Multiple addition theorem on arrangements of hyperplanes and a proof of the Shapiro-Steinberg-Kostant-Macdonald dual-partition formula

Hiroaki Terao

(Hokkaido University, Sapporo, Japan)

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Credit

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with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

Torsten Hoge (Leibniz Universität Hannover)

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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free S-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

Free Arrangements and their Exponents

Example.
(the braid arrangement (Weyl arrangement of type A_3))

$$\mathcal{A} := \{\ker(x_i - x_j) \mid 1 \leq i < j \leq 4\}$$

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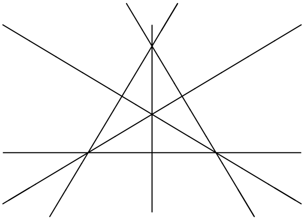

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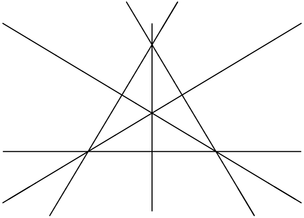

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The exponents are

$$(0, 1, 2, 3)$$

because ...

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The S-module $D(\mathcal{A})$ is a free module with a basis

$$\begin{aligned}
\theta_0 &= (\partial/\partial x_1) + (\partial/\partial x_2) + (\partial/\partial x_3) + (\partial/\partial x_4)\\
\theta_1 &= x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3) + x_4(\partial/\partial x_4)\\
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\end{aligned}$$

Thus the exponents are:

$$(\deg\theta_0, \deg\theta_1, \deg\theta_2, \deg\theta_3) = (0, 1, 2, 3).$$

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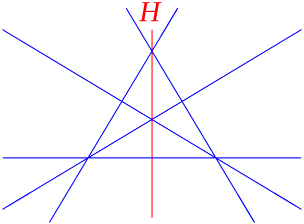

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braid arrangement A_3

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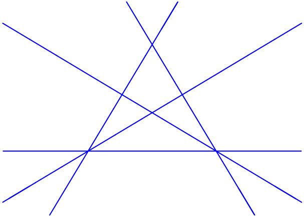

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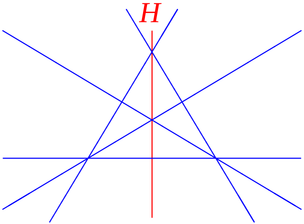

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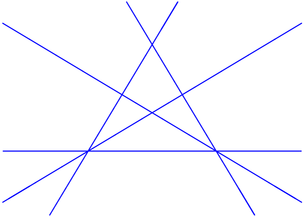

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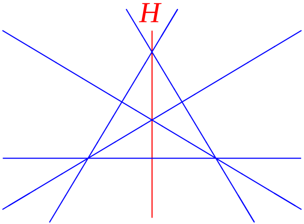

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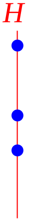

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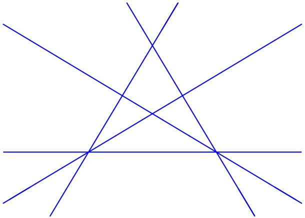

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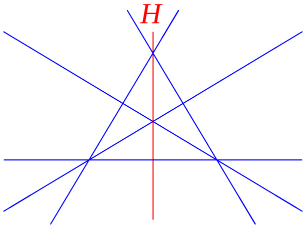

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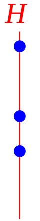

$\mathcal{A}''$

In this case we have:

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}), \quad \exp(\mathcal{A}'') = (0, 1, 2).$$

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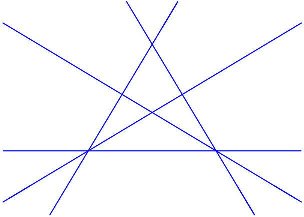

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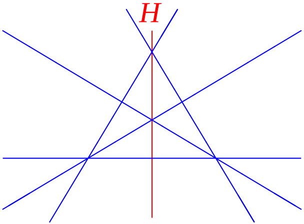

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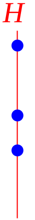

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This example is generalized into the Addition Theorem (AT) ....

Addition Theorem (AT)

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Theorem

(H. T.(1980)) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$*, suppose that* $\mathcal{A}'$ *is free with* $\exp(\mathcal{A}') = (d_1, d_2, \ldots, d_{\ell-1}, \underline{d_\ell})$ *and* $\mathcal{A}''$ *is free with* $\exp(\mathcal{A}'') = (d_1, d_2, \ldots, d_{\ell-1})$*. Then* $\mathcal{A}$ *is also free with* $\exp(\mathcal{A}) = (d_1, d_2, \ldots, \underline{d_\ell + 1})$.

