Verification of Hyperbolicity and Non-Hyperbolicity via Topological Methods

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1 Motivation

Problem: Given a family of dynamical systems with some parameters, determine the set of hyperbolic and non-hyperbolic parameters.

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However, proving hyperbolicity is a difficult problem even for simple low-dimensional dynamical systems.

2 Results

Example: the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x)$$

where $a, b \in \mathbb{R}$ are the parameters.

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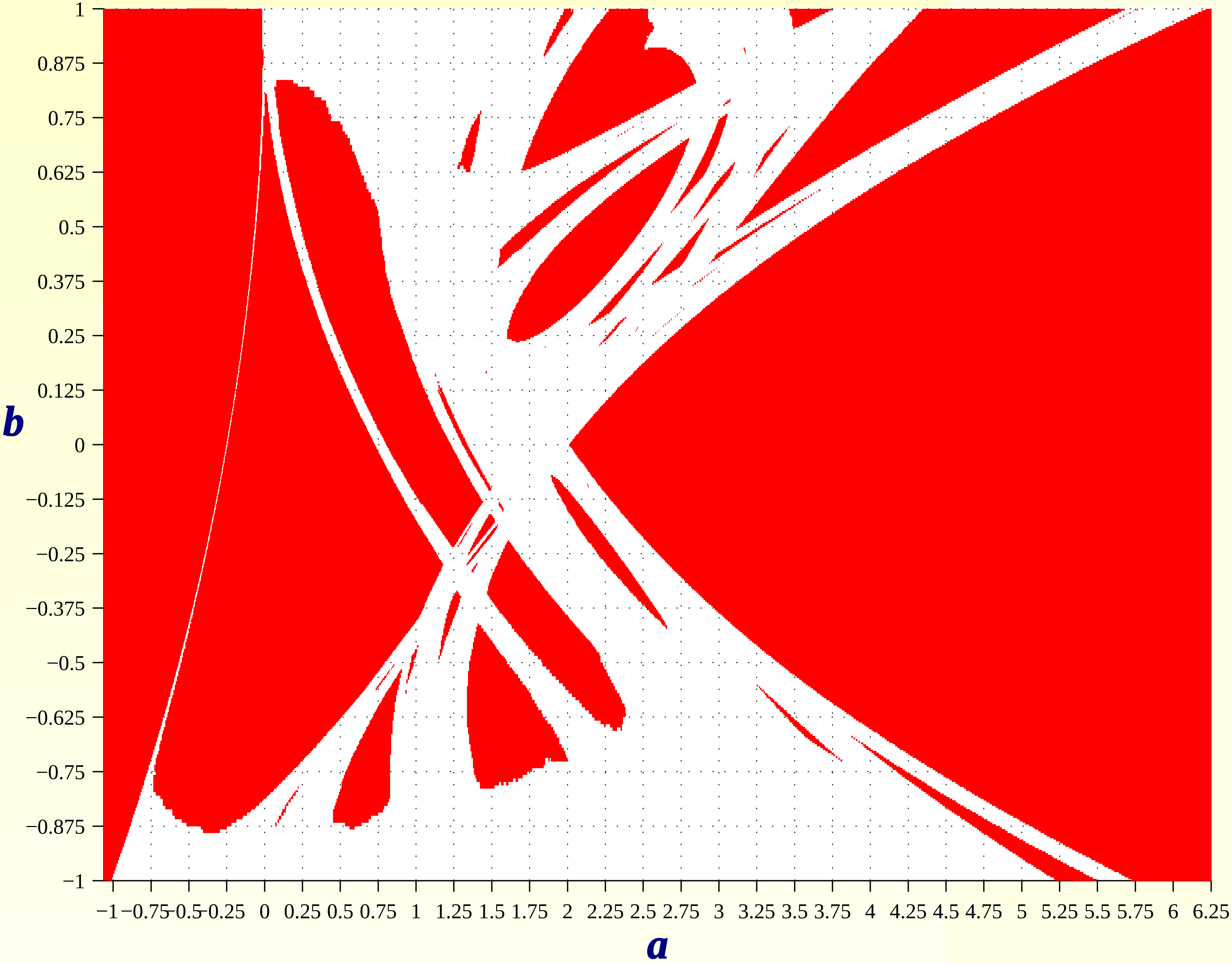
where $a, b \in \mathbb{R}$ are the parameters.

Denote by $\mathcal{R}(H_{a,b})$ the chain recurrent set of $H_{a,b}$.

