An Algorithm for Hyperbolicity Verification and its Application

May 19, 2006

Lorenz Center

Zin ARAI

Kyoto University

arai@math.kyoto-u.ac.jp

1 Introduction

Problem: Given a family of dynamical systems, determine the set of hyperbolic and non-hyperbolic parameters.

1 Introduction

Problem: Given a family of dynamical systems, determine the set of hyperbolic and non-hyperbolic parameters.

uniform hyperbolicity $\Rightarrow$ Markov partition, symbolic dynamics, topological entropy...

1 Introduction

Problem: Given a family of dynamical systems, determine the set of hyperbolic and non-hyperbolic parameters.

uniform hyperbolicity $\Rightarrow$ Markov partition, symbolic dynamics, topological entropy...

tangency $\Rightarrow$ Newhouse phenomena, strange attractor...

1 Introduction

Problem: Given a family of dynamical systems, determine the set of hyperbolic and non-hyperbolic parameters.

uniform hyperbolicity $\Rightarrow$ Markov partition, symbolic dynamics, topological entropy...

tangency $\Rightarrow$ Newhouse phenomena, strange attractor...

However, proving hyperbolicity is a very difficult problem even for simple low-dimensional dynamical systems.

2 Results

Example: the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x)$$

where $a, b \in \mathbb{R}$ are the parameters.

2 Results

Example: the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x)$$

where $a, b \in \mathbb{R}$ are the parameters.

History:
[Devaney and Nitecki 79] For any fixed $b \neq 0$, if a is sufficiently large then $\Omega(H_{a,b})$ is uniformly hyperbolic.

2 Results

Example: the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x)$$

where $a, b \in \mathbb{R}$ are the parameters.

History:

[Devaney and Nitecki 79] For any fixed $b \neq 0$, if a is sufficiently large then $\Omega(H_{a,b})$ is uniformly hyperbolic.

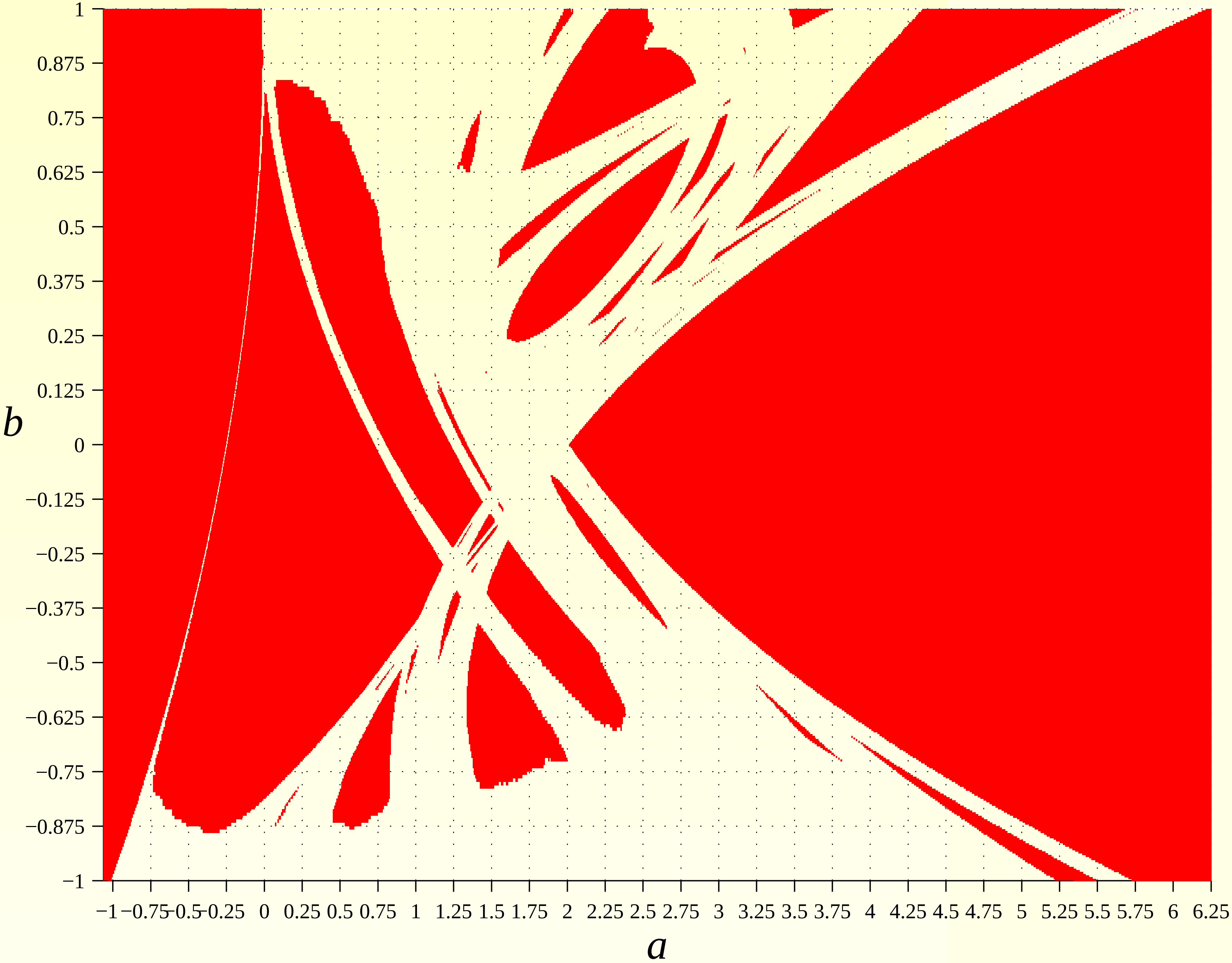
[Davis, MacKay and Sannami 91] Conjectured that $\Omega(H_{a,-1})$ is uniformly hyperbolic if a is in some intervals, which are not included in the region given by DN.