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Recall that, for the braid arranegemnt (the Weyl arrangement of type A_3,

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Remark. In the AT, d_ℓ is not necessarily the maximum exponent in $\exp(\mathcal{A}')$.

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(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ $(j = 1, \ldots, q)$

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Then (a) $q \le p$

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(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ $(j = 1, \ldots, q)$ (**Remark:** *$\le$ always holds true)* .
Then (a) $q \le p$ and (b) $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.

Addition Theorem (AT)

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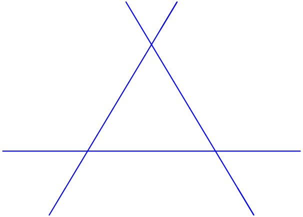

$\mathcal{A}'$

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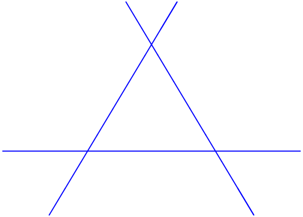

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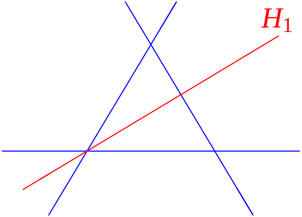

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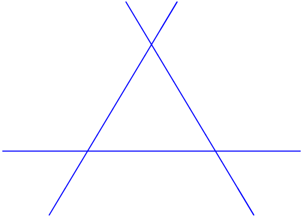

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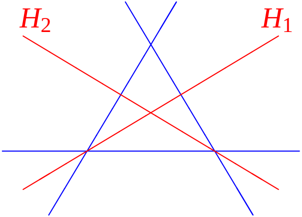

$\mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\}$

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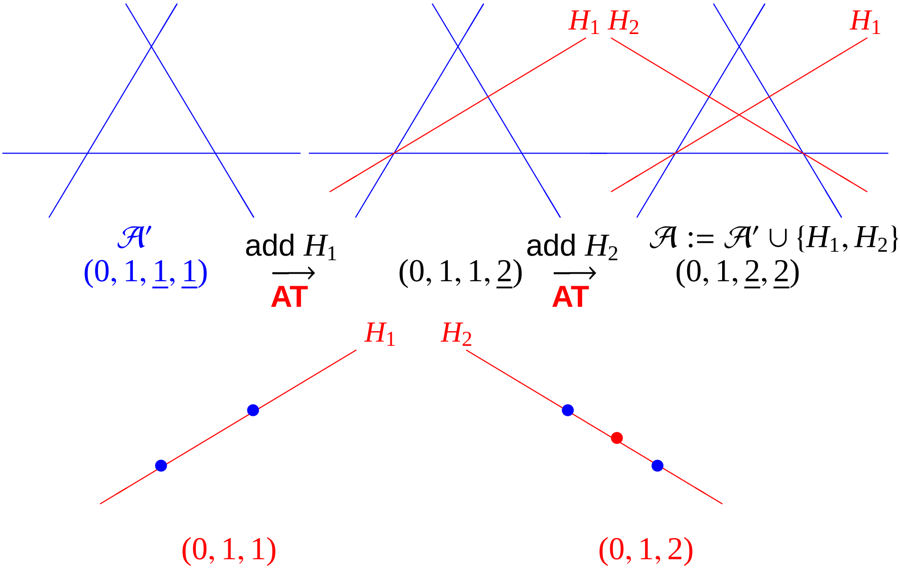

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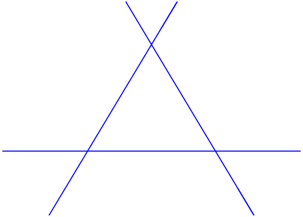

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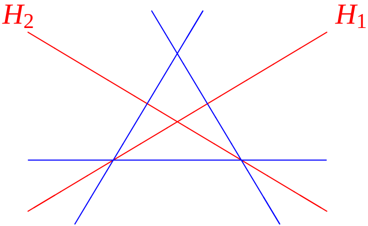

$\mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\}$

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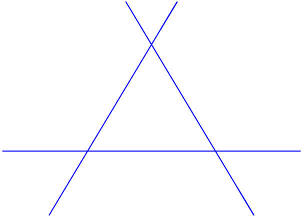
Multiple Addition Theorem (MAT)



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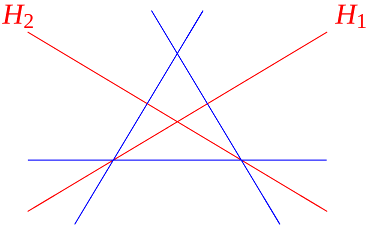
add 2 hyperplanes

