

Hyperbolicity, Stability and Monodromy of Dynamical Systems

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We propose two methods for the study of dynamical systems:

Stability Analysis

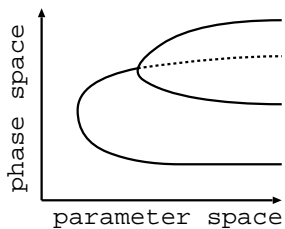
- For the study of the parameter space
- Find structurally stable parameter values
- Complementary to "bifurcation analysis"

Monodromy Analysis

- For the study of the phase space
- Complete description of chaotic invariant sets
- Analog of the monodromy theory for linear differential equations

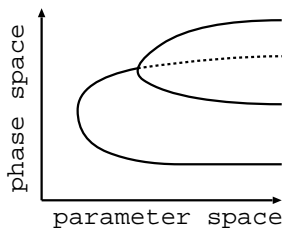
Complement of Bifurcations

A typical picture of bifurcation analysis is like:



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Question

- What happens away from bifurcation curves?
- How do we find important objects to be investigated (periodic orbits, homoclinic orbits, etc..)?

Problems

- Determine the structurally stable parameter set without any a priori knowledge on dynamics.
- Describe the global dynamics for these parameter set.

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EXAMPLE: We consider the real Hénon map

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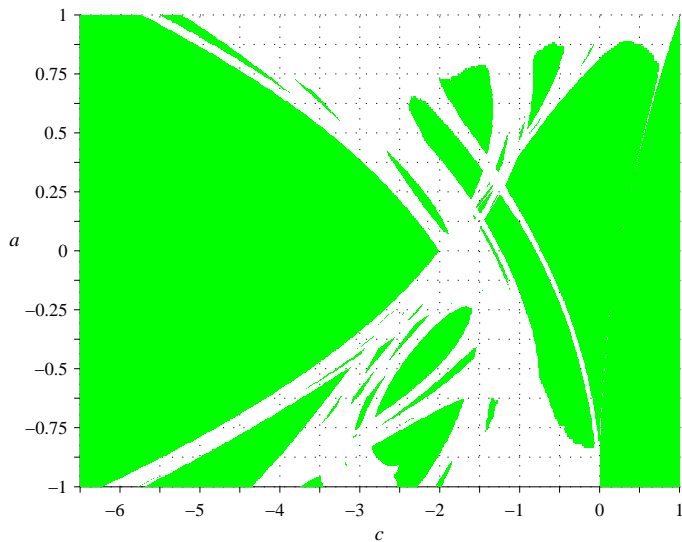
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Applying our method, we obtain the following figure:

Hyperbolic Plateau of the real Hénon Map



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Λ : compact invariant set of f , $T\Lambda := TM|_{\Lambda}$.

Definition

Λ is *uniformly hyperbolic* if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf -invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n|_{E^s}\| < c\lambda^n \quad \text{and} \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n$$

hold for all $n \geq 0$. Here $\|\cdot\|$ denotes a metric on M .

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However,

Theorem (Churchill-Franke-Selgrade, Sacker-Sell)

Assume $f|_{\Lambda}$ is chain recurrent. Then

quasi-hyperbolicity \Leftrightarrow *uniform hyperbolicity*

Quasi-hyperbolicity Criterion

A compact set N is an *isolating neighborhood* if $\text{Inv}_f N := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\} \subset \text{int } N$.

An invariant set S is called an *isolated invariant set* if there is an isolating nbd N such that $\text{Inv}_f N = S$.

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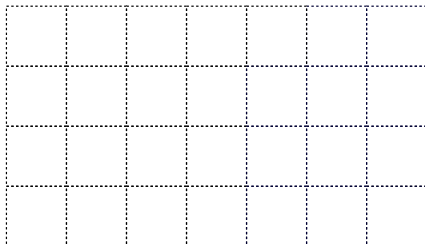
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Proposition

Let $N \subset T\Lambda$ be an isolating neighborhood of $Tf : T\Lambda \rightarrow T\Lambda$ and assume that N contains the 0-section of $T\Lambda$. Then Λ is quasi-hyperbolic.