Denote by $\mathcal{R}(H_{a,b})$ the chain recurrent set of $H_{a,b}$.

Note that $\Omega(H_{a,b}) \subset \mathcal{R}(H_{a,b})$ and all of the chaotic orbits are contained in $\mathcal{R}(f)$.

Denote by $\mathcal{R}(H_{a,b})$ the chain recurrent set of $H_{a,b}$.
Note that $\Omega(H_{a,b}) \subset \mathcal{R}(H_{a,b})$ and all of the chaotic orbits are contained in $\mathcal{R}(f)$.

Theorem 1 (hyperbolicity of the real Hénon maps).
If (a, b) is in the red region of the figure, then $\mathcal{R}(H_{a,b})$ is uniformly hyperbolic.

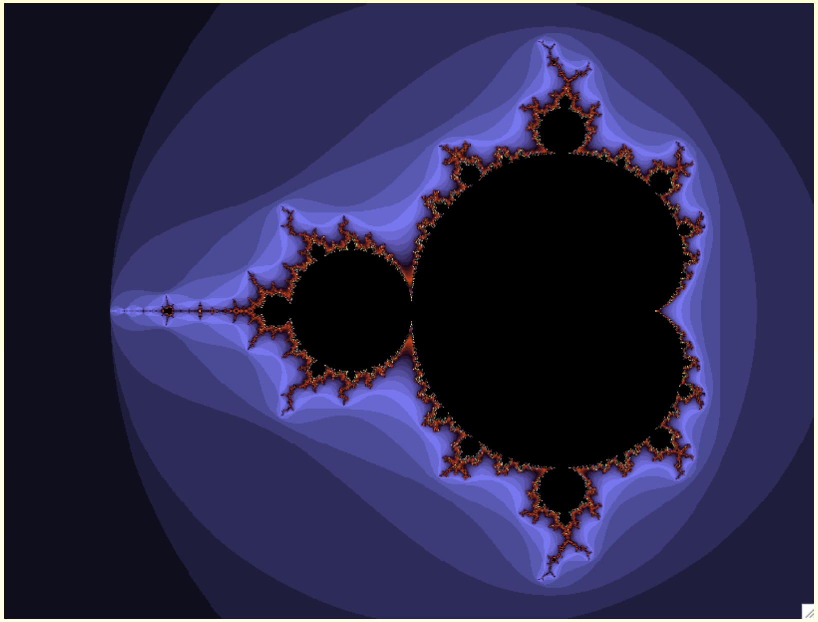

b
1
0.875
0.75
0.625
0.5
0.375
0.25
0.125
0
−0.125
−0.25
−0.375
−0.5
−0.625
−0.75
−0.875
−1
−1 −0.75 −0.5 −0.25 0 0.25 0.5 0.75 1 1.25 1.5 1.75 2 2.25 2.5 2.75 3 3.25 3.5 3.75 4 4.25 4.5 4.75 5 5.25 5.5 5.75 6 6.25
a