$\Longrightarrow$

MAT



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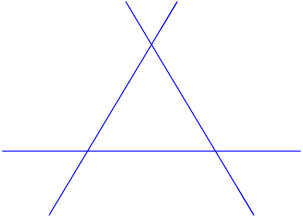

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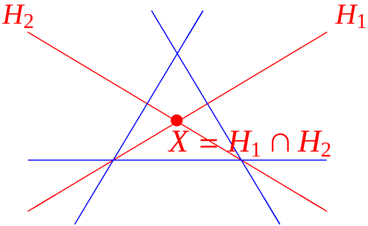

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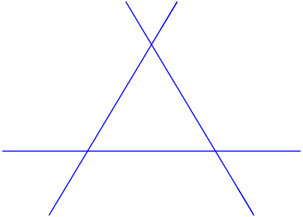

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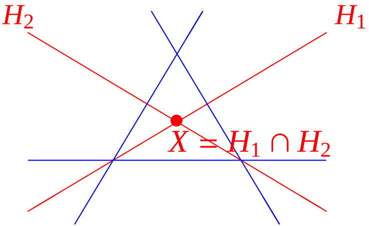

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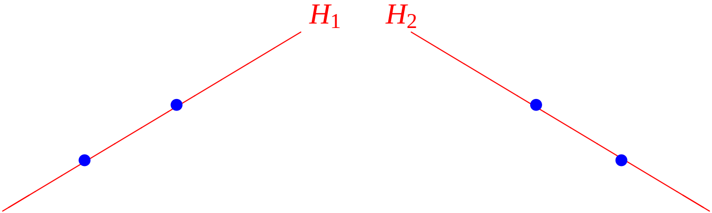

$\mathcal{A}''_1$ $\mathcal{A}''_2$

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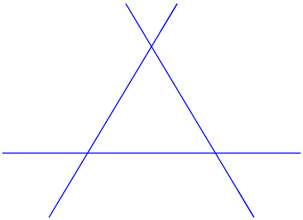

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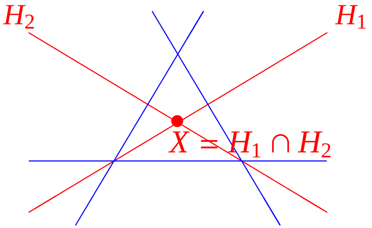

$\mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\}$

$(0, 1, \underline{2}, \underline{2})$

$|\mathcal{A}'| - |\mathcal{A}''_1| = 3 - 2 = \underline{1}$

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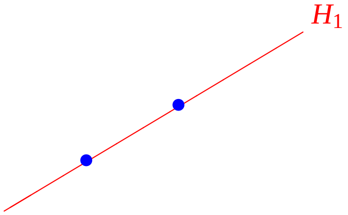

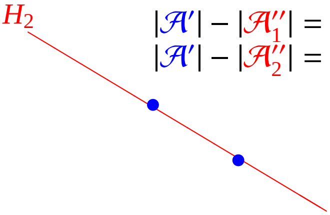

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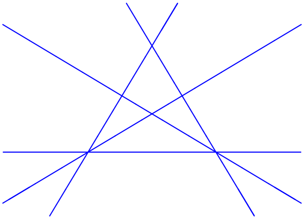

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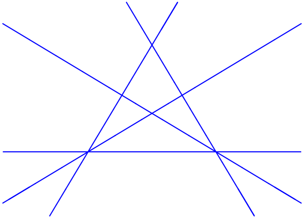

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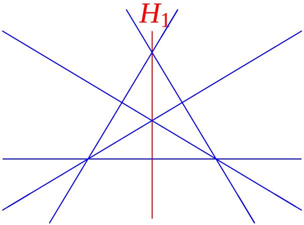

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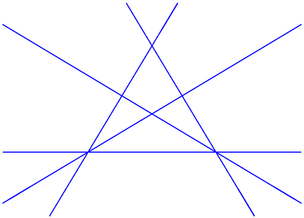

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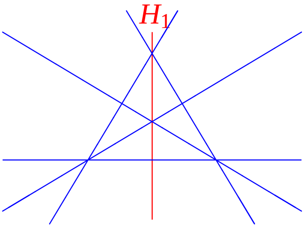

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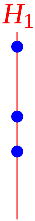
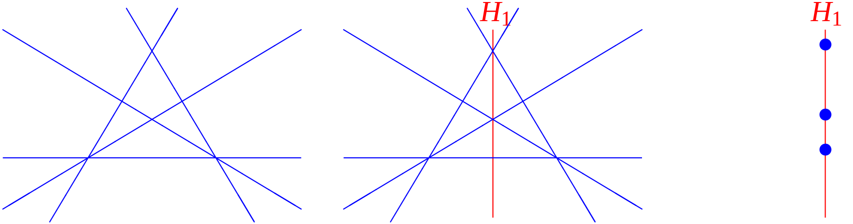


$\mathcal{A}_1''$

Multiple Addition Theorem (MAT)



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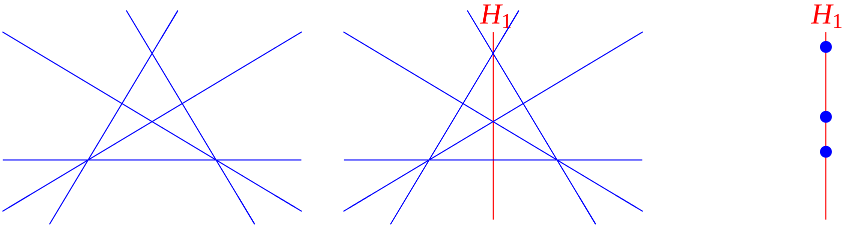

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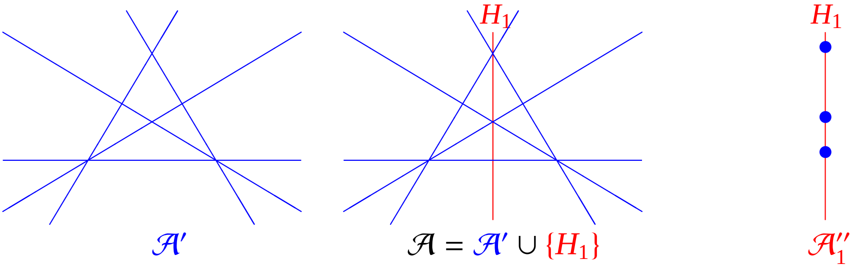

$\mathcal{A}'$ is free with exponents $(1, \underline{2}, \underline{2})$, $\quad d = 2$ (the max exponent).
$\mathcal{A}''_1 := \{H \cap H_1 \mid H \in \mathcal{A}'\}, |\mathcal{A}''_1| = 3$ and $|\mathcal{A}'| - |\mathcal{A}''_1| = 5 - 3 = \underline{2}$.
Thus $\mathcal{A} = \mathcal{A}' \cup \{H_1\}$ with exponents $(1, 2, \underline{3})$

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Question. Is there any significant application of MAT?

Contents

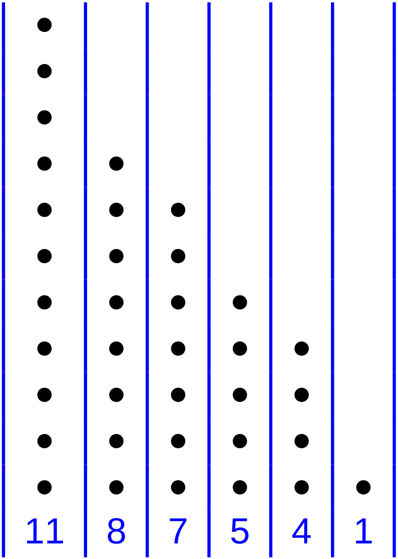
What are Dual Partitions?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are Dual Partitions?



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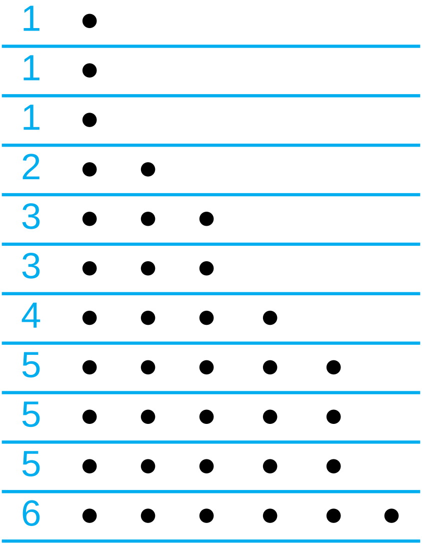

What are Dual Partitions?

1	•					
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
	11	8	7	5	4	1

What are these numbers?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are these numbers?

$(1, 4, 5, 7, 8, 11)$ is the exponents of the root system of the type E_6

$\updownarrow$ Dual Partitions

$(1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 6)$ is the height distribution of the positive roots of the type E_6

the dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald

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Theorem

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(1) This theorem can be (was) regarded as a method to "reading off" the exponents from the root structure.