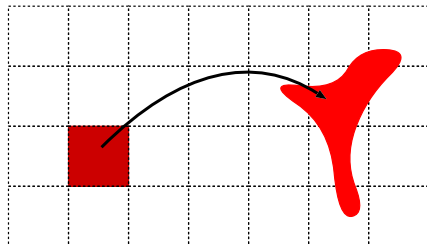
From Dynamics to Directed Graph

- 1 Subdivide the phase space into small cubes.
- 2 Using rigorous interval arithmetics, enclose the image of each cube.
- 3 Define a directed graph G with
 - vertices = cubes
 - \exists edge $v_i \rightarrow v_j \iff$
cubical enclosure of the image of v_i intersects v_j



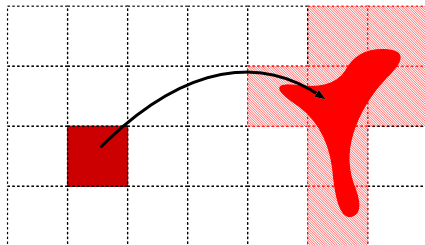
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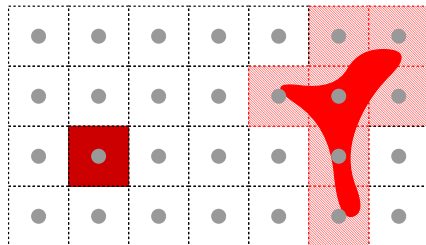
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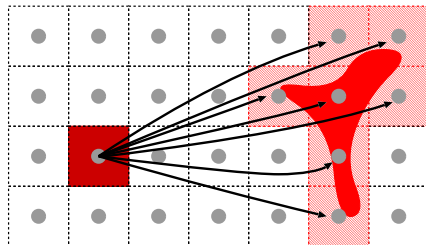
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Graph Invariant Set

For any directed graph G , define

$$\text{Inv } G := \{v \in G \mid \exists \text{ bi-infinitely long path through } v_i\}$$

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Proposition

If G is a directed graph associated to f and $|G| = N$,

$$\mathcal{R}(f, N) \subset |\text{Scc } G|$$

$$\text{Inv}_f(N) \subset |\text{Inv } G|.$$

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Hyperbolicity Verification for parametrized families

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- 2 Choose a cube P from the set of parameter cubes.
- 3 Try to prove hyperbolicity on P .
- 4 If P is hyperbolic, throw P away and then goto 2.
else subdivide P into smaller cubes and then goto 2.

Describing Chaotic Dynamics

The Conley Index Theory is a powerful tool to understand the dynamics on hyperbolic plateaus:

- Count the number of periodic points
- Estimate the topological entropy
(cf. 1st talk by Day, Frongillo and Trevino)
- Find a semi-conjugacy to symbolic dynamics
- Applicable to a wide class of dynamical systems

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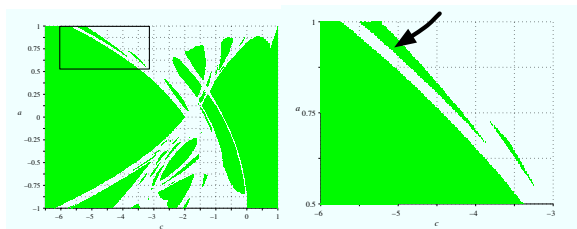
On the other hand,

Monodromy Theory:

- Applicable to polynomial maps
- Construct a **conjugacy** to symbolic dynamics

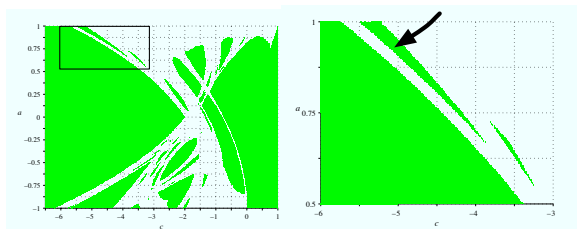
Complex Hénon Maps

Problem: describe the dynamics on this island:



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Extend the Hénon map to complex variables:

$$H_{a,c} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (x, y) \mapsto (x^2 + c - ay, x); (a, c) \in \mathbb{C}^2.$$

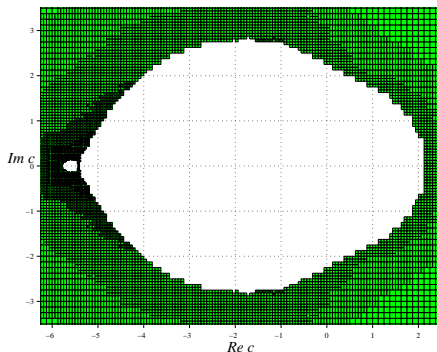
Let $K_{a,c}^{\mathbb{C}} := \{ \text{bdd orbits of } H_{a,c} : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \}$.

