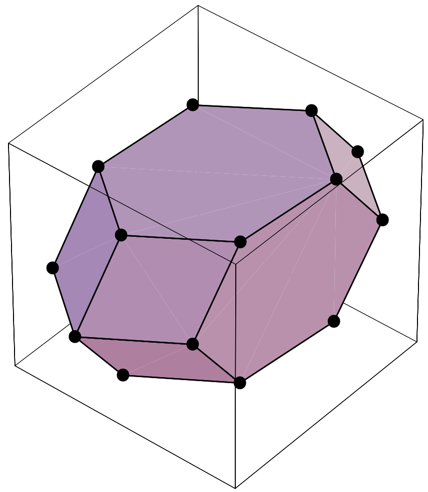
$$\textbf{For } \boldsymbol{B} = \begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

$V(0)$ is a Dodecahedron (Garnet)



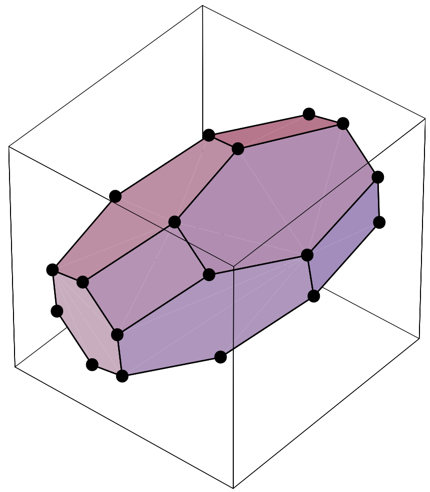
$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Apophyllite $KCa_4(Si_4O_{10})_2F \cdot 8H_2O$



$$B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

A Truncated Octahedron (Zinc Blende ZnS)



Thank you for your attention