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Remark

(1) This theorem can be (was) regarded as a method to "reading off" the exponents from the root structure.
(2) The other methods to find the exponents include: (a) from the degrees of basic invariants, (b) from the eigenvalues of a Coxeter transformation, etc.

Exponents

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Dynkin diagrams (root systems) and exponents

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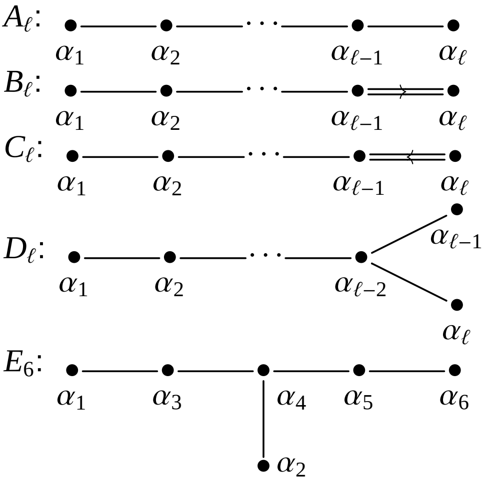


A_ℓ: $(1, 2, \ldots, \ell)$

B_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

C_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

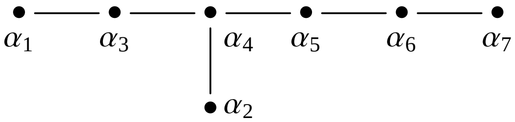
D_ℓ: $(1, 3, 5, \ldots, 2\ell - 3, \ell - 1)$

E_6: $(1, 4, 5, 7, 8, 11)$

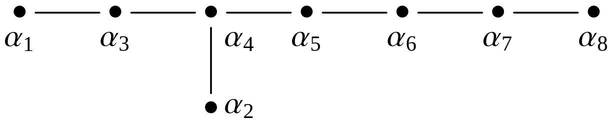
Exponents

Dynkin diagrams (root systems) and exponents

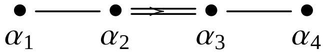
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

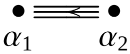


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

Height of positive roots

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- $\mathrm{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (height) for a positive root $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ $(c_i \in \mathbb{Z}_{\geq 0})$
- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}|$ $(1 \leq j \leq m)$

Height of positive roots (E_6)

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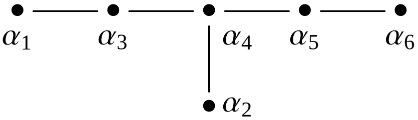
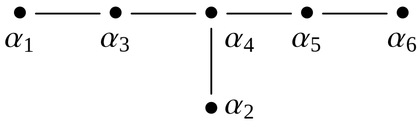
E_6:



Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:

E_6:



Exponents: $(1, 4, 5, 7, 8, 11)$

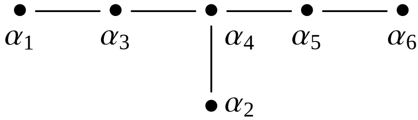
List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

E_6:



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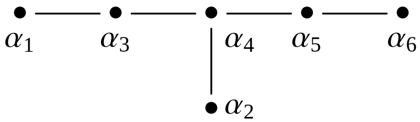

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.

.

.

.

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

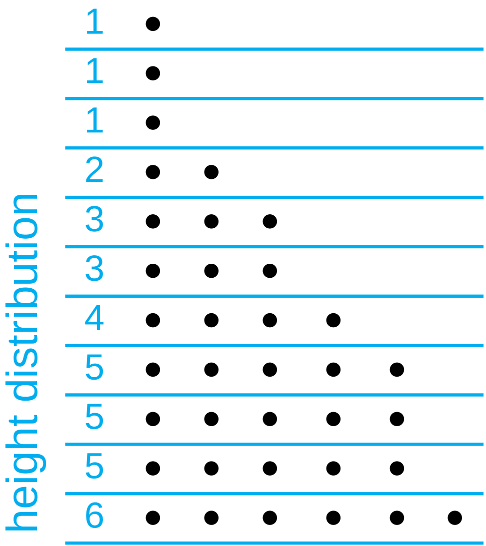
Height of positive roots (E_6)