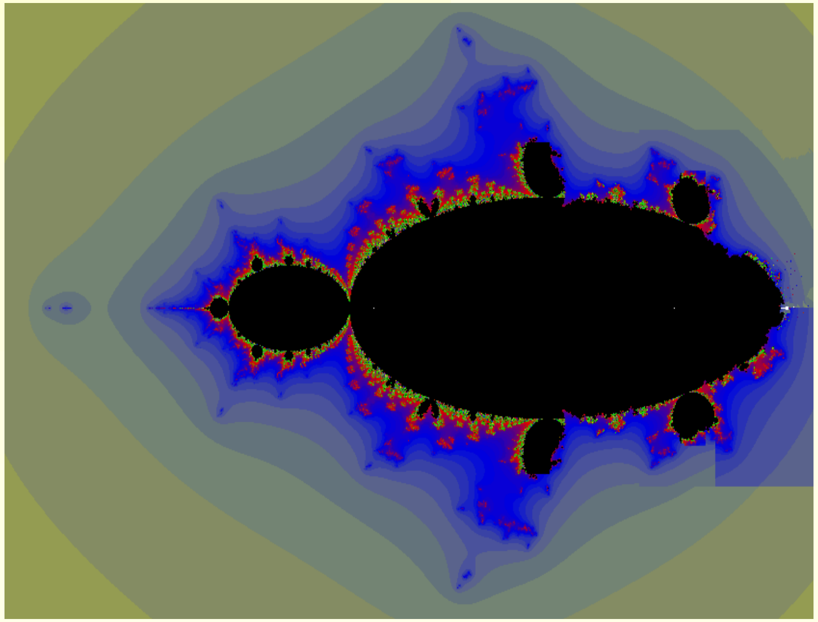
2.1 Complex Hénon Maps

If $b = 0$, the complex Hénon map $H_{a,0} : \mathbb{C}^2 \to \mathbb{C}^2$ reduces to the 1-dimensional map $z \mapsto z^2 + c : \mathbb{C} \to \mathbb{C}$.

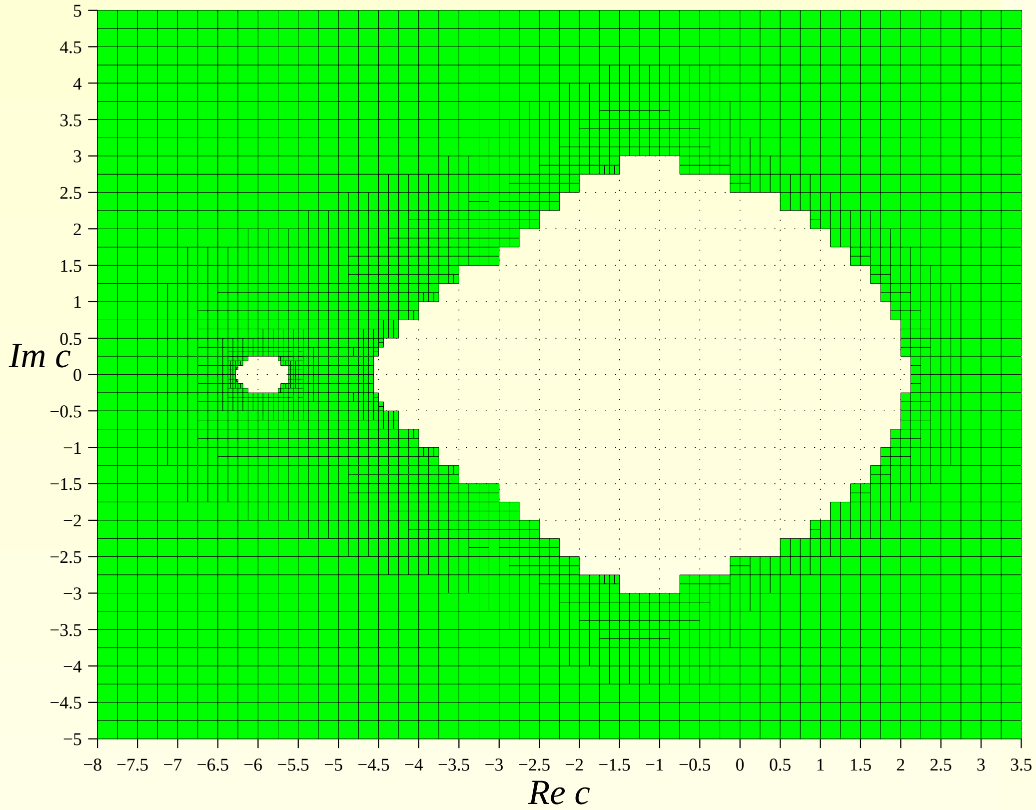
2.1 Complex Hénon Maps

If $b = 0$, the complex Hénon map $H_{a,0} : \mathbb{C}^2 \to \mathbb{C}^2$ reduces to the 1-dimensional map $z \mapsto z^2 + c : \mathbb{C} \to \mathbb{C}$.





Theorem 2. *There exists a non-trivial loop in the hyperbolic horseshoe locus for the family* $H_{a,1}$.



Hyperbolic Horseshoe Loci for $(x, y) \mapsto (x^2 + c - y, x)$

3 Hyperbolicity & Quasi-Hyperbolicity

3.1 Hyperbolic Sets

$f : M \to M$: a diffeomorphism on a manifold M

Λ: compact invariant set of f, $T\Lambda := TM|_\Lambda$.