If $(a, c) \in \mathbb{R}^2$ then $K_{a,c}^{\mathbb{R}} := K_{a,c}^{\mathbb{C}} \cap \mathbb{R}^2$ is invariant.

Loops in the complex Hyperbolic locus

Fix $a = 1$ and search hyperbolic parameters of $H_{1,c}$.

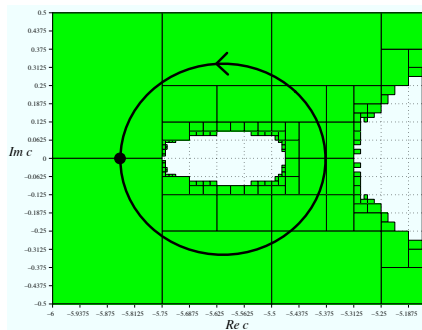
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We consider the loop γ_p as above.

Monodromy Homomorphism

Given a loop $\gamma : [0, 1] \rightarrow \mathcal{H}_0^{\mathbb{C}}$ based at (α_0, c_0) , we construct a continuous family of conjugacies

$$h_t : K_{\gamma(t)}^{\mathbb{C}} \rightarrow \Sigma_2.$$

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By setting $\rho(\gamma) := h_1 \circ h_0^{-1} : \Sigma_2 \rightarrow \Sigma_2$, we define

$$\rho : \pi_1(\mathcal{H}_0^{\mathbb{C}}, (a_0, c_0)) \rightarrow \text{Aut}(\Sigma_2).$$

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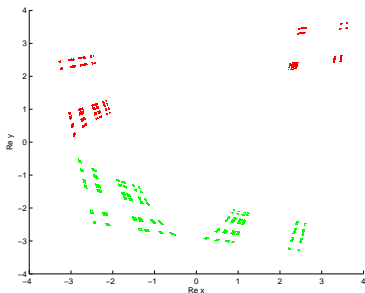
Theorem

If (α, c) is real and contained in $\mathcal{H}_0^{\mathbb{C}}$ there exists a loop γ thorough (α, c) such that we have a topological conjugacy

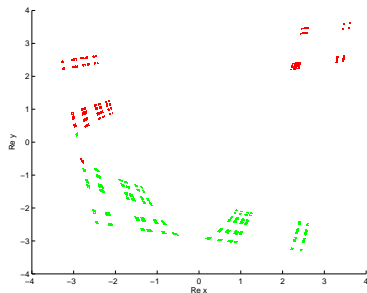
$$K_{\alpha, c}^{\mathbb{R}} \cong \text{Fix}(\rho(\gamma)).$$

Compute the Monodromy

For each $t \in [0, 1]$, we **rigorously** compute the generating partition of the isolating neighbourhood of $K_{\gamma_p(t)}^{\mathbb{C}}$ so that these partitions satisfy the continuation property.



$t = 0$

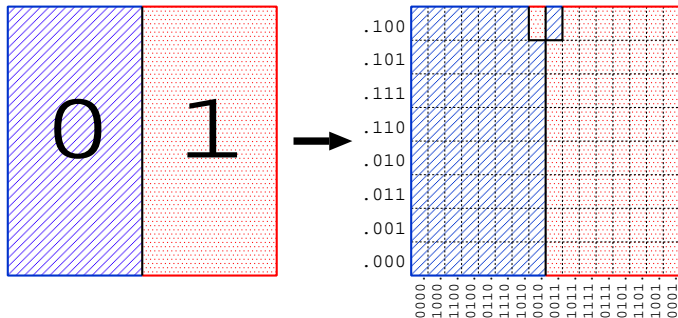


$t = 1$

Symbolic Coding Exchange

Lemma

The partition generating the symbolic coding interchanges along γ_p as follows:



$\rho(\gamma_p)$ interchanges 0010100 and 0011100

Corollary

For $\alpha = 1$ and $c = -5.4$, the real Hénon map

$$H_{\alpha,c} : K_{\alpha,c}^{\mathbb{R}} \rightarrow K_{\alpha,c}^{\mathbb{R}}$$

is conjugate to a subshift of Σ_2 whose forbidden words are

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Thank you for your attention!