heights

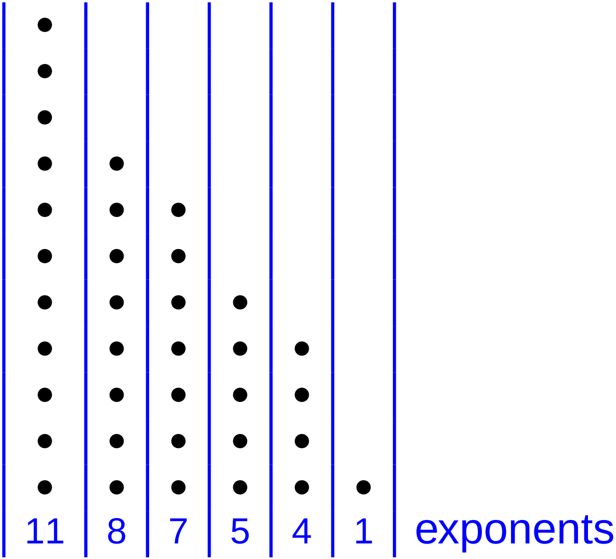
ht=11	$\tilde{\alpha}$					
ht=10	•					
ht=9	•					
ht=8	•	•				
ht=7	•	•	•			
ht=6	•	•	•			
ht=5	•	•	•	•		
ht=4	•	•	•	•	•	
ht=3	•	•	•	•	•	
ht=2	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_4$	$\alpha_3 + \alpha_4$	•	•	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)



Exponents (E_6)



The Dual-Partition Formula (E_6)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

History of the Dual-Partition Formula

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By Bertram Kostant.

...... we shall presently describe, of "reading off" the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. ...... However, even though one verifies that the numbers produced by this procedure agree with the exponents ...... the important question of proving that this "agreement" is more than just a coincidence remained open.

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Contents

Weyl arrangements

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The poset is called the (positive) root poset.
A subset I of Φ^+ is called an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$,

$$\beta_1 \geq \beta_2,\ \beta_1 \in I \Rightarrow\ \beta_2 \in I.$$

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Definition

Introduce a partial order $\geq$ into the set Φ^+ of positive roots by

$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i.$$

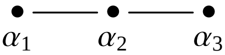
The poset is called the (positive) root poset.
A subset I of Φ^+ is called an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$,

$$\beta_1 \geq \beta_2,\ \beta_1 \in I \Rightarrow\ \beta_2 \in I.$$

Definition

When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker\alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.

Examples of ideals/non-ideals of the root poset of A_3

A_3: 

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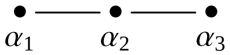
A_3: 

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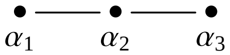
A_3: 

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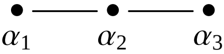

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Note that the entire set Φ^+ is always an ideal.

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For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^{\perp}$. Then Φ_X is a root system of rank $\operatorname{codim} X$.

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(3) $|\mathcal{A}'| - |\mathcal{A}_j''| = d$ $(j = 1, \ldots, q)$ (**Remark:** $\leq$ *always holds true)* .
Then (a) $q \leq p$ *and (b)* $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ *is free with exponents* $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.

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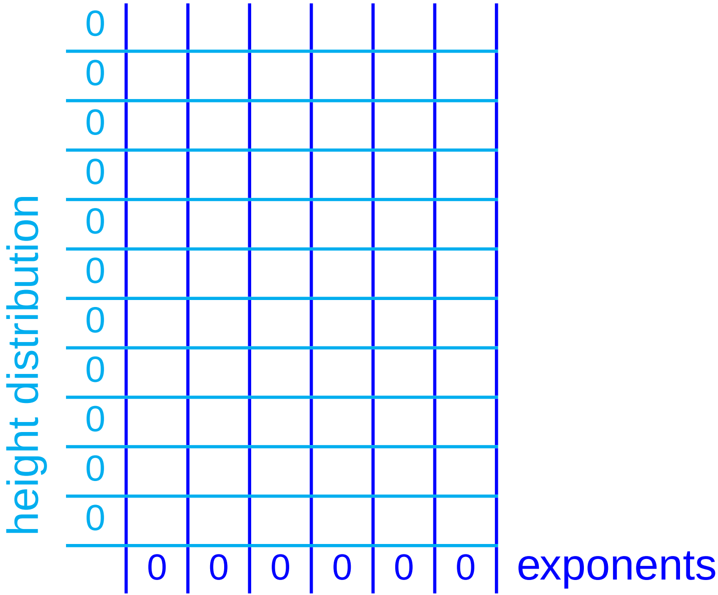

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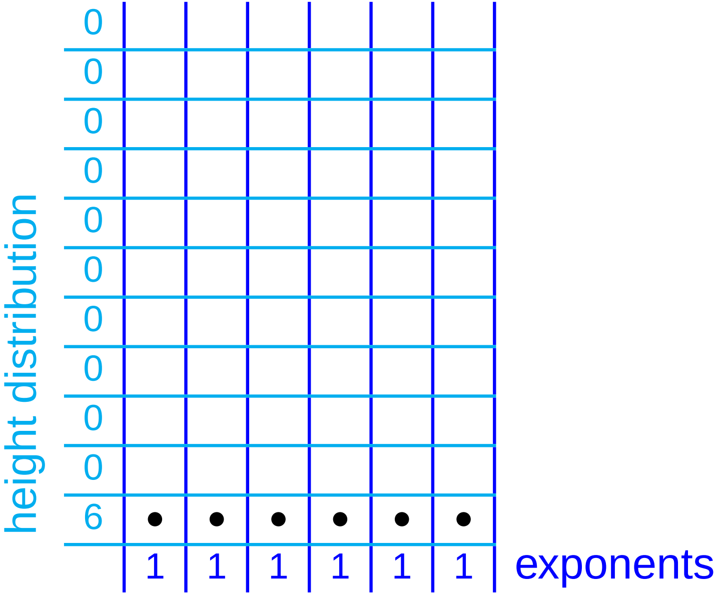

Inductive use of MAT (E_6) : $I = \Phi_2^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
0						
0						
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	2	2	2	2	2	1

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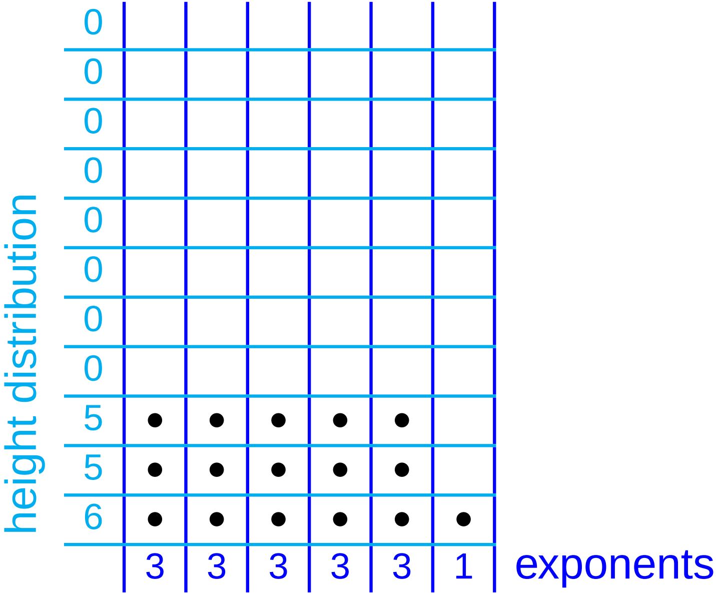

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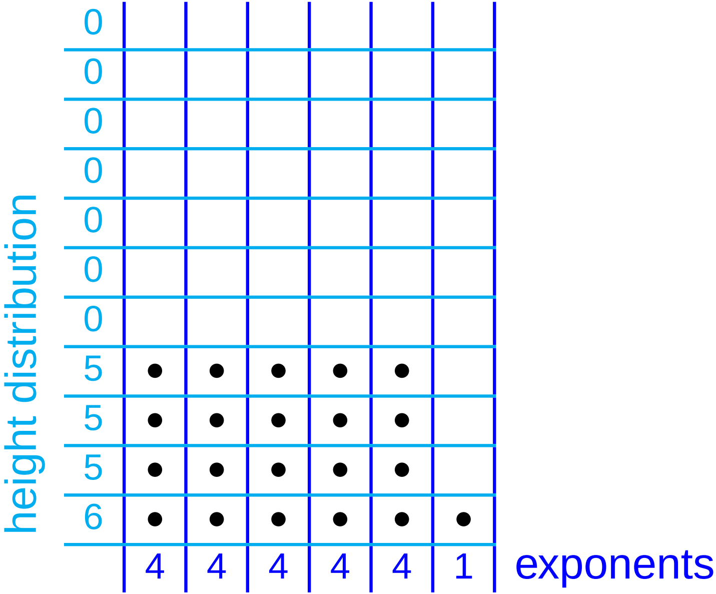

Inductive use of MAT (E_6) : $I = \Phi_5^+$

height distribution						
0						
0						
0						
0						
0						
0						
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	5	5	5	5	4	1

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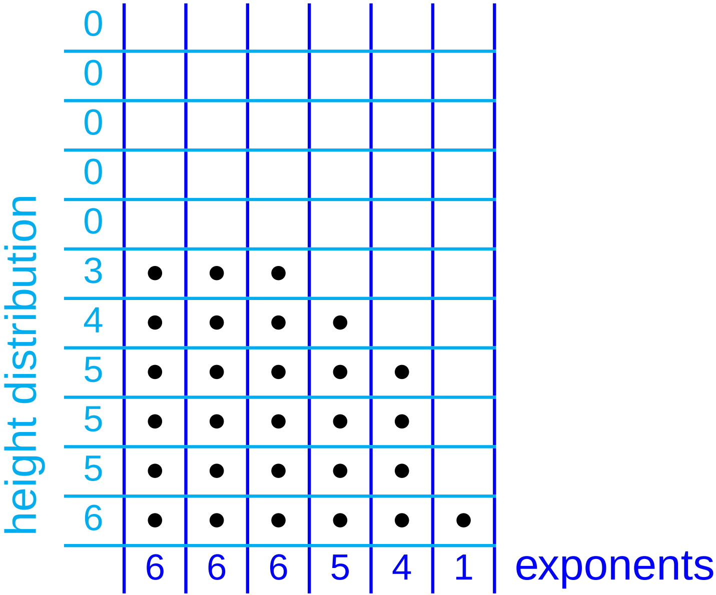
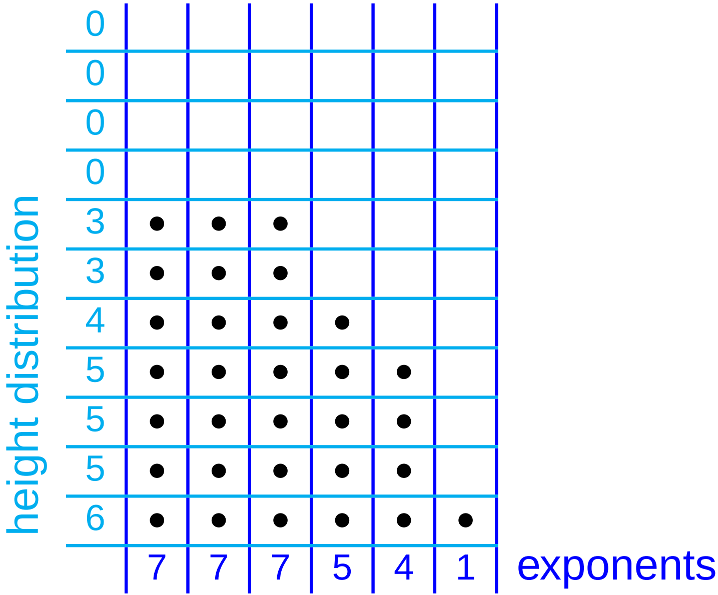
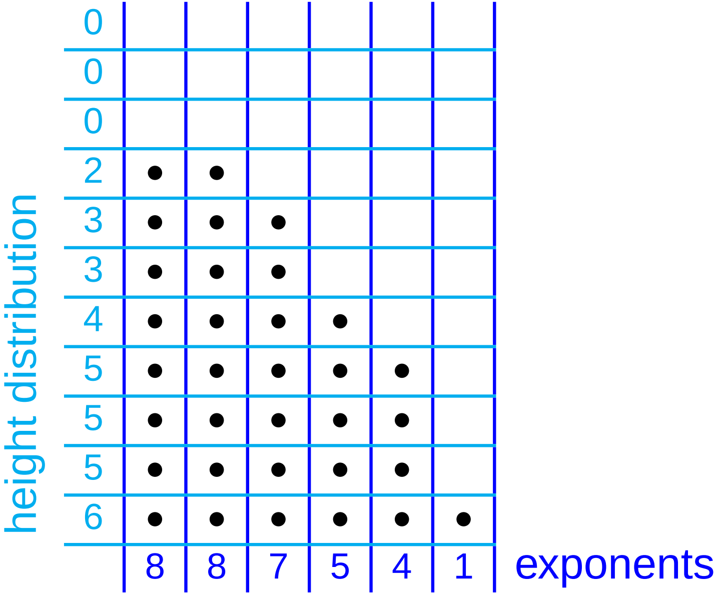


Inductive use of MAT (E_6) : $I = \Phi_7^+$



Inductive use of MAT (E_6) : $I = \Phi_8^+$



Inductive use of MAT (E_6) : $I = \Phi_9^+$

height distribution						
0						
0						
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	9	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi^+_{10}$

height distribution						
0						
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	10	8	7	5	4	1

The Dual-Partition Formula (E_6) (again)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

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2. Although the MAT is similar to the old addition theorem (AT) (1980), it does not generalize the AT .
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4. Moreover, we have the dual-partion formula for any ideal subarrangements.

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Thanks for your attention!