3 Hyperbolicity & Quasi-Hyperbolicity

3.1 Hyperbolic Sets

$f : M \to M$: a diffeomorphism on a manifold M
Λ: compact invariant set of f, $T\Lambda := TM|_\Lambda$.

Definition. Λ is *uniformly hyperbolic* if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$||Tf^n|_{E^s}|| < c\lambda^n \quad \text{and} \quad ||Tf^{-n}|_{E^u}|| < c\lambda^n$$

hold for all $n \geq 0$. Here $|| \cdot ||$ denotes a metric on M.

3.2 Quasi-Hyperbolic Sets

Definition. We say f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

3.2 Quasi-Hyperbolic Sets

Definition. We say f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

hyperbolicity $\Rightarrow$ quasi-hyperbolicity

3.2 Quasi-Hyperbolic Sets

Definition. We say f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

hyperbolicity $\Rightarrow$ quasi-hyperbolicity

but the converse is not true in general

3.2 Quasi-Hyperbolic Sets

Definition. We say f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

hyperbolicity $\Rightarrow$ quasi-hyperbolicity

but the converse is not true in general

However,

Theorem 3 (Churchill-Franke-Selgrade, Sacker-Sell). *Assume $f|_\Lambda$ is chain recurrent. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

3.3 Isolating Neighborhood

A compact set N is an *isolating neighborhood* if $\mathrm{Inv}_f N := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\} \subset \mathrm{int}\, N$.
An invariant set S is called an *isolated invariant set* if there is an isolating nbd N such that $\mathrm{Inv}_f N = S$.

3.3 Isolating Neighborhood

A compact set N is an *isolating neighborhood* if $\mathrm{Inv}_f N := \{x \in N \mid f^n(x) \in N$ for all $n \in \mathbb{Z}\} \subset \mathrm{int}\, N$.
An invariant set S is called an *isolated invariant set* if there is an isolating nbd N such that $\mathrm{Inv}_f N = S$.

Λ is quasi-hyperbolic
$\Leftrightarrow$ the 0-section $\subset T\Lambda$ is an isolated invariant set of Tf.

3.3 Isolating Neighborhood

A compact set N is an *isolating neighborhood* if $\mathrm{Inv}_f N := \{x \in N \mid f^n(x) \in N$ for all $n \in \mathbb{Z}\} \subset \mathrm{int}\, N$.
An invariant set S is called an *isolated invariant set* if there is an isolating nbd N such that $\mathrm{Inv}_f N = S$.

Λ is quasi-hyperbolic
$\Leftrightarrow$ the 0-section $\subset T\Lambda$ is an isolated invariant set of Tf.

Proposition 4. *Let $N \subset T\Lambda$ be an isolating neighborhood of $Tf : T\Lambda \to T\Lambda$ and assume that N contains Λ. Then Λ is quasi-hyperbolic.*

3.4 Chain Recurrent Set of the Hénon map

$$r(a, b) := \frac{1}{2}(1 + |b| + \sqrt{(1 + |b|)^2 + 4a})$$

$$S(a, b) := \{(x, y) : |x| \leq r(a, b), |y| \leq r(a, b)\}$$

3.4 Chain Recurrent Set of the Hénon map

$$r(a,b) := \frac{1}{2}(1 + |b| + \sqrt{(1+|b|)^2 + 4a})$$

$$S(a,b) := \{(x,y) : |x| \le r(a,b), |y| \le r(a,b)\}$$

Lemma. $\mathcal{R} = \mathcal{R}(H_{a,b}) \subset S(a,b)$ *and* $\mathcal{R}(H_{a,b}|_{\mathcal{R}}) = \mathcal{R}$.

3.5 What we want to show

Since the inverse of $H_{a,b}$ is conjugate to $H_{a/b^2,1/b}$, we only need to consider the case $b \in [-1, 1]$.

3.5 What we want to show

Since the inverse of $H_{a,b}$ is conjugate to $H_{a/b^2,1/b}$, we only need to consider the case $b \in [-1, 1]$.

Further, by the result of Devaney and Nitecki, we only need to consider the case $(a, b) \in [-1, 12] \times [-1, 1]$.

3.5 What we want to show

Since the inverse of $H_{a,b}$ is conjugate to $H_{a/b^2,1/b}$, we only need to consider the case $b \in [-1, 1]$.

Further, by the result of Devaney and Nitecki, we only need to consider the case $(a, b) \in [-1, 12] \times [-1, 1]$.

Let $K := [-8, 8] \times [-8, 8] \subset M = \mathbb{R}^2$. ($\Rightarrow \mathcal{R}(H_{a,b}) \subset K$.) (We will inductively replace K later.)

Let $N = B(K) := K \times [-1, 1]^2 \subset TM = \mathbb{R}^2 \times \mathbb{R}^2$.

3.5 What we want to show

Since the inverse of $H_{a,b}$ is conjugate to $H_{a/b^2,1/b}$, we only need to consider the case $b \in [-1, 1]$.

Further, by the result of Devaney and Nitecki, we only need to consider the case $(a, b) \in [-1, 12] \times [-1, 1]$.

Let $K := [-8, 8] \times [-8, 8] \subset M = \mathbb{R}^2$. ($\Rightarrow \mathcal{R}(H_{a,b}) \subset K$.)
(We will inductively replace K later.)

Let $N = B(K) := K \times [-1, 1]^2 \subset TM = \mathbb{R}^2 \times \mathbb{R}^2$.

We want to show N is an isolating neighborhood.
That is, we want to check the inclusion $\mathrm{Inv}_{TH_{a,b}} N \subset \mathrm{int}\, N$.

4 How to Compute the Invariant Set

4.1 Interval Arithmetic

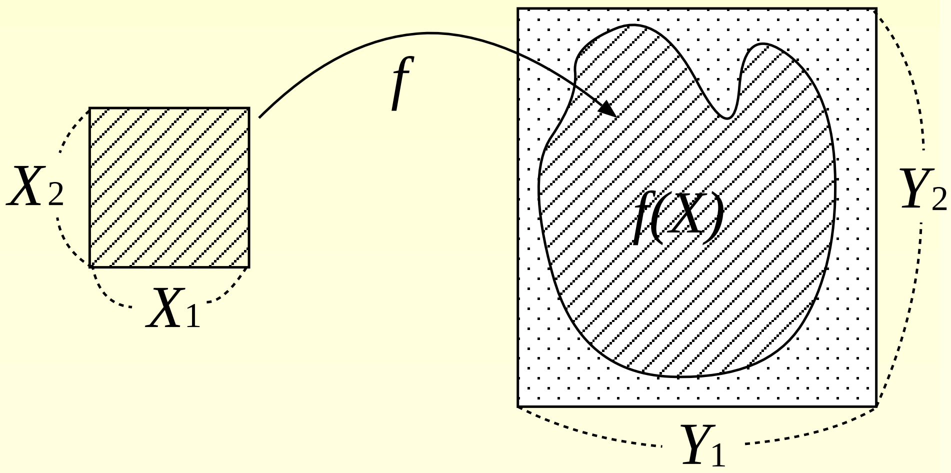
Let $\mathbb{F}$ be the set of floating point numbers and let

$$\mathcal{I} := \{I = [s,t] \subset \mathbb{R} : s,t \in \mathbb{F}\},$$
$$\mathcal{I}^n := \{I_1 \times \cdots \times I_n \subset \mathbb{R}^n : I_i \in \mathcal{I}\}.$$

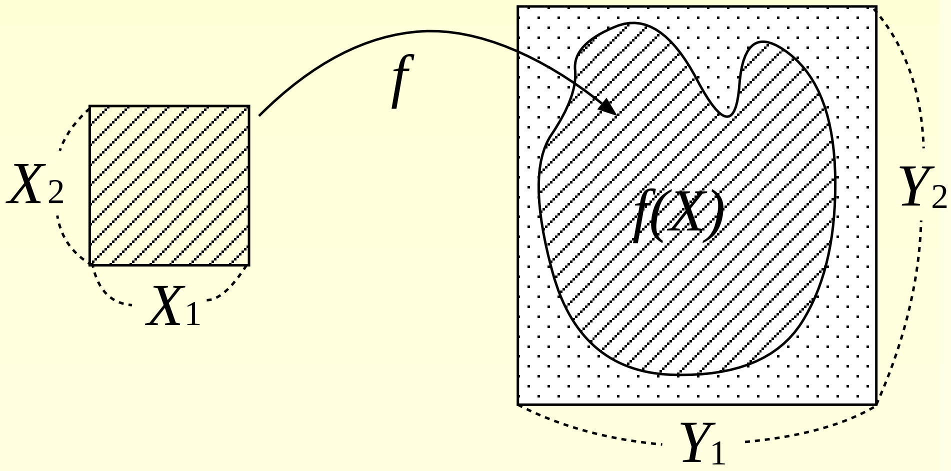
Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and
$X = X_1 \times \cdots \times X_n \in \mathcal{I}^n$ be a cube in its domain.

We can not compute the exact image $f(X)$ by a computer.

However, by applying interval arithmetic, we can find $Y = Y_1 \times \cdots \times Y_n \in \mathcal{I}^n$ such that $f(X) \subset \operatorname{int}(Y)$ rigorously holds.



