Modular Varieties and Hecke Symmetry

Ching-Li Chai

Sapporo, September 2007

§1 Hecke symmetry on modular varieties

Unramified PEL data $(B, *, \mathcal{O}_B, V, V_{\mathbb{Z}_p}, \langle\cdot,\cdot\rangle, h)$:

- B – a finite dim. semisimple $\mathbb{Q}$-algebra unramified at p,
- $\mathcal{O}_B$ – a maximal order of B maximal at p,
- $*$ – a positive involution on B preserving $\mathcal{O}_B$,
- V – a B-module of finite dimension over $\mathbb{Q}$,
- $\langle\cdot,\cdot\rangle$ – a $\mathbb{Q}$-valued nondegen. alternating form on V compatible with $(B, *)$,
- $V_{\mathbb{Z}_p}$ – a self-dual $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ stable under $\mathcal{O}_B$
- $h : \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$, a $*$-homomorphism s.t.

$$(v, w) \ \mapsto \ \langle v, h(\sqrt{-1})w\rangle$$

is a pos. definite symmetric form on $V_\mathbb{R}$

Modular varieties of PEL type

Given an unramified PEL data $\rightsquigarrow$

- G = unitary group attached to $(\mathrm{End}_B(V), *)$,
- $\widetilde{\mathcal{M}} = \left(\mathcal{M}_{K^p}\right)$, a tower of modular varieties over $\mathbb{F}$ indexed by the set of all compact open subgroups K^p of $G(\mathbb{A}_f^p)$, where
  - $\mathbb{A}_f^p = \prod'_{\ell \neq p} \mathbb{Q}_\ell$
  - $\mathcal{M}_{K^p}$ classifies abelian varieties with endomorphisms by $\mathcal{O}_B$, plus prime-to-p polarization and level structure, whose H_1 is modeled on the given PEL datum.

HECKE SYMMETRIES

(1) The group $G(\mathbb{A}_f^p)$ operates on the projective system $\widetilde{\mathcal{M}}$.

(2) If a level subgroup K_0^p is fixed, then on $\mathcal{M}_{K_0^p}$ the remnant from the action of $G(\mathbb{A}_f^p)$ takes the form of a family of finite étale algebraic correspondences on $\mathcal{M}_{K_0^p}$; they are known as *Hecke correspondences*.

(3) Given a point $x \in \mathcal{M}_{K_0^p}(\mathbb{F})$, let $\tilde{x}$ be a lift of x in $\widetilde{\mathcal{M}}(\mathbb{F})$. Define the *prime-to-$p$ Hecke orbit* $\mathcal{H}^p \cdot x$ of x to be the image in $\mathcal{M}_{K_0^p}(\mathbb{F})$ of the $G(\mathbb{A}_f^p)$-orbit of $\tilde{x}$; it is a countable set.

EXAMPLE. Siegel modular varieties $\mathcal{A}_{g,n}$, $(n,p)=1$, $n\geq 3$

- $\mathcal{A}_{g,n}$ classifies g-dimensional principally polarized abelian varieties (A,λ) with a symplectic level-n structure η.

- Two $\mathbb{F}$-points $[(A_1,\lambda_1,\eta_1)]$, $[(A_2,\lambda_2,\eta_2)]$ in $\mathcal{A}_{g,n}$ are in the same prime-to-p Hecke orbit iff $\exists$ a prime-to-p quasi-isogeny β (="$\beta_2\circ\beta_1^{-1}$")

$$\beta : A_1 \xleftarrow{\beta_1} A_3 \xrightarrow{\beta_2} A_2$$

  defined by prime-to-p isogenies β_1 and β_2 s.t. β respects the principal polarizations λ_1 and λ_2, i.e. $\beta_1^*(\lambda_1)=\beta_2^*(\lambda_2)$.

PEL datum:

$B=\mathbb{Q}$, V = $2g$-dim. v.s. over $\mathbb{Q}$, $G=\mathrm{Sp}_{2g}$.

EXAMPLE. Hilbert modular varieties $\mathcal{M}_{E,d,n}$

$F_1, \ldots, F_r$: totally real number fields,

$E = F_1 \times \cdots \times F_r,\ \mathcal{O}_E = \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_r}$,

$d, n \geq 1$, integers, $\gcd(dn, p) = 1$.

Hilbert modular variety $\mathcal{M}_{E,d,n}$ over $\mathbb{F}$:
classifies quadruples $(A \to S, \iota, \lambda, \eta)$, where

- $A \to S$ is an abelian scheme,
  $\dim(A \to S) = [E : \mathbb{Q}]$,
- $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ is a ring homomorphism,
- λ is an $\mathcal{O}_E$-linear polarization on A of degree d,
- η is a level-n structure.

PEL datum:
$B = E$, V = free E-module of rank two,
$G = \prod_{E/\mathbb{Q}} \mathrm{SL}_2$.

