Numerical Detection of Structurally Unstable Connecting Orbits via the Conley Index Theory

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START:▷

1 Examples

Result 1. *Let*

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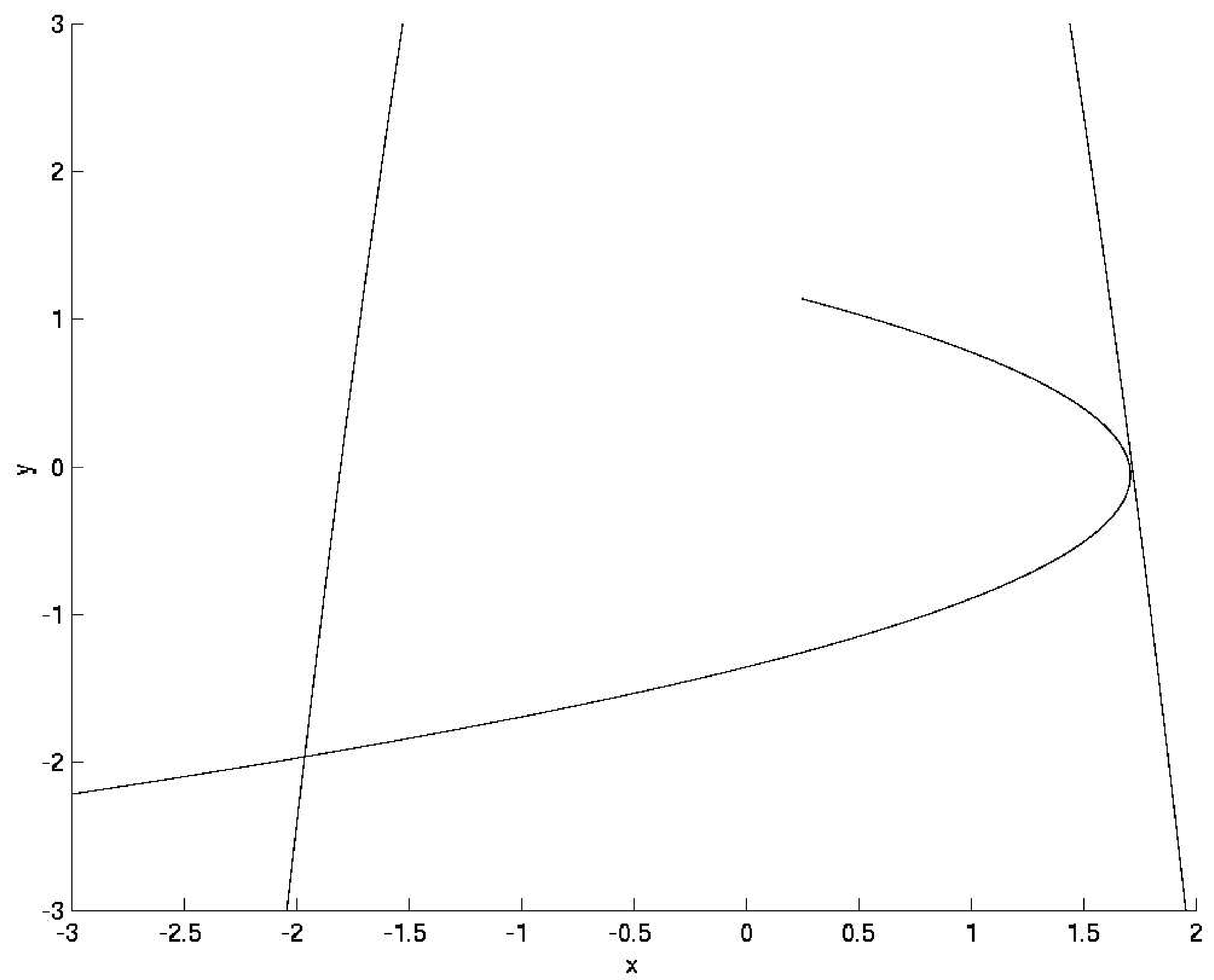
There exist $\lambda_0 \in [0.9305, 0.9335]$ *such that* f_{λ_0} *has a connecting orbit from* $(0,0)$ *to* $(1,0)$*.*

Result 2. *For the Hénon family*

$$H_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix},$$

there exist $\varepsilon > 0$ *such that for every* $b \in [-0.3 - \varepsilon, -0.3 + \varepsilon]$, *there exist* $a \in [1.313, 1.316]$ *such that* $H_{a,b}$ *has a homoclinic tangency with respect to the lower left saddle fixed point.*


3
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1.5
2
x

2 Definition of the Conley index for maps

The *homology Conley index* for an isolated invariant set S is the shift equivalent class of the pair

$$CH_*(S) = H_*(N/L, [L]) \quad \text{and}$$

$$\chi(S) : CH_*(S) \to CH_*(S)$$

where (N, L) is an index pair for S and $\chi(S)$ is the homomorphism induced from f.

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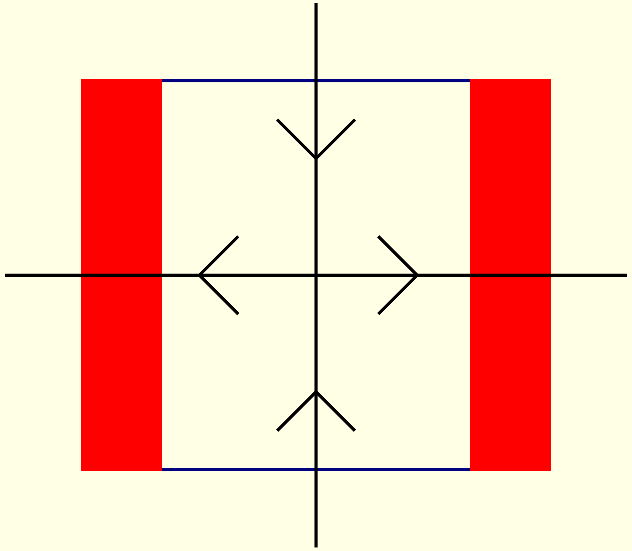

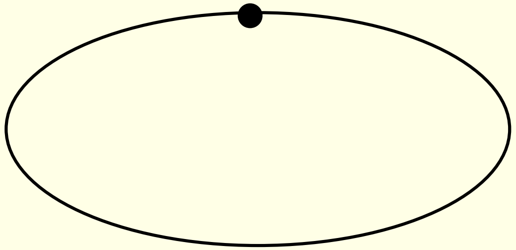

$f : X \to X$ and $g : Y \to Y$ is said to be *shift equivalent* if there is $m \in \mathbb{Z}_{\geq 0}$ and $s : X \to Y$, $r : Y \to X$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & X \arrow[d, "s"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & X \\ Y \arrow[r, "g"'] \arrow[u, "r"] & Y \arrow[u, "r"'] \end{tikzcd}$$

is commutative and $r \circ s = g^m$, $s \circ r = f^m$.

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If $f(N_2) \cap N_1 = \emptyset$ *and*

$$\mathrm{Con}_*(\mathrm{Inv}(N)) \not\cong \mathrm{Con}_*(\mathrm{Inv}(N_1)) \oplus \mathrm{Con}_*(\mathrm{Inv}(N_2)),$$

then there exists a connecting orbit from $\mathrm{Inv}(N_1)$ *to* $\mathrm{Inv}(N_2)$.

Proof. Since $f(N_2) \cap N_1 = \emptyset$,

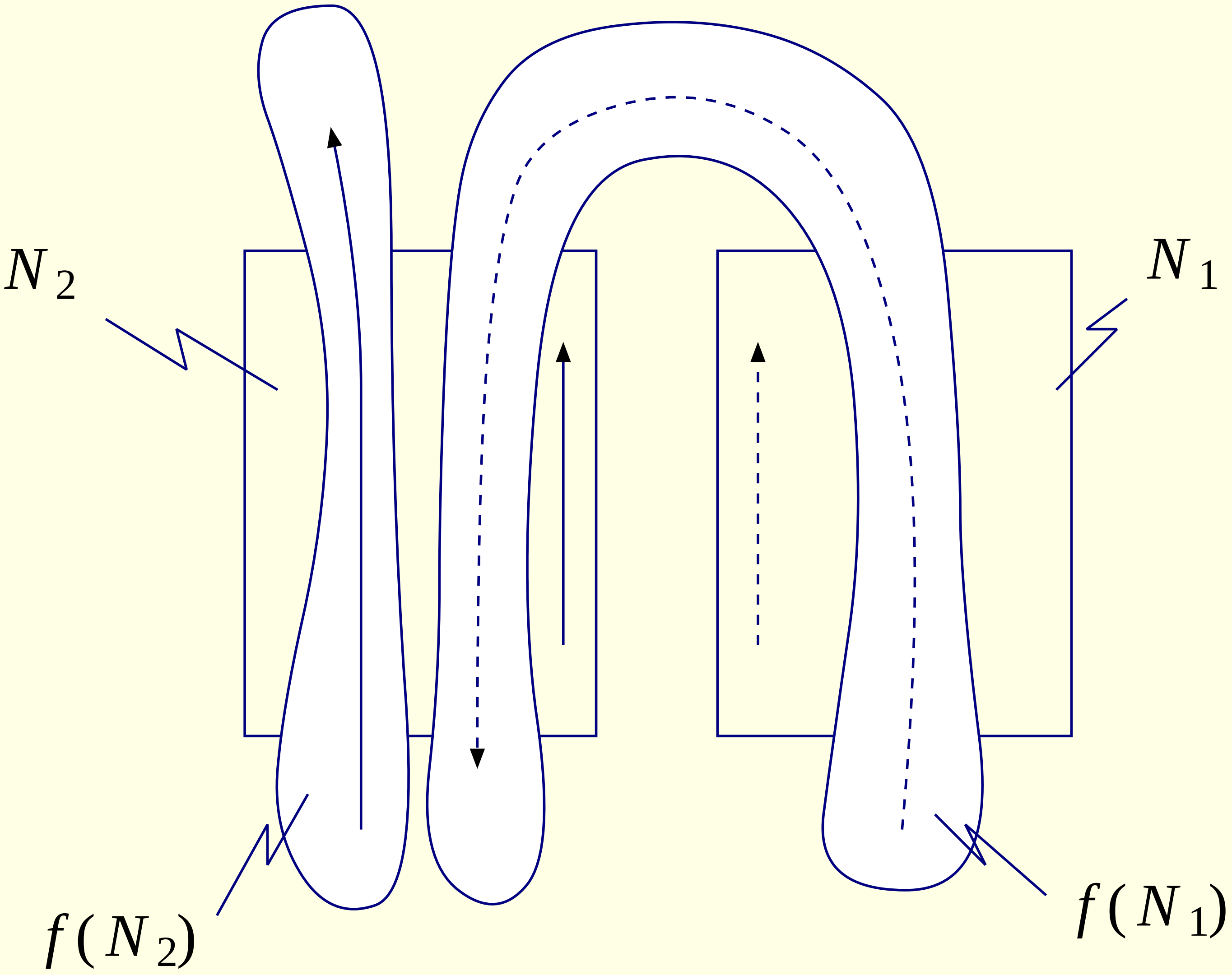
$$\operatorname{Inv}(N) = \operatorname{Inv}(N_1) \cup \operatorname{Inv}(N_2) \cup C(\operatorname{Inv}(N_1), \operatorname{Inv}(N_2)).$$

Thus, if $C(\operatorname{Inv}(N_1), \operatorname{Inv}(N_2)) = \emptyset$ then

$$\operatorname{Con}_*(\operatorname{Inv}(N)) \cong \operatorname{Con}_*(\operatorname{Inv}(N_1)) \oplus \operatorname{Con}_*(\operatorname{Inv}(N_2))$$

by the additivity of the Conley index. □

3.1 Example



Since

$$\mathrm{Con}_2(S) = \left(\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}\right) \underset{\text{shift}}{\cong} \left(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$$

$$\mathrm{Con}_2(S_1)\oplus\mathrm{Con}_2(S_2) \underset{\text{shift}}{\cong} \left(\mathbb{Z}\oplus\mathbb{Z}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

Theorem imply that there exists a connecting orbit form S_1 to S_2.

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Theorem imply that there exists a connecting orbit form S_1 to S_2.

Remark: Only structurally stable connecting orbits can be found by this theorem.

4 One-parameter families

$f_\lambda : X \to X$, $\lambda \in \Lambda$

$$F(x,\lambda) := (f_\lambda(x),\lambda) : X \times \Lambda \to X \times \Lambda$$

$S_1(\lambda)$, $S_2(\lambda) \subset S(\lambda)$: isolated invariant sets

$\bigcup_\lambda S(\lambda)$ is an isolated invariant set w.r.t. F.

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$\bigcup_\lambda S(\lambda)$ is an isolated invariant set w.r.t. F.

$C(\bigcup_\lambda S_1(\lambda), \bigcup_\lambda S_2(\lambda); F) \neq \emptyset$ implies that there is λ_0 such that $C(S_1(\lambda_0), S_2(\lambda_0); f_{\lambda_0}) \neq \emptyset$.

Remark: Simple application of the theorem to F is not sufficient

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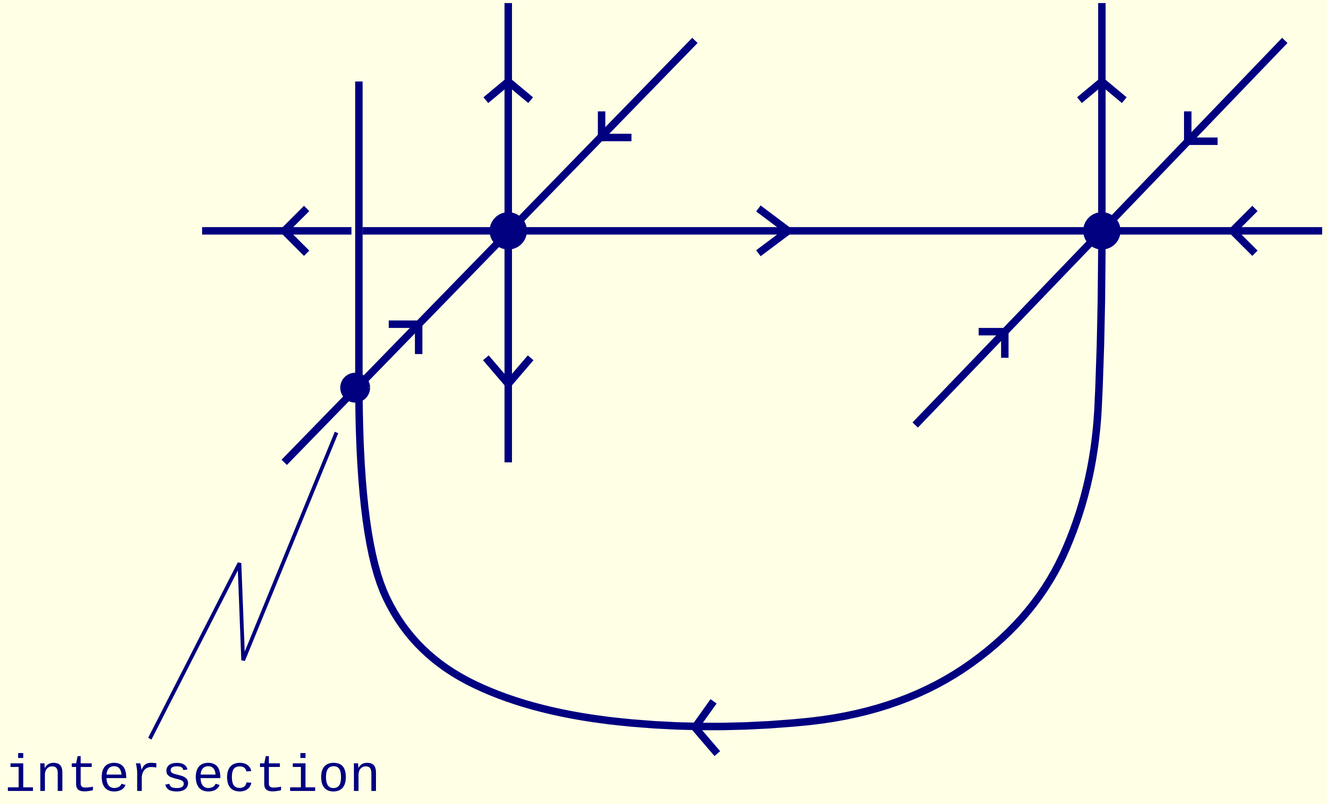

5 Suspension

Setting: N, N_1, N_2: iso. nbds for $\bigcup_\lambda S(\lambda)$, etc.

$N = N_1 \sqcup N_2$ and $F(N_2) \cap N_1 = \emptyset$

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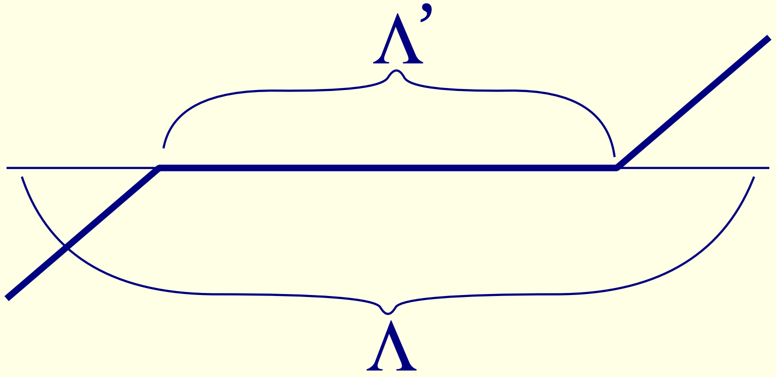
$N = N_1 \sqcup N_2$ and $F(N_2) \cap N_1 = \emptyset$

$$F'(x,\lambda) = \begin{cases} (f_\lambda(x), \lambda + g(\lambda)) & x \in N_1 \\ (f_\lambda(x), \lambda - g(\lambda)) & x \in N_2 \end{cases}$$



Let $S_1' = \mathrm{Inv}(N_1, F')$, $S_2' = \mathrm{Inv}(N_2, F')$.

Then by suspension isomorphism theorem and homotopy invariance of the Conley index,

$$\mathrm{Con}_*(S_1', F') = \mathrm{Con}_{*-1}(S_1, F)$$

$$\mathrm{Con}_*(S_2', F') = \mathrm{Con}_*(S_2, F).$$

6 Numerical Example

$$f_\lambda(x, y) := (x^2 + \lambda y,\ 5y(1 - y))$$

Simple numerical computation suggests that there should be a connecting orbit from $(0, 0)$ to $(1, 1)$ when λ is close to 0.93.

$$S_1(\lambda) = (0, 0), \qquad S_2(\lambda) = (1, 0)$$

$$\Lambda = [0.75, 1.15], \qquad \Lambda' = [0.85, 1.05]$$

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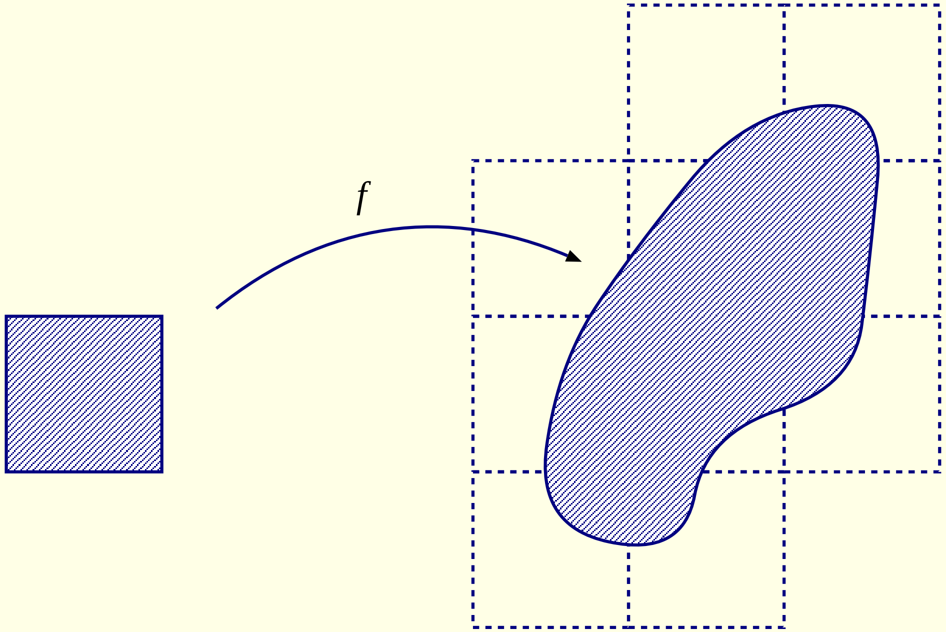
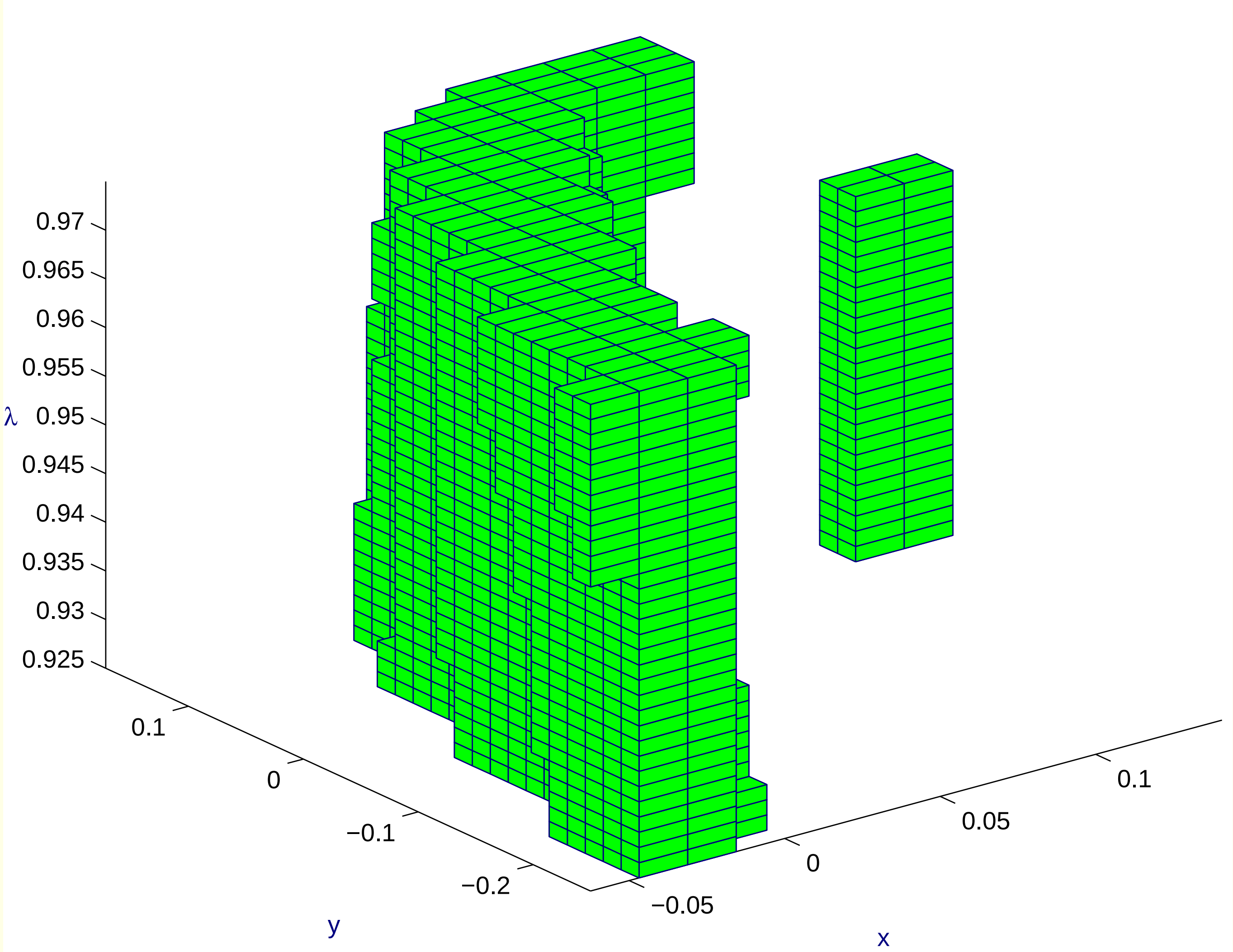
Step 5. Show that $S'_1 = (0, 0)$, $S'_2 = (1, 0)$

6.2 Representation by cubical sets



Let Ω be a set of “cubes” in $\mathbb{R}^n$ and let $\mathcal{F} : \Omega \to 2^{\Omega}$ be a function which maps $K \in \Omega$ to a set of boxes containing $f(K)$.

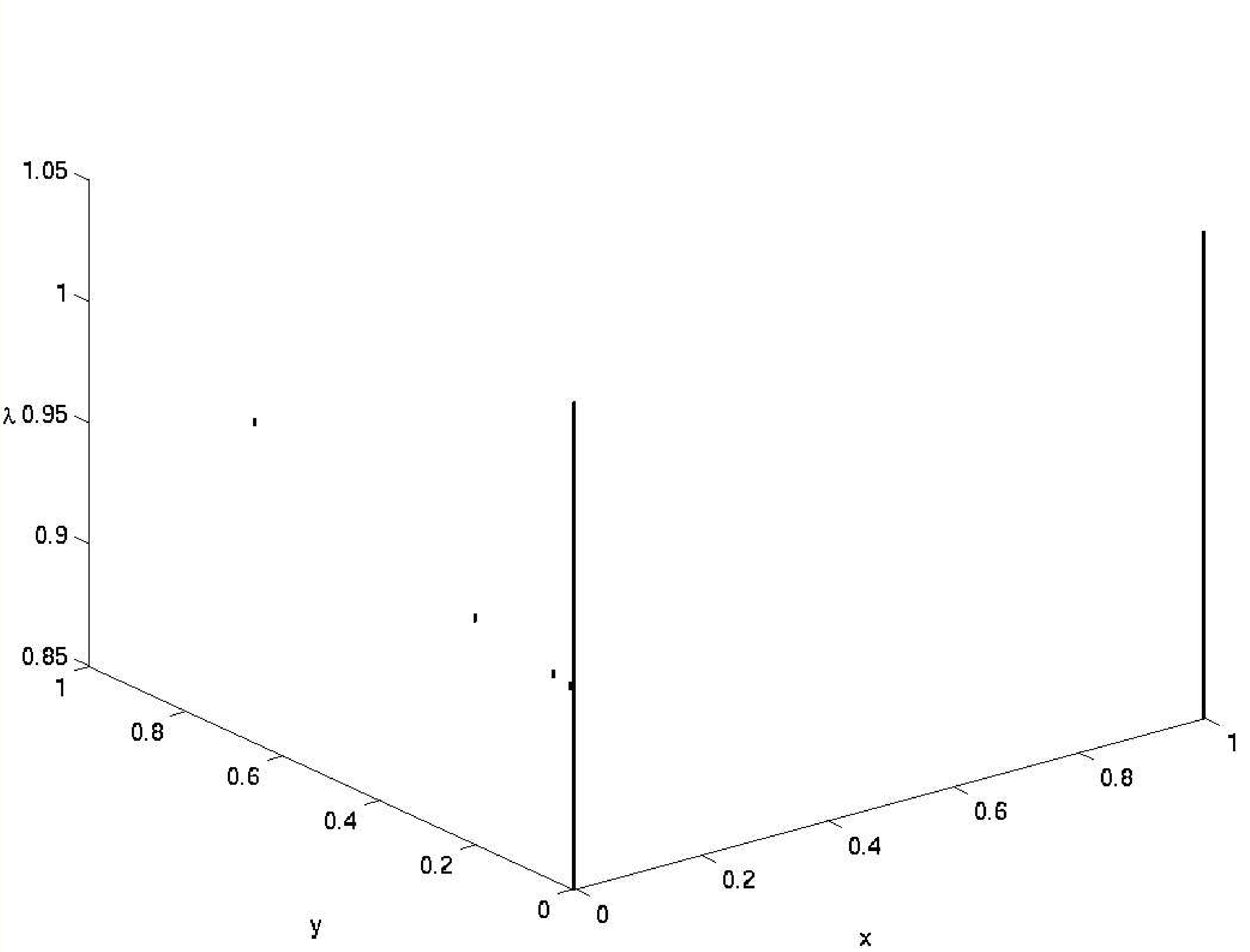
6.3 Step 1: Initial Guess



6.4 Step 2: Modify the initial guess

Algorithm (O. Junge)

```
$\mathcal{I} = \mathtt{make\_isolated}\,(\tilde{\mathcal{I}})$
    $\mathcal{I} := \mathrm{Inv}(\tilde{\mathcal{I}}, \mathcal{F})$
    **while** $o(\mathcal{I}) \not\subset \tilde{\mathcal{I}}$
        $\tilde{\mathcal{I}} := o(\mathcal{I})$
        $\mathcal{I} := \mathrm{Inv}(\tilde{\mathcal{I}}, \mathcal{F})$
    **if** $|\mathcal{I}| \subset \mathrm{int}\,|o(\mathcal{I})|$ **return** $\mathcal{I}$
    **else return** $\emptyset$
```


1.05
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λ 0.95
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0.85
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0.8
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0.4
0.2
0
y
0
0.2
0.4
0.6
0.8
1
x

6.5 Step 3: Construct the Index pair

Apply A. Szymczak's algorithm. Namely,

$$(Q_1, Q_0) = \Big(|(d(\mathcal{B}) \cap \mathcal{F}(\mathcal{B})) \cup \mathcal{B}|,\ |(d(\mathcal{B}) \cap \mathcal{F}(\mathcal{B}))|\Big)$$

is our index pair for S' where $\mathcal{B} = \mathrm{Inv}(\mathcal{I}, \mathcal{F})$.

6.6 Step 4

Applying the program `homcubes` written by P. Pilarczyk, we get

$$H_*(Q_1/Q_0) = \begin{cases} 0 & \text{if } * \neq 2 \\ \mathbb{Z}^7 & \text{if } * = 2 \end{cases}$$

$$\chi_2(S') = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \cong$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

6.7 Step 5

This step is an easy exercise in this example.
In general cases, one can the
Hartman-Grobman theorem for this step.

7 Tangencies

Let $f : M \to M$ be a diffeomorphism.

$$PM = \coprod_{x \in M} P_x M$$

$$Pf([v]) := [df(v)], \qquad 0 \neq v \in TM$$

$$\begin{tikzcd}
TM \setminus M \arrow[r, "df|_{TM\setminus M}"] \arrow[d, "\pi"'] & TM \setminus M \arrow[d, "\pi"] \\
PM \arrow[r, "Pf"] & PM
\end{tikzcd}$$

Let $p \in M$: f be a hyperbolic fixed point,

$T_pM = \tilde{E}^s_p \oplus \tilde{E}^u_p$.

$$E^s_p := \pi(\tilde{E}^s_p \setminus \{0\})$$

$$E^u_p := \pi(\tilde{E}^u_p \setminus \{0\})$$

(These are iso. inv. set w.r.t. Pf)

Theorem 5. *Let p, q be hyperbolic fixed points of f such that*

$$\dim W^u(p) + \dim W^s(q) \leq n.$$

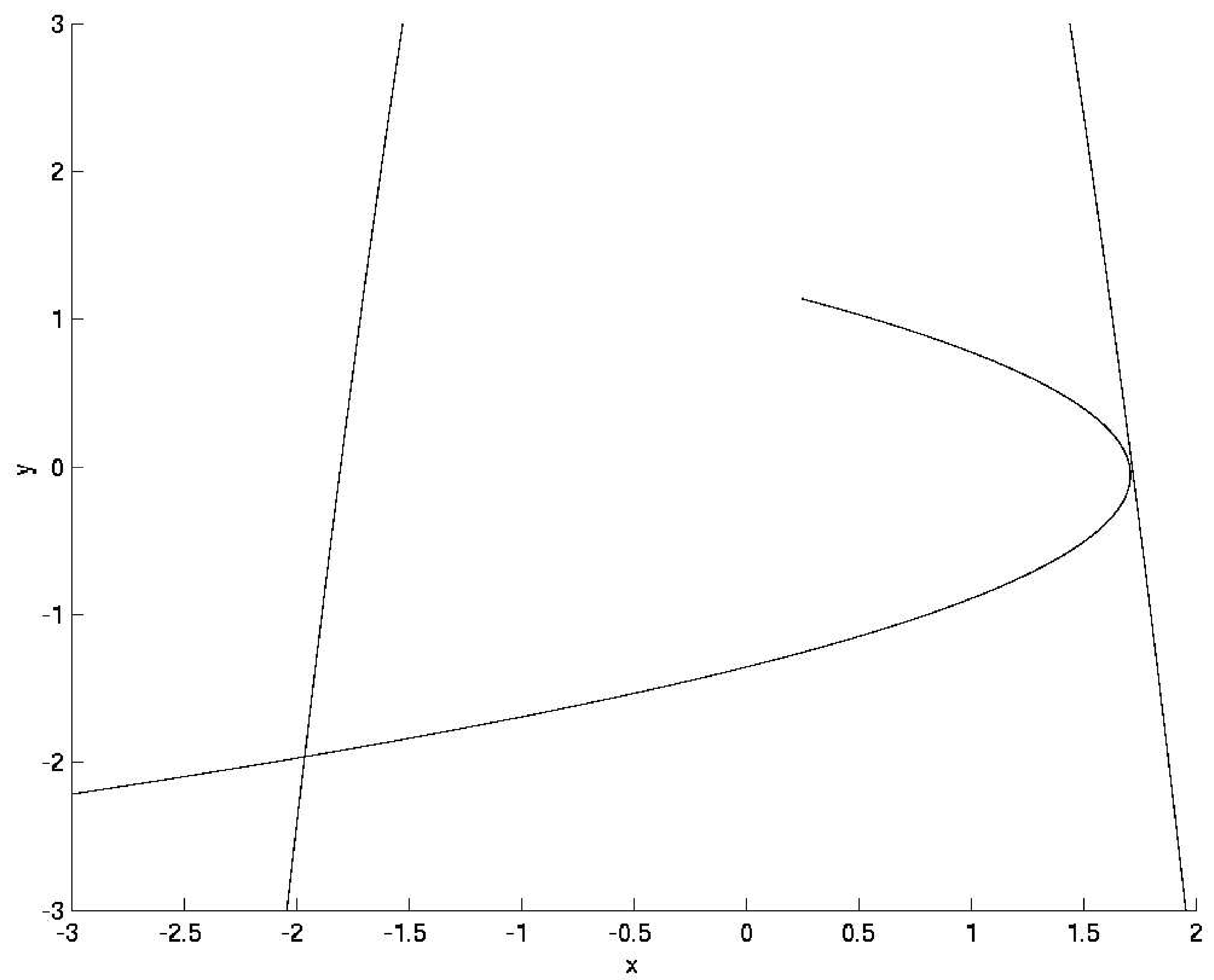
If there exists a connecting orbit from E_p^u *to* E_q^s *with respect to* Pf*, then* $W^u(p)$ *and* $W^s(q)$ *have a tangency.*

$$H_{a,b}(x,y) = (a - x^2 + by, x)$$

If there exists a connecting orbit from E_p^u *to* E_q^s *with respect to* Pf, *then* $W^u(p)$ *and* $W^s(q)$ *have a tangency.*

$$H_{a,b}(x,y) = (a - x^2 + by, x)$$

It seems that there exists a homoclinic tangency around $a = 1.3$, $b = -0.3$.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-2
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-3
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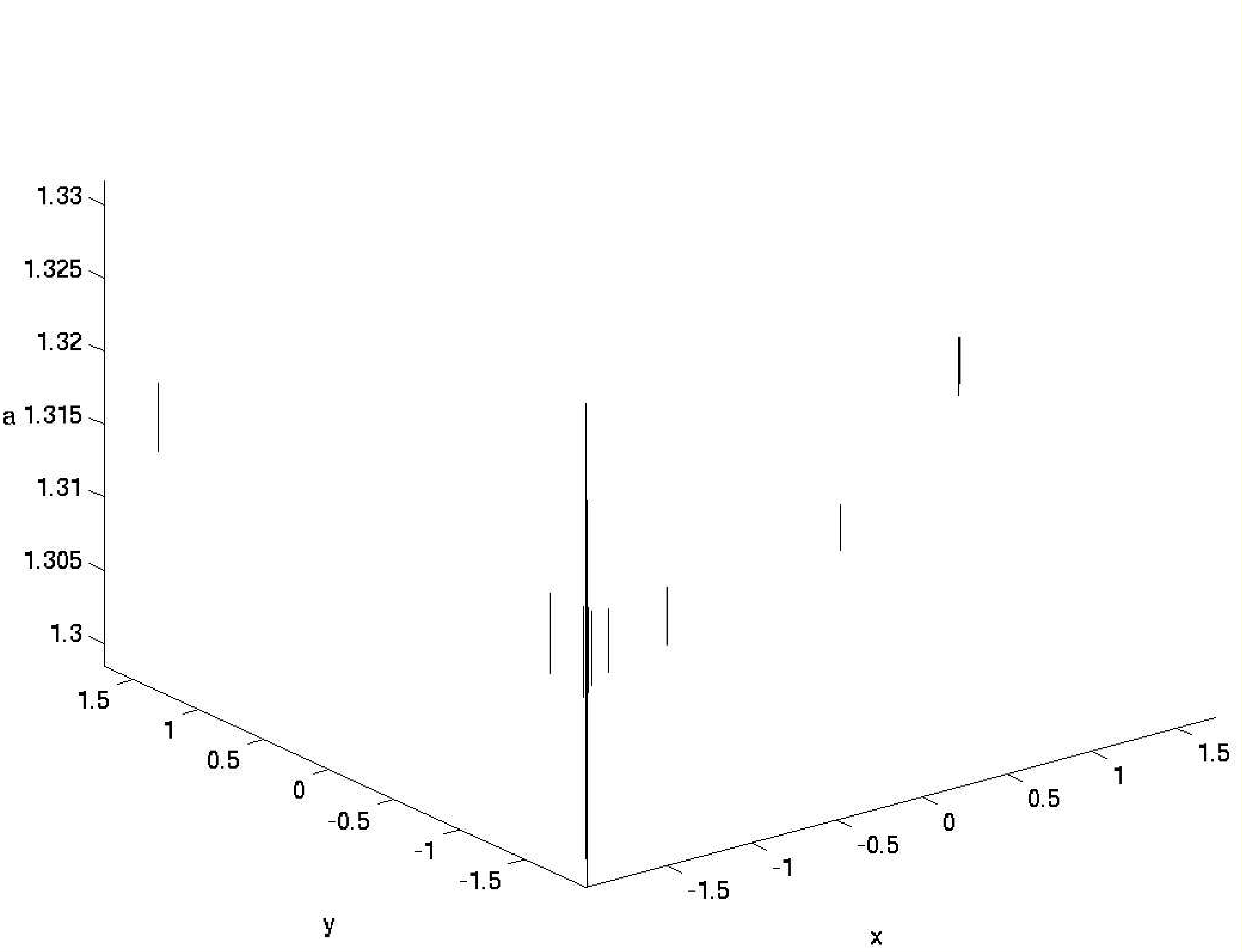
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

We fix b and consider $H_{a,b}$ as a one-parameter family with parameter a.

Now $M = \mathbb{R}^2$, $PM = \mathbb{R}^2 \times \mathbb{S}^1$. The fixed of our interest is $p(a, b)$ located at

$$x = y = \frac{b - 1 - \sqrt{(b-1)^2 + 4a}}{2}.$$

Our goal is to show the existence of connecting orbit form $E^u_a := E^u_{p(a,-0.3)}$ to $E^s_a := E^s_{p(a,-0.3)}$.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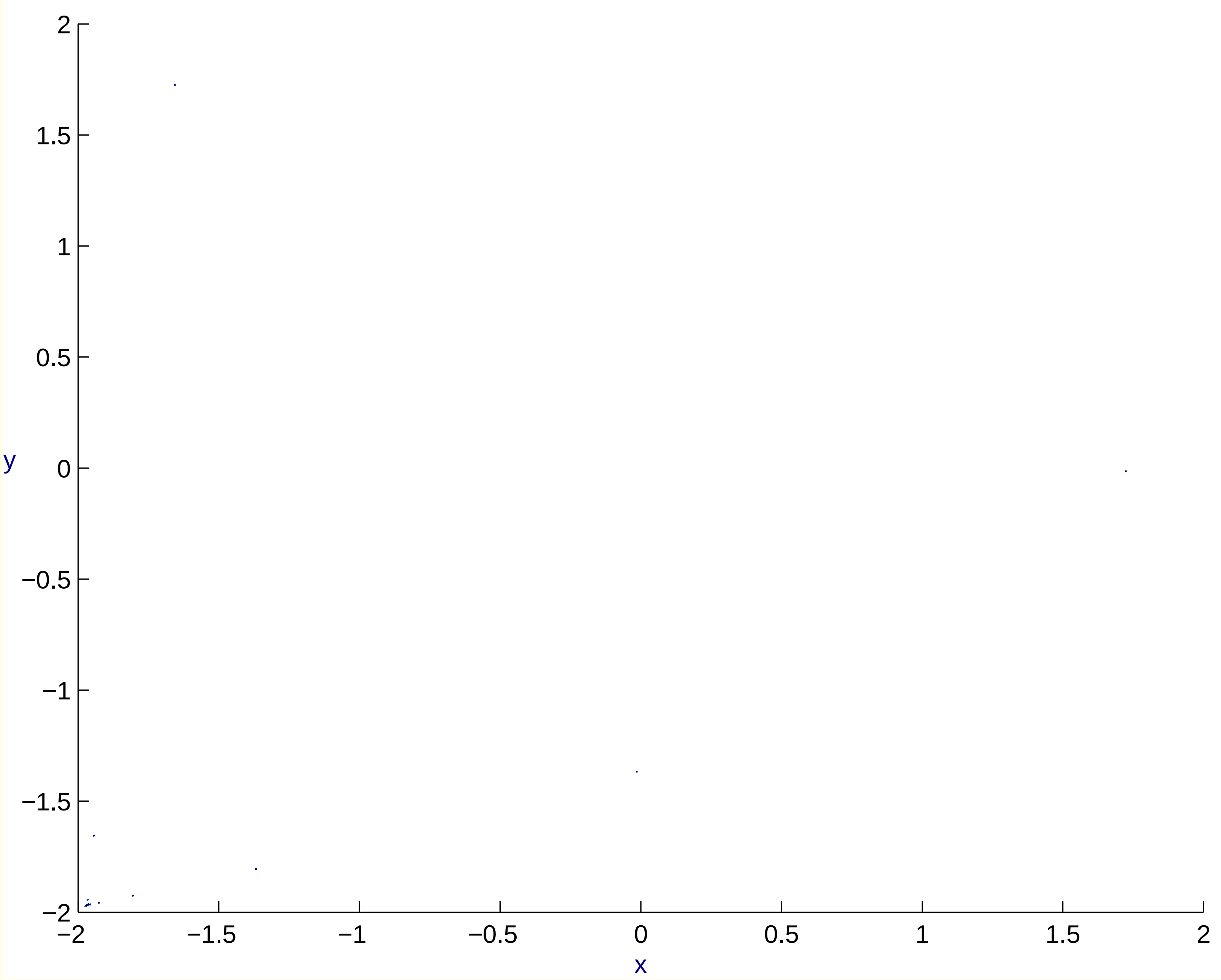
2
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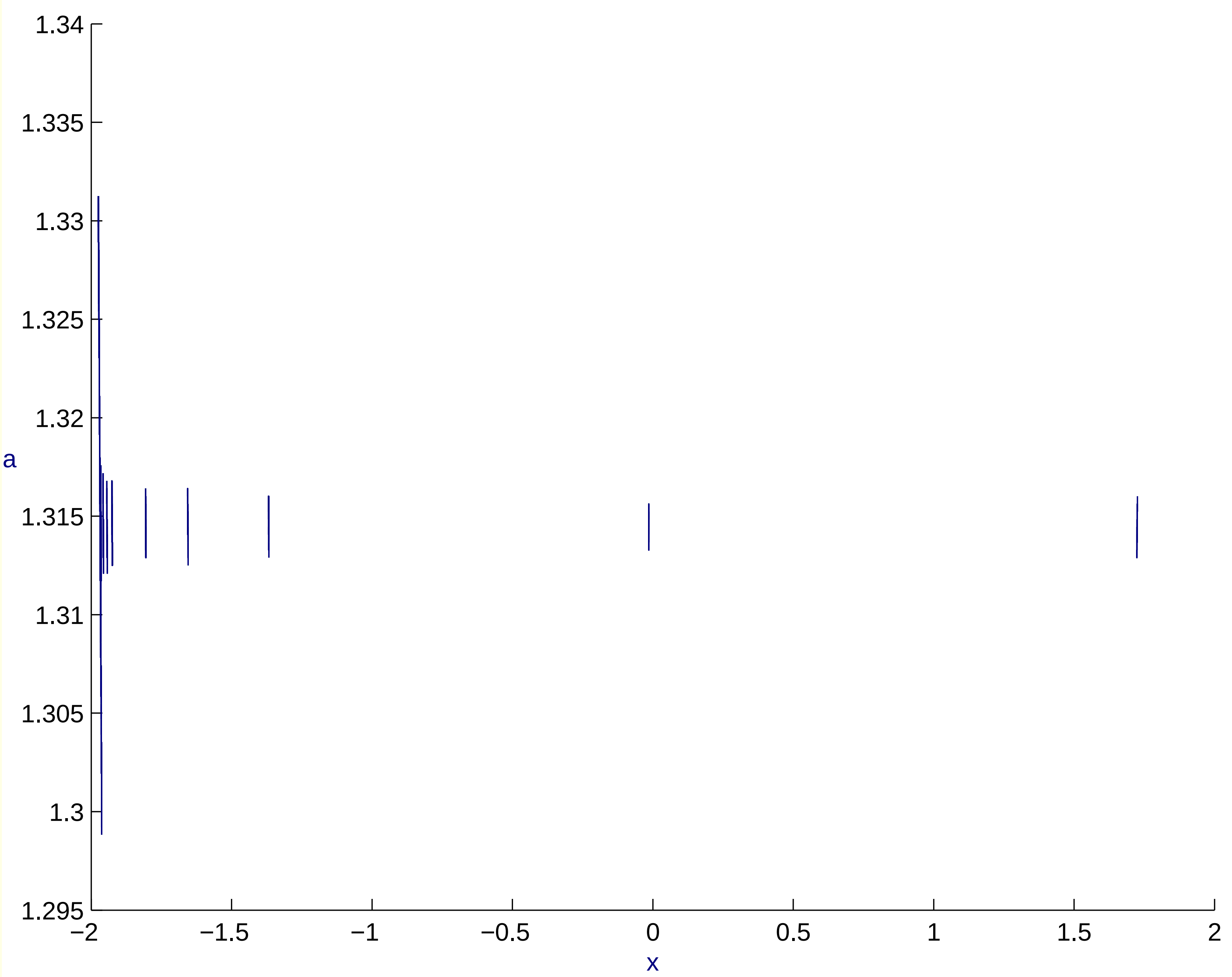
1.34
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x