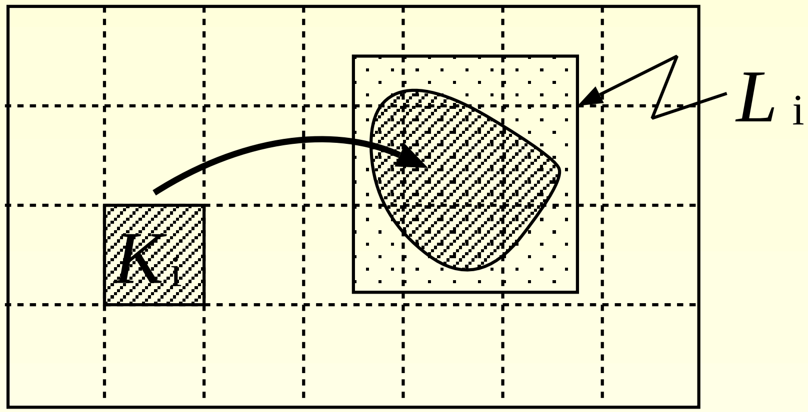
However, by applying interval arithmetic, we can find $Y = Y_1 \times \cdots \times Y_n \in \mathcal{I}^n$ such that $f(X) \subset \operatorname{int}(Y)$ rigorously holds.



Remark: For a family f_λ where $\lambda \in \Lambda$ is a parameter, we can also find $Y \in \mathcal{I}^n$ such that $f_\lambda(X) \subset \operatorname{int}(Y)$ rigorously holds for all $\lambda \in \Lambda$.

4.2 Cubical Grid

Let $K = \bigcup_{i=1}^{k} K_i \ (K_i \in \mathcal{I}^n)$ be a decomposition of $K \in \mathcal{I}^n$. Using interval arithmetic, we compute L_i for each i, such that $f(K_i) \subset \mathrm{int}\, L_i$ holds.



4.3 Graph Representation

Define a directed graph $G(f,K)$ by

- ⋆ it has k vertices: $\{v_1, v_2, \ldots, v_k\}$
- ⋆ $\exists$ edge from v_i to $v_j \Leftrightarrow L_i \cap K_j \neq \emptyset$

4.3 Graph Representation

Define a directed graph $G(f, K)$ by

- ⋆ it has k vertices: $\{v_1, v_2, \ldots, v_k\}$
- ⋆ $\exists$ edge from v_i to $v_j \Leftrightarrow L_i \cap K_j \neq \emptyset$

If $x \in K_i$ is mapped to $f(x) \in K_j$, then there must be an edge of $G(f, K)$ from v_i to v_j since $f(x) \in L_i \cap K_j$.

4.3 Graph Representation

Define a directed graph $G(f, K)$ by

- ⋆ it has k vertices: $\{v_1, v_2, \ldots, v_k\}$
- ⋆ $\exists$ edge from v_i to $v_j \Leftrightarrow L_i \cap K_j \neq \emptyset$

If $x \in K_i$ is mapped to $f(x) \in K_j$, then there must be an edge of $G(f, K)$ from v_i to v_j since $f(x) \in L_i \cap K_j$.

If there is a periodic point of period p in K, then $G(f, K)$ must have a cycle of length p.

4.3 Graph Representation

Define a directed graph $G(f, K)$ by

- ⋆ it has k vertices: $\{v_1, v_2, \ldots, v_k\}$
- ⋆ $\exists$ edge from v_i to $v_j \Leftrightarrow L_i \cap K_j \neq \emptyset$

If $x \in K_i$ is mapped to $f(x) \in K_j$, then there must be an edge of $G(f, K)$ from v_i to v_j since $f(x) \in L_i \cap K_j$.

If there is a periodic point of period p in K, then $G(f, K)$ must have a cycle of length p.

We decompose N into $2n$-dimensional cubes and compute the graph $G(Tf, N)$ of Tf in the same way.

4.4 Graph Invariant Set

For any directed graph G, define

$$\operatorname{Inv} G := \{v \in G \mid \exists \text{ bi-infinitely long path through } v_i\}$$

$$\operatorname{Scc} G := \{v \in G \mid \exists \text{ path from } v_i \text{ to itself }\}$$

4.4 Graph Invariant Set

For any directed graph G, define

$$\operatorname{Inv} G := \{v \in G \mid \exists \text{ bi-infinitely long path through } v_i\}$$

$$\operatorname{Scc} G := \{v \in G \mid \exists \text{ path from } v_i \text{ to itself }\}$$

For subgraphs $G \subset G(f, K)$ and $G' \subset G(Tf, N)$,

$$\text{define} \quad |G| := \bigcup_{v_i \in G} K_i, \quad |G'| := \bigcup_{v_i \in G'} N_i.$$

4.4 Graph Invariant Set

For any directed graph G, define

$$\operatorname{Inv} G := \{v \in G \mid \exists \text{ bi-infinitely long path through } v_i\}$$

$$\operatorname{Scc} G := \{v \in G \mid \exists \text{ path from } v_i \text{ to itself }\}$$

For subgraphs $G \subset G(f, K)$ and $G' \subset G(Tf, N)$,

define $\quad |G| := \bigcup_{v_i \in G} K_i, \quad |G'| := \bigcup_{v_i \in G'} N_i.$

Proposition 5.

$$\mathcal{R}(f) \subset |\operatorname{Scc} G(f, K)|$$

$$\operatorname{Inv}_{Tf} N \subset |\operatorname{Inv} G(Tf, N)|$$

5 Algorithms

Algorithm for a fixed parameter value

5 Algorithms

Algorithm for a fixed parameter value

1. Find K such that $\mathcal{R}(f) \subset \operatorname{int} K$ and let $N := B(K)$.

5 Algorithms

Algorithm for a fixed parameter value

1. Find K such that $\mathcal{R}(f) \subset \operatorname{int} K$ and let $N := B(K)$.
2. Compute $\operatorname{Scc} G(f, K)$ and replace K with $|\operatorname{Scc} G(f, K)|$, then replace N with $N \cap B(K)$.

5 Algorithms

Algorithm for a fixed parameter value

1. Find K such that $\mathcal{R}(f) \subset \operatorname{int} K$ and let $N := B(K)$.
2. Compute $\operatorname{Scc} G(f, K)$ and replace K with $|\operatorname{Scc} G(f, K)|$, then replace N with $N \cap B(K)$.
3. Compute $\operatorname{Inv} G(Tf, N)$.

5 Algorithms

Algorithm for a fixed parameter value

1. Find K such that $\mathcal{R}(f) \subset \operatorname{int} K$ and let $N := B(K)$.
2. Compute $\operatorname{Scc} G(f, K)$ and replace K with $|\operatorname{Scc} G(f, K)|$, then replace N with $N \cap B(K)$.
3. Compute $\operatorname{Inv} G(Tf, N)$.
4. `if` $|\operatorname{Inv} G(Tf, N)| \subset \operatorname{int} B(K)$ `then` the algorithm stops. `else` subdivide the cubes in K, N and `goto` 2.

5 Algorithms

Algorithm for a fixed parameter value

1. Find K such that $\mathcal{R}(f) \subset \operatorname{int} K$ and let $N := B(K)$.
2. Compute $\operatorname{Scc} G(f, K)$ and replace K with $|\operatorname{Scc} G(f, K)|$, then replace N with $N \cap B(K)$.
3. Compute $\operatorname{Inv} G(Tf, N)$.
4. `if` $|\operatorname{Inv} G(Tf, N)| \subset \operatorname{int} B(K)$ `then` the algorithm stops. `else` subdivide the cubes in K, N and `goto` 2.

Theorem 6. *If the algorithm stops, then $\mathcal{R}(f)$ is uniformly hyperbolic.*

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

1. Let $\Lambda_0 = \Lambda$, $\mathcal{L} = \{\Lambda_0\}$ and $K_0 = K$, $N_0 = N$.

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

1. Let $\Lambda_0 = \Lambda$, $\mathcal{L} = \{\Lambda_0\}$ and $K_0 = K$, $N_0 = N$.
2. Choose Λ_i from $\mathcal{L}$ and load the data of K_i and N_i.

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

1. Let $\Lambda_0 = \Lambda$, $\mathcal{L} = \{\Lambda_0\}$ and $K_0 = K$, $N_0 = N$.
2. Choose Λ_i from $\mathcal{L}$ and load the data of K_i and N_i.
3. Apply step 2 and 3 of the previous algorithm for $\{f_\lambda : \lambda \in \Lambda_i\}$ with respect to K_i and N_i.

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

1. Let $\Lambda_0 = \Lambda$, $\mathcal{L} = \{\Lambda_0\}$ and $K_0 = K$, $N_0 = N$.
2. Choose Λ_i from $\mathcal{L}$ and load the data of K_i and N_i.
3. Apply step 2 and 3 of the previous algorithm for $\{f_\lambda : \lambda \in \Lambda_i\}$ with respect to K_i and N_i.
4. `if` $|\mathrm{Inv}\, G(\{Tf_\lambda : \lambda \in \Lambda_i\}, N_i)| \subset \mathrm{int}\, B(K_i)$ `then` remove Λ_i from $\mathcal{L}$ and `goto` step 2.

Algorithm for a family of maps

Consider f_λ where $\lambda \in \Lambda$ is a parameter.

1. Let $\Lambda_0 = \Lambda$, $\mathcal{L} = \{\Lambda_0\}$ and $K_0 = K$, $N_0 = N$.
2. Choose Λ_i from $\mathcal{L}$ and load the data of K_i and N_i.
3. Apply step 2 and 3 of the previous algorithm for $\{f_\lambda : \lambda \in \Lambda_i\}$ with respect to K_i and N_i.
4. `if` $|\mathrm{Inv}\, G(\{T f_\lambda : \lambda \in \Lambda_i\}, N_i)| \subset \mathrm{int}\ B(K_i)$ `then` remove Λ_i from $\mathcal{L}$ and `goto` step 2.
5. `else` bisect Λ_i into Λ_{i0} and Λ_{i1}. Subdivide the cubes in N_i, K_i and let $N_{i0} = N_{i1} = N_i$, $K_{i0} = K_{i1} = K_i$, $\mathcal{L} = (\mathcal{L} \setminus \Lambda_i) \cup \{\Lambda_{i0}, \Lambda_{i1}\}$. Then `goto` step 2.

6 Computational Costs

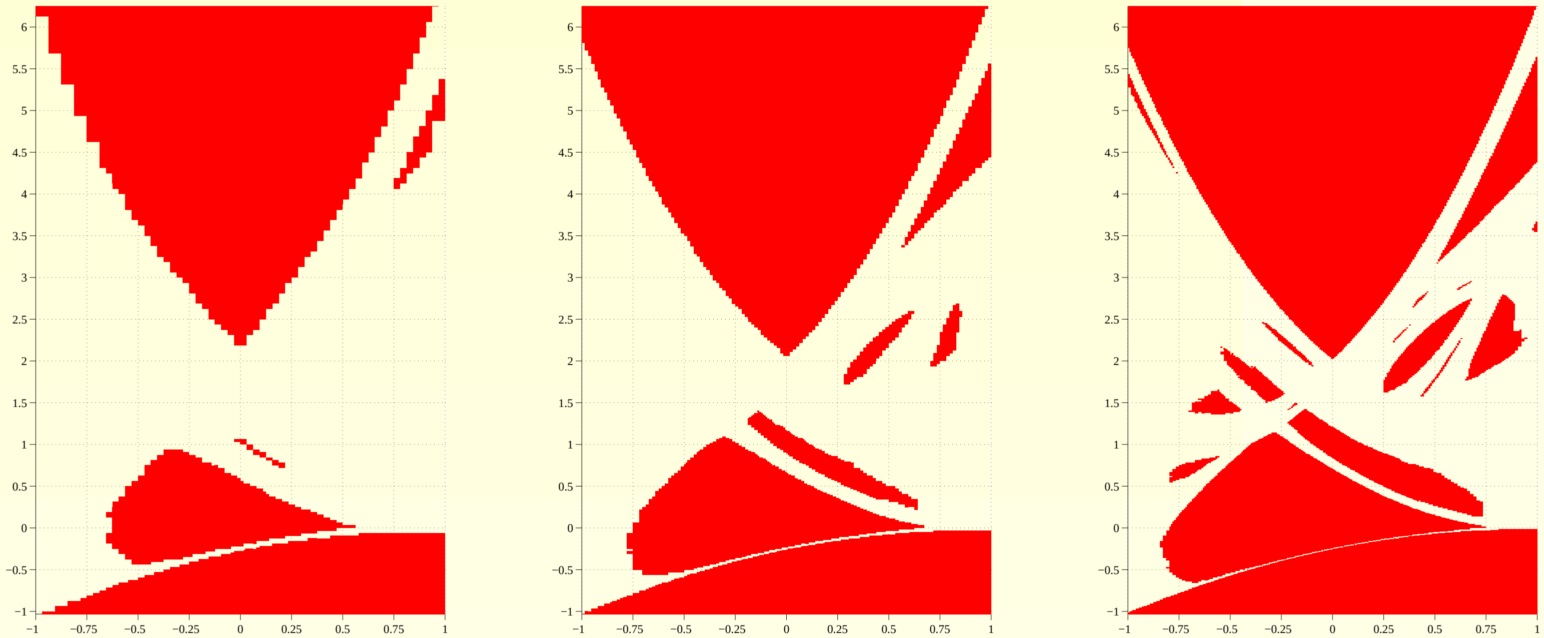


Figure 1: Results after 1, 10 and 100 hour computation

7 Softwares

GAIO

(Global Analysis of Invariant Objects)

http://math-www.uni-paderborn.de/˜agdellnitz/gaio/

PROFIL/BIAS

(Programmer's Runtime Optimized Fast Interval Library)

http://www.ti3.tu-harburg.de/knueppel/profil/

CAPD

(Computer Assisted Proofs in Dynamics)

http://capd.wsb-nlu.edu.pl/

Home page of Zin ARAI

http://www.math.kyoto-u.ac.jp/˜arai/