$\mathcal{M} = \mathcal{M}_{K_0^p}$, a modular variety of PEL type over $\mathbb{F}$

$x_0 = [(A_0, \lambda_0, \iota_0, \eta_0)] \in \mathcal{M}(\mathbb{F})$

DEF 1. The *leaf* $\mathcal{C}_{\mathcal{M}}(x_0)$ in $\mathcal{M}$ passing through x_0 is the reduced locally closed subscheme of $\mathcal{M}$ *smooth* over $\mathbb{F}$ such that $\mathcal{C}_{\mathcal{M}}(x_0)(\mathbb{F})$ consists of all points $x = [(A, \lambda, \iota, \eta)] \in \mathcal{C}_{\mathcal{M}}(x_0)(\mathbb{F})$ s.t.

$$(A, \lambda, \iota)[p^\infty]) \cong (A_0, \lambda_0, \iota_0)[p^\infty]\,,$$

where $(A, \lambda, \iota)[p^\infty]$ is the $\mathcal{O}_B$-linear polarized p-divisible group attached to (A, λ, ι).

Oort's Hecke orbit conjecture

CONJ 1 (HO). *Every prime-to-p Hecke orbit in a modular variety of PEL type $\mathcal{M}$ over $\mathbb{F}$ is dense in the leaf in $\mathcal{M}$ containing it.*

CONJ ($\mathrm{HO}_{\mathrm{ct}}$). The closure of any prime-to-p Hecke orbit in the leaf $\mathcal{C}$ containing it is an open-and-closed subset of $\mathcal{C}$, i.e. it is a union of irreducible components of the smooth variety $\mathcal{C}$.

CONJ ($\mathrm{HO}_{\mathrm{dc}}$). Every prime-to-$p$ Hecke orbit in a leaf $\mathcal{C}$ meets every irreducible component of $\mathcal{C}$.

Clearly HO $\Longleftrightarrow$ $\mathrm{HO}_{\mathrm{ct}}$ + $\mathrm{HO}_{\mathrm{dc}}$.

Evidence of HO: Known for Siegel modular varieties (F. Oort, C.-F. Yu and CLC).

Need new ideas for the general case of HO.

Question 2. *(B. Poonen) Let* $x_0 \in \mathcal{M}(\mathbb{C}_p)$ *be a* $\mathbb{C}_p$*-point of* $\mathcal{M}$*, where* $\mathbb{C}_p$ *is the completion of* $\overline{\mathbb{Q}_p}$*. Is the Hecke orbit of* x_0 **nowhere dense** *in* $\mathcal{M}(\mathbb{C}_p)$*?*

We discuss two topics related to Hecke symmetry

- monodromy
- CM-lifting

The first is closely related to the Conj. HO.

§2. Monodromy

2A. ℓ-adic monodromy

Let $Z(x_0)$ be the Zariski closure of the prime-to-p Hecke orbit of x_0 for the group $G_{\mathrm{der}}^{\mathrm{sc}}$ in the leaf $\mathcal{C}(x_0)$.

THM 1. *Assume that the prime-to-p Hecke orbit of x_0 with respect to every simple factor of $G_{\mathrm{der}}^{\mathrm{sc}}$ is infinite. Then $Z(x_0)$ is irreducible, and the Zariski closure of the ℓ-adic monodromy group of $Z(x_0)$ is $G_{\mathrm{der}}(\mathbb{Q}_\ell)$ for every prime number $\ell \neq p$.*

Note. Irreducibility of $Z(x_0)$ uses: $G_{\mathrm{der}}^{\mathrm{sc}}(\mathbb{Q}_\ell)$ has no proper subgroup of finite index.

We restrict to the Siegel modular case. Let $Z \subseteq \mathcal{C} = \mathcal{C}(x_0)$ be an irreducible smooth subvariety contained in a leaf $\mathcal{C} \subset \mathcal{A}_{g,n}$, stable under all prime-to-p Hecke correspondences. Write $x_0 = [(A_0, \lambda_0, \eta_0)] \in \mathcal{A}_{g,n}(\mathbb{F})$. The p-adic monodromy for Z is a homomorphism

$$\rho_{Z,x_0} : \pi_1(Z', x_0) \to \mathrm{Aut}((A_0, \lambda_0)[p^\infty])$$

Let U_{x_0} be the unitary group attached to $(\mathrm{End}^0(A_0), *_0)$, and denote by H_{x_0} the subgroup consisting of all $U_{x_0}(\mathbb{Q}_p)$ which preserves the lattice $\mathrm{End}(A_0) \otimes \mathbb{Z}_p$ in $\mathrm{End}(A_0) \otimes \mathbb{Q}_p$.

THM 2. *The image of ρ_{Z,x_0} contains the image of H_{x_0} in $\mathrm{Aut}((A_0, \lambda_0)[p^\infty])$.*

B: a simple algebra over $\mathbb{Q}$, $\mathcal{O}_B$: an order of B.

$k \supset \mathbb{F}_p$, $k = k^{\mathrm{alg}}$.

DEF 2. (i) An $\mathcal{O}_B$-linear abelian variety (A, ι) over k is B-*hypersymmetric*, or *hypersymmetric* for short, if the canonical map

$$\mathrm{End}_{\mathcal{O}_B}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_{\mathcal{O}_B}(A[p^\infty])$$

is an isomorphism.

