Workshop on The Algebraic Geometry and Topology of Hyperplane Arrangements,
Northeastern University, Boston, April 8, 2011.

Irreducibility of Moduli Spaces of Line Arrangements

(Joint work with Shaheen Nazir)

Masahiko Yoshinaga (Kyoto)

Contents

§1 Introduction: moduli space $R(I)$.
(What and Why?)

§2 (Dis)connectivity:
when is $R(I)$ connected/disconnected?

§3 Classification up to 9 lines.

1 Intro: moduli space $R(I)$

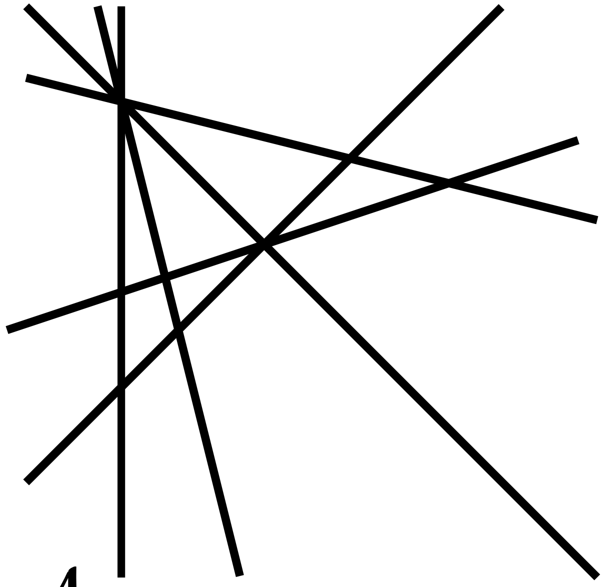
Basic Setting:

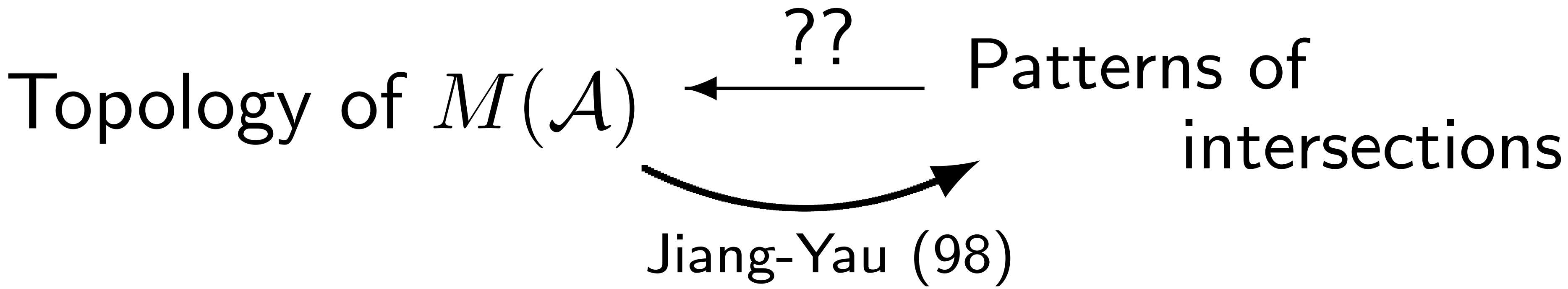
$$\mathcal{A} = \{H_1, H_2, \ldots, H_n\},$$

($H_i \subset \mathbb{P}^2_{\mathbb{C}}$: a line on $\mathbb{P}^2_{\mathbb{C}}$).

$$M(\mathcal{A}) = \mathbb{P}^2_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} H_i,$$

the complement of the arrangement $\mathcal{A}$.





Topology of $M(\mathcal{A})$

??

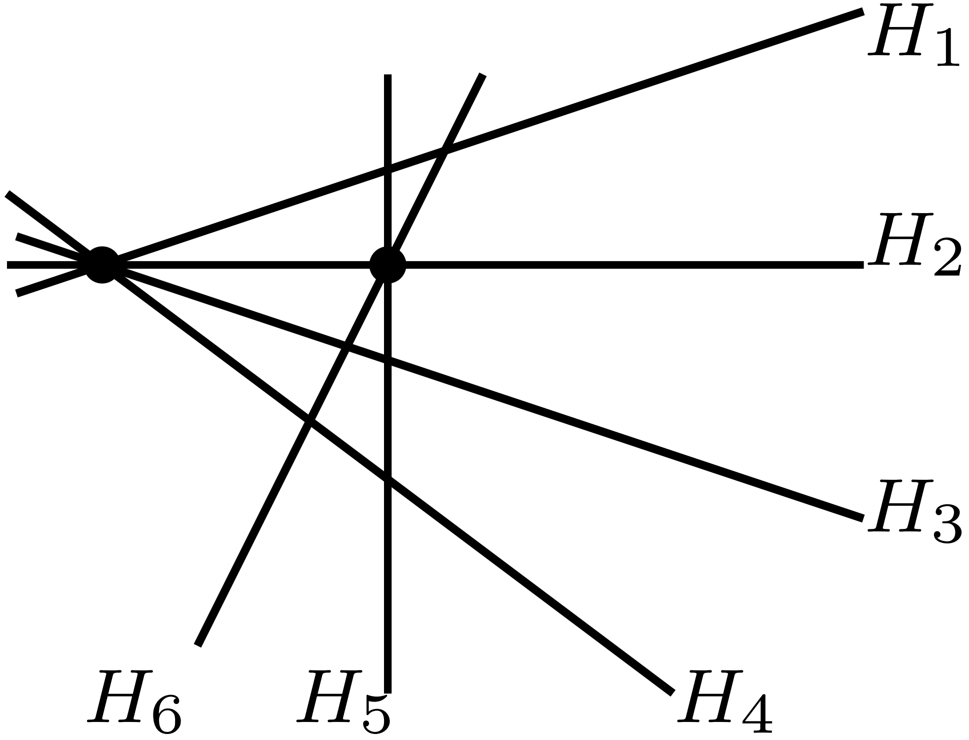
Patterns of intersections

Jiang-Yau (98)

1 Intro: moduli space $R(I)$

Def. (Incidence $I(\mathcal{A})$ of $\mathcal{A}$)

$$I(\mathcal{A}) := \left\{ \{i,j,k\} \in 2^{\{1,\ldots,n\}} \mid H_i \cap H_j \cap H_k \neq \emptyset \right\}$$



Example:

$$I(\mathcal{A}) = \{123, 124, 134, 234, 256\}$$

1 Intro: moduli space $R(I)$

Notation: $\mathcal{A} = \{H_1, \dots, H_n\}$, lines on $\mathbb{P}^2_{\mathbb{C}}$,

$$M(\mathcal{A}) