Zariski N-plets for arrangements of plane curves of low degree and rational elliptic sufaces

Hiro-o Tokunaga

Tokyo Metropolitan University

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References:

[1] Sections of elliptic surfaces and Zariski pairs for conic-line arrangements via dihedral covers, J. of MSJ **66**(2014), 613-640.

[2] (with S. Bannai) Geometry of bisections of elliptic surfaces and Zariski N-plets for conic arrangements, preprint.

[3] (with S. Bannai, E. Yorisaki) Zariski N-plets for arrangement of curves of low degrees and rational elliptic sufaces, in preparation.

In this talk, we consider some arrangements of plane curves of low degree:

- ‘conic-line arrangements’
- ‘conic arrangements’
- ‘cubic-conic-line arrangements’

Let $\mathcal{B} = \sum_{i=1}^{r} \mathcal{B}_i$ be a reduced plane curve with irreducible components $\mathcal{B}_i$. We assume that $\deg \mathcal{B}_i \leq 3$

How do we give arrangements of curves with the same combinatorics so that they have different geometric backgrounds or features?

Here ‘different geometric feature’ means, for example,

Example(E. Artal Bartolo '94)

$\mathcal{B}_1, \mathcal{B}_2$: reduced plane curves of degree 6

The combinatorics of $\mathcal{B}_i$: A cubic C and its three inflectional tangent line. Let

$$\mathcal{B}_1 = C + L_1 + L_2 + L_3, \quad \mathcal{B}_2 = C + L_1 + L_2 + L_4.$$

The difference between $\mathcal{B}_1$ and $\mathcal{B}_2$ is as follows:

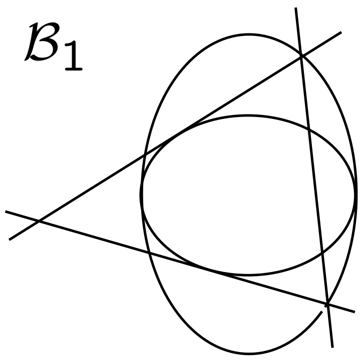
Put $P_i = C \cap L_i$, i.e., the tangent point of L_i.

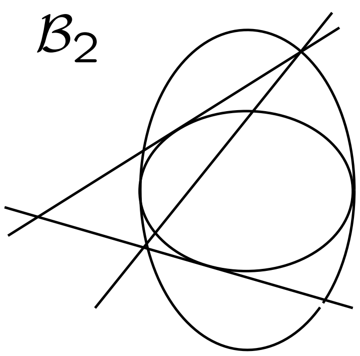
P_1, P_2, P_3 are collinear, while P_1, P_2, P_4 are <u>NOT</u>

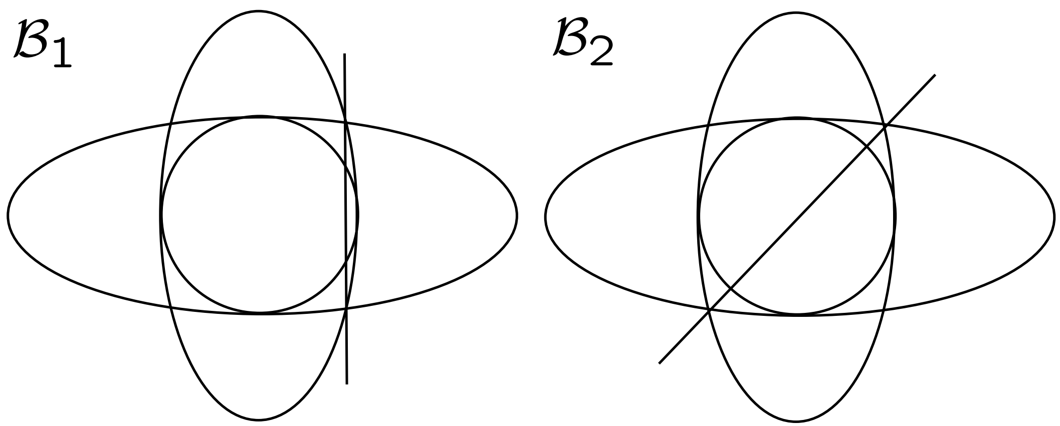
Then, $(\mathbb{P}^2, \mathcal{B}_1)$ and $(\mathbb{P}^2, \mathcal{B}_2)$ are not homeomorphic to each other.

Some more examples:

Example






$\mathcal{B}_1$
$\mathcal{B}_2$

Position of singularities or intersections can be seen in previous examples.

Here we introduce a ‘new point of view’ based on branched Galois covers:

Double cover and elliptic surfaces

Let $\mathcal{B} = \sum_i \mathcal{B}_i$ be as above.

Consider a double cover $f : S' \to \mathbb{P}^2$ whose branch curve is a part of $\mathcal{B}$, say $\mathcal{B}_1 + \ldots + \mathcal{B}_s$, and $\mu : S \to S'$ denotes the canonical resolution of singluarities of S'.

In order to study the topology of $(\mathbb{P}^2, \mathcal{B})$, we consider S and the preimages of $\mathcal{B}_{s+1}, \ldots, \mathcal{B}_r$.

The case we consider:

$$\mathcal{B} = \mathcal{Q} + \sum_i \mathcal{L}_i + \sum_j \mathcal{C}_j$$

where $\mathcal{Q}$, $\mathcal{L}_i$ and $\mathcal{C}_j$ denote a reducible quartic, a line and an irreducible conic, respectively, and $f' : S' \to \mathbb{P}^2$ is a double cover branched along $\mathcal{Q}$.

More precisely, we modify S so that S has a relatively minimal elliptic fibration.

We make use of the arithmetic properties of the elliptic fibration to study the preimages of $\mathcal{L}_i$ and $\mathcal{C}_j$, or to construct such curves (vice versa).

Before we go on further, let us recall the precise definition of ellitpic surfaces. $\varphi : S \to \mathbb{P}^1$: a rational elliptic surface with a section O, that is

- φ is a relatively minimal elliptic fibration,
- S is a rational surface.

E_S: the generic fiber of S. $\leftarrow$ an elliptic curve over $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$. It is well-known that we can endow E_S with the structure of abelian group.

Conversely, given an elliptic curve E over $\mathbb{C}(t)$, we have an elliptic surface S_E over $\mathbb{P}^1$ canonically, which is called the Kodaira-Néron model of E.

Note that S_E is not necessarily rational.

We here introduce some terminologies and notation:

- $\mathrm{Sing}(\varphi) := \{v \in C \mid \varphi^{-1}(v) \neq \text{a smooth curve of genus 1.}\}$

- $\mathrm{Red}\,\varphi := \{v \in \mathrm{Sing}(\varphi) \mid \varphi^{-1}(v) \text{ is reducible.}\}$

- For $v \in \mathrm{Red}(\varphi)$, we put

$$F_v = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} a_{v,i}\Theta_{v,i},$$

where m_v := the number of irreducible components of F_v and $\Theta_{v,0}$:= the irreducible component with $\Theta_{v,0}O = 1$ ($\leftarrow$ we call this *the identity component*.

- NS(S): the Néron-Severi group of S $\leftarrow$ A free group generated by irreducible curves on S modulo algebraic equivalence.

- T_φ := the subgroup of NS(S) generated by O, a fiber F, and $\Theta_{v,i}$'s ($v \in \mathrm{Red}(\varphi)$, $i = 1, \ldots, m_v - 1$).

- $E_S(\mathbb{C}(t))$: the set of $\mathbb{C}(t)$-rational points of E_S.

Under our settings:

Theorem (Shioda '90)

$$E_S(\mathbb{C}(t)) \cong \mathsf{NS}(S)/T_\varphi.$$

Roughly speaking, the isomorphism is given in the following way:

- Given a horizontal irreducible divisor D, $DF = d$, D meets a general fiber at d points.

- Adding up these points, we have one points. Collecting these points, we have a unique section $s(D)$.

- $s(D)$ gives rise to a rational point P_D.

- $E_S(\mathbb{C}(C))$ can be identified with the set of sections of $\varphi : S \to C$, MW(S). This means the LHS of Shioda's Theorem has both arithmetic and geometric aspects

- The RHS of Shioda's Theorem is determined by S and φ.

$\dot{+}$: the addition on $E_S(\mathbb{C}(C))$ (or MW(S)).

$+$: the addition on NS(S) (or that of divisors).

Example 1 Consider

$$E : y^2 = f_E(t, x) = (x - t^2)(x - 3t + 2)(x + 3t + 2).$$

(The Kodaira-Néron model S_E is rational.)

We have

$$P_1 = (t + 2, 2\sqrt{2}(t - 2)(t + 1)) \in E(\mathbb{C}(t)),$$

and

$$[2]P_1 = \left(\frac{9}{8}t^2, \frac{1}{32}\sqrt{2}t(9t^2 - 16)\right).$$

Moreover,

Consider a line (over $\mathbb{C}(t)$):

$$L : y = \left(\frac{1}{2\sqrt{2}}t + c\right)\left(x - \frac{9}{8}t^2\right) + \frac{1}{32}\sqrt{2}t(9t^2 - 16) \quad c \in \mathbb{C}.$$

Eliminating y from E and L, we have

$$\begin{aligned} & (x - t^2)(x - 3t + 2)(x + 3t + 2) \\ & - \left\{\left(\frac{1}{2\sqrt{2}}t + c\right)\left(x - \frac{9}{8}t^2\right) - \frac{1}{32}\sqrt{2}t(9t^2 - 16)\right\}^2 \\ = \; & \frac{1}{64}(4c\sqrt{2}tx - 9c^2t^2 - 8\sqrt{2}ct + 8c^2x + 64t^2 - 8x^2 - 32x - 32) \\ & \times(9t^2 - 8x) \end{aligned}$$

Now the conic $\mathcal{C}_{c,1}$ given by

$$\mathcal{C}_{c,1} : 4c\sqrt{2}tx - 9c^2t^2 - 8\sqrt{2}ct + 8c^2x + 64t^2 - 8x^2 - 32x - 32 = 0$$

is tangent to $\{f_E(t, x) = 0\}$ at 4 distinct points for general c.

By using $P_2 = (1, 3(t+1)(t-1)), [2]P_2 = (t^2+1/4, 1/2t^2-9/8)$, we have another family of conics $\mathcal{C}_{c,2}$ $c \in \mathbb{C}$ given by

$$-140t^2+16c^2t^2-16ct^2+68x+4c^2+16x^2+81+36c-16c^2x = 0.$$

For general $c \in \mathbb{C}$,

$\mathcal{C}_{c,2}$ is also tangent to $\{f_E(t,x) = 0\}$ at 4 distinct points.

Note that $P_1, [2]P_1, P_2, [2]P_2$ give plane curves

$$x - (t + 2) = 0, x - \frac{9}{8}t^2 = 0, x - 1 = 0, x - (t^2 + 1/4) = 0.$$

"Given a plane curve, one can construct another plane curve by using *arithmetic* properties, $\dot{+}$ or $s(\bullet)$, of S "

$$\begin{array}{ccc} \mathsf{Div}(S) & \longrightarrow & \mathsf{Div}(S) \\ \downarrow & & \downarrow \\ \mathsf{Div}(\mathbb{P}^2) & \longrightarrow & \mathsf{Div}(\mathbb{P}^2) \end{array}$$

Moreover, one can prove certain topological statements by using arithmetic properties of S via dihedral covers. In fact, we have:

Theorem (- [1]) Let $\mathcal{B} = \mathcal{Q} + \sum_i \mathcal{L}_i + \sum_j \mathcal{C}_j$ and $f : S \to \mathbb{P}^2$ as before. Choose $\mathcal{L}_{i_1}, \dots, \mathcal{L}_{i_s}$ and $\mathcal{C}_{j_1}, \dots, \mathcal{C}_{j_t}$.

Then $\exists$ a D_{2p}-cover of $\mathbb{P}^2$ branched at $2\mathcal{Q}+p(\sum_{k=1}^{s} \mathcal{L}_{i_k}+\sum_{l=1}^{t} \mathcal{C}_{j_l})$

<u>if and only if</u>

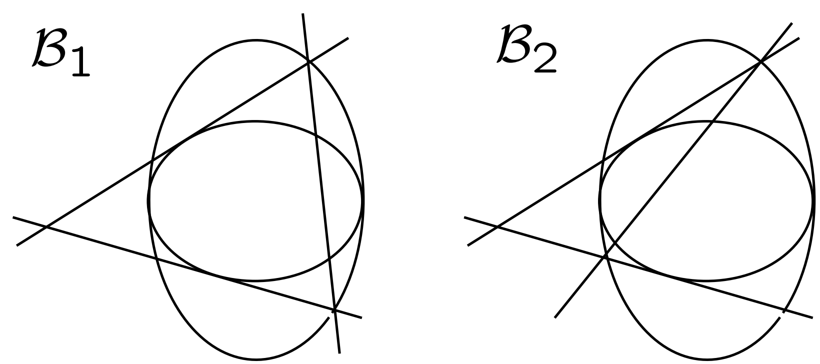
- $f^*\mathcal{L}_{i_k}\setminus\{\text{exceptional set}\}$ and $f^*\mathcal{C}_{j_l}\setminus\{\text{exceptional set}\}$ consist of two irreducible components $\mathcal{L}_{i_k}^{\pm}, \mathcal{C}_{j_l}^{\pm}$ for all i_k, j_l. (i.e., $\mathcal{L}_{i_k}, \mathcal{C}_{j_l}$ do not meet $\mathcal{Q}$ with *odd* intersection multiplicity.)

- $\sum_k [a_k] P_{\mathcal{L}_{i_k}^+} \dot{+} \sum_l [b_l] P_{\mathcal{Q}_{j_l}^+} \in [p]E_S(\mathbb{C}(t))$ for some $0 < a_k, b_l < p$

By Theorem, we have Zariski pairs as folllows:

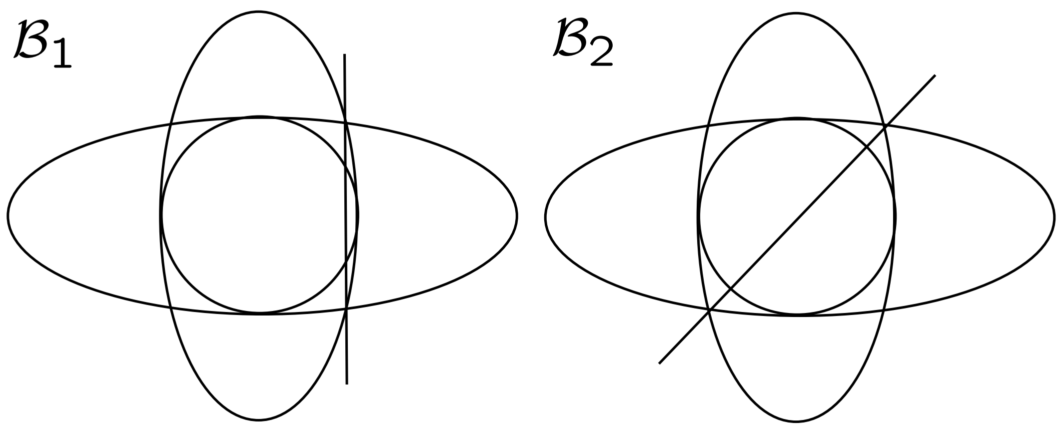
From Example 1 before

Example 2 Two conics +three lines



Moreover, by the same argument, we have

Example 3 Three conics + a line



In both of Examples 2 and 3, **existence of a special conic** makes the difference:

- $\exists$ a conic C_o passing through the 4 tacnodes and 2 triple points for $\mathcal{B}_1$ (resp. $\mathcal{B}_2$)

- $\not\exists$ a conic C_o passing through the 4 tacnodes and 2 triple points for $\mathcal{B}_2$ (resp. $\mathcal{B}_1$).

The existence of C_o is equivalent to that of the D_{2p}-cover in Theorem.

In the case of Example 1:

The tangent line at P_1 is given by

$$y = -\frac{\sqrt{2}}{4}(t+8)(x-t-2) + 2\sqrt{2}(t-2)(t+1)$$

The LHS $= 0$ gives a conic $\mathcal{C}_o$ and we have

$$\begin{aligned} & (x-t^2)(x-3t+2)(x+3t+2) \\ & - \left\{ -\frac{\sqrt{2}}{4}(t+8)(x-t-2) + 2\sqrt{2}(t-2)(t+1) \right\}^2 \\ = \ & \left(x - \frac{9}{8}t^2 \right) \{x-(t+2)\}^2 \end{aligned}$$

Thus $\mathcal{C}_o$ is the conic described above.

The method using arithmetic of elliptic curves over $\mathbb{C}(t)$ gives more examples of Zariski N-plets for

- conic-line arrangements,
- conic-arrangements and
- cubic-conic-line arrangements.

(see [2], [3])

Problems

- Example 3 can be a candidate for a Zariski 3-plet. (But our method does work at this point.) Is it a Zariski 3-plet or not.

- Proof without using arithmetic of elliptic curves over $\mathbb{C}(t)$.

Thank you for your attention!