Theorem 1 (2-parameter family). *If (a, b) is in the red region of the figure, then $\mathcal{R}(H_{a,b})$ is uniformly hyperbolic.*


1
0.875
0.75
0.625
0.5
0.375
0.25
0.125
0
−0.125
−0.25
−0.375
−0.5
−0.625
−0.75
−0.875
−1
b
−1 −0.75 −0.5 −0.25 0 0.25 0.5 0.75 1 1.25 1.5 1.75 2 2.25 2.5 2.75 3 3.25 3.5 3.75 4 4.25 4.5 4.75 5 5.25 5.5 5.75 6 6.25
a

Theorem 2 (Area Preserving Family). *If a is in one of the following intervals:*

$$
\begin{array}{ll}
[4.5659179687500, 4.5898437500000], & [4.6240234375000, 4.6406250000000] \\
[4.6748046875000, 4.6845703125000], & [4.7729492187500, 4.7734375000000] \\
[4.7739257812500, 4.7963867187500], & [4.8525390625000, 4.8530273437500] \\
[4.8540039062500, 4.8613281250000], & [4.9672851562500, 4.9682617187500] \\
[5.1474609375000, 5.1494140625000], & [5.1914062500000, 5.5363769531250] \\
[5.5668945312500, 5.6075439453125], & [5.6352539062500, 5.6766357421875] \\
[5.6767578125000, 5.6768798828125], & [5.6821289062500, 5.6857910156250] \\
[5.6859130859375, 5.6860351562500], & [5.6916503906250, 5.6951904296875]
\end{array}
$$

$$[5.6999511718750, \infty)$$

then $\mathcal{R}(H_{a,-1})$ is uniformly hyperbolic.

2.1 First Bifurcation Problem

Using a method of ZA and K. Mischaikow, we can prove

Theorem 3. *There exists*

$$a \in [5.6993102, 5.6993113]$$

such that $H_{a,-1}$ *has a homoclinic tangency with respect to the saddle fixed point on the third quadrant.*

3 hyperbolicity & quasi-hyperbolicity

3.1 Hyperbolic Sets

$f : M \to M$: a diffeomorphism on a manifold M

Λ: compact invariant set of f, $T\Lambda := TM|_\Lambda$.

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Λ: compact invariant set of f, $T\Lambda := TM|_\Lambda$.

Definition. Λ is *uniformly hyperbolic* if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$||Tf^n|_{E^s}|| < c\lambda^n \quad \text{and} \quad ||Tf^{-n}|_{E^u}|| < c\lambda^n$$

hold for all $n \geq 0$. Here $|| \cdot ||$ denotes a metric on M.

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However,

Theorem 4 (Churchill-Franke-Selgrade, Sacker-Sell). *Assume $f|_\Lambda$ is chain recurrent. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

3.3 Isolating Neighborhood

A compact set N is an *isolating neighborhood* if $\operatorname{Inv}_f N := \{x \in N \mid f^n(x) \in N$ for all $n \in \mathbb{Z}\} \subset \operatorname{int} N$.
An invariant set S is called an *isolated invariant set* if there is an isolating nbd N such that $\operatorname{Inv}_f N = S$.

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Proposition 5. *Let $K \subset M$ be a compact set containing Λ and $N \subset TK$ be a compact neighborhood of the zero section of TK. If N is an isolating neighborhood with respect to $Tf : TM \to TM$, then Λ is quasi-hyperbolic.*

3.4 Chain Recurrent Set of the Hénon map

$$R(a,b) := \frac{1}{2}(1 + |b| + \sqrt{(1+|b|)^2 + 4a})$$

$$S(a,b) := \{(x,y) : |x| \leq R(a,b), |y| \leq R(a,b)\}$$

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Lemma. $\mathcal{R} = \mathcal{R}(H_{a,b}) \subset S(a,b)$ *and* $\mathcal{R}(H_{a,b}|_{\mathcal{R}}) = \mathcal{R}$.

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Let $K := [-8,8] \times [-8,8] \subset M = \mathbb{R}^2$. ($\Leftrightarrow \mathcal{R}(H_{a,b}) \subset K$.)
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(We will inductively replace K later.)

Let $N = B(K) := K \times [-1, 1]^2 \subset TM = \mathbb{R}^2 \times \mathbb{R}^2$.

We want to show N is an isolating neighborhood.
That is, we want to check the inclusion

$$\mathrm{Inv}_{TH_{a,b}} N \subset \mathrm{int}\, N.$$

4 How to Compute the Invariant Set

4.1 Interval Arithmetics

Let $\mathbb{F}$ be the set of floating point numbers and

$$\mathcal{I} := \{I = [a,b] \subset \mathbb{R} : a,b \in \mathbb{F}\}$$

be the set of intervals with end-points in $\mathbb{F}$.

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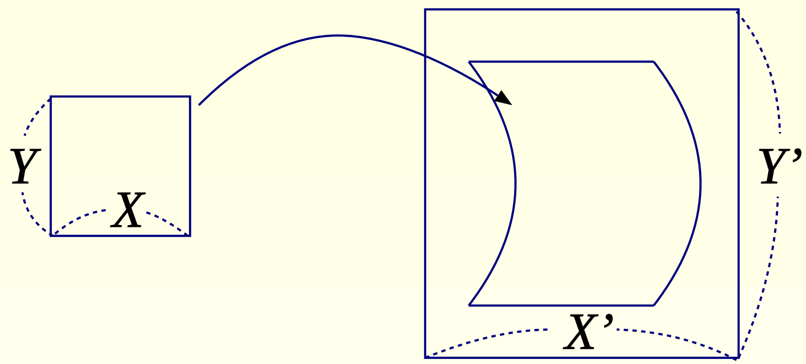
If a CPU satisfies IEEE754 standard, it can give us the intervals $X', Y', V', W' \in \mathcal{I}$ such that for any $(a, b) \in A \times B$,

$$H_{a,b}(X \times Y) \subset \mathrm{int}(X' \times Y')$$

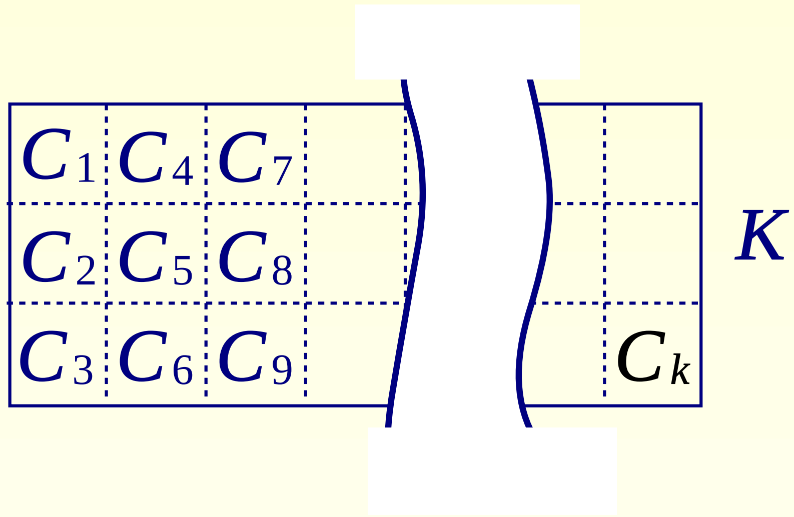
$$TH_{a,b}(X \times Y \times V \times W) \subset \mathrm{int}(X' \times Y' \times V' \times W').$$



4.3 Cubical Grid

Define $\mathcal{I}^n := \{I_1 \times \cdots \times I_n \subset \mathbb{R}^n : I_i \in \mathcal{I}\}$.

Let $K = \bigcup_{i=1}^{k} C_i \ \ (C_i \in \mathcal{I}^2)$ be a decomposition of K.
Using the interval arithmetic, we can compute D_i such that $H_{a,b}(C_i) \subset \operatorname{int} D_i$ holds for for every $a \in A$ and $b \in B$.



4.4 Graph Representation

From this information, we define a directed graph $G(K)$:

- ⋆ vertices: $\{v_1, v_2, \ldots, v_k\}$
- ⋆ edges: $\exists$ edge from v_i to $v_j \Leftrightarrow D_i \cap C_j$

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Similarly we decompose N into 4-dimensional cubes and make graph $G(N)$ of $TH_{a.b}$ using that decomposition.

4.5 Graph Invariant Set

For any directed graph G, define

$$\mathrm{Inv}\,G := \{v_i \mid \exists \text{ edge starts at } v_i, \exists \text{ edge ends at } v_i\}$$

$$\mathrm{Scc}\,G := \{v_i \mid \exists \text{ path from } v_i \text{ to itself }\}$$

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Proposition 6. *For any $a \in A, b \in B$,*

$$\mathcal{R}(H_{a,b}) \subset |\operatorname{Scc} G(K)|, \quad \operatorname{Inv}_{TH_{a,b}} N \subset |\operatorname{Inv} G(N)|.$$

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4. `if` $|\mathrm{Inv}\, G(N)| \subset \mathrm{int}\, N_0$ `then` the algorithm stops.
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Theorem 7. *If the algorithm stops, then $\mathcal{R}H_{a,b}$ is hyperbolic for all $(a,b) \in A \times B$.*

6 Softwares

GAIO

(Global Analysis of Invariant Objects)

`http://math-www.uni-paderborn.de/~agdellnitz/gaio`

PROFIL/BIAS

(Programmer's Runtime Optimized Fast Interval Library)

`http://www.ti3.tu-harburg.de/Software/PROFILEnglisch.html`

CHomP

(Computational Homology Program)

`http://www.math.gatech.edu/~chom/`

Home page of Zin ARAI

`http://www.math.kyoto-u.ac.jp/~arai`