COR 3. *Let x_0 be a hypersymmetric point in a leaf $\mathcal{C}$, and let $Z \subset \mathcal{C}(x_0)$ be an irreducible smooth subvariety containing x_0 and stable under all prime-to-p Hecke correspondences. Then the image of the p-adic monodromy for Z is equal to $\mathrm{Aut}(A_0[p^\infty], \lambda_0[p^\infty])$.*

§3. CM-lifting

Let $\mathbb{F}_q$ be a finite field of size q, and let B be an abelian variety of dimension $g > 0$ over $\mathbb{F}_q$. Assume that B is isotypic over $\mathbb{F}_q$. Consider the following four assertions concerning the existence of a CM-lifting of B.

(CMLR) *CM-lifting after finite residue field extension*: $\exists$ a local domain R with char. 0 and finite residue field $\kappa \supset \mathbb{F}_q$, an abelian scheme A over R of rel. dim. g plus an action with $[K : \mathbb{Q}] = 2g$, and an isom. $\phi : A \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\kappa) \simeq B_\kappa$ over κ.

(CMLI) *CM-lifting up to isogeny*: $\exists$ a local domain R with char. 0 and residue field $\mathbb{F}_q$, an abelian scheme A over R with rel. dim. g plus an action by a CM field K with $[K : \mathbb{Q}] = 2g$, and an isogeny $A \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\mathbb{F}_q) \sim B$ over $\mathbb{F}_q$.

(CMLNI) *CM-lifting to normal domains up to isogeny*: $\exists$ a normal local domain R with char. 0 and residue field $\mathbb{F}_q$ such that (CMLI) is satisfied for B using R.

(CMLNIR) *CM-lifting to normal domains up to isogeny after finite residue field extension*: $\exists$ a normal local domain R with char. 0 and finite residue field $\kappa \supset \mathbb{F}_q$ such that (CMLR) is satisfied for B using R except that ϕ is only required to be an isogeny over κ rather than an isomorphism.

KNOWN: (CMLNIR) is true, (CMLR) is false.

Will explain an obstruction to (CMLNI). This obstruction can be non-trivial, but it is the only obstruction.

RESIDUAL REFLEX CONDITION: If (CMLNI) holds for a g-dimensional abelian variety B over $\mathbb{F}_q$, then there is a CM subfield $K \subseteq \mathrm{End}^0_{\mathbb{F}_q}(B)$ with $[K:\mathbb{Q}] = 2g$ and a p-adic CM type $\Phi \subseteq \mathrm{Hom}_{\mathrm{ring}}(K, \overline{\mathbb{Q}}_p)$ s.t.

(i) The slopes of B are given in terms of (K, Φ) by the Shimura–Taniyama formula

$$\frac{\mathrm{ord}_v(\mathrm{Fr}_{B,q})}{\mathrm{ord}_v(q)} = \frac{\#\{\phi \in \Phi : \phi \text{ induces } v \text{ on } K\}}{[K_v : \mathbb{Q}_p]}$$

for every place v of K above p.

(ii) Let $E \subseteq \overline{\mathbb{Q}}_p$ be the reflex field attached to (K, Φ), and let w be the induced p-adic place of E. The residue field κ_w of $\mathcal{O}_{E,w}$ can be realized as a subfield of $\mathbb{F}_q$.

THM 4. (F. Oort, B. Conrad, CLC) *Let B be an abelian variety of dimension $g > 0$ over $\mathbb{F}_q$ and let $K \subseteq \mathrm{End}^0_{\mathbb{F}_q}(B)$ be a CM field with $[K : \mathbb{Q}] = 2g$. Let $\Phi \subseteq \mathrm{Hom}_{\mathrm{ring}}(K, \overline{\mathbb{Q}}_p)$ be a p-adic CM type on K, and let $E \subseteq \overline{\mathbb{Q}}_p$ be the associated reflex field. Assume that (K, Φ) satisfies the residual reflex condition.*

There exists a finite extension E'/E inside of $\overline{\mathbb{Q}}_p$, a g-dimensional abelian variety A over E' with good reduction at the p-adic place w' on E' induced by $\overline{\mathbb{Q}}_p$, and an inclusion $K \hookrightarrow \mathrm{End}^0_{E'}(A)$ with associated p-adic CM-type Φ such that the reduction of A at w' is K-linearly isogenous to B over an isomorphism of finite fields $\kappa_{w'} \simeq \mathbb{F}_q$. In particular, B satisfies (CMLNI) *using a lifting of the K-action over a p-adic integer ring with residue field $\mathbb{F}_q$.*

Remark. An analog of Thm. 4 holds for modular varieties of PEL-type: a suitable residual reflex condition is both necessary and sufficient for the existence of CM lifting over normal domains up to Hecke correspondence.

Question 3. *Does (CMLI) hold for every isotypic abelian variety over a finite field?*

No counter-example is known for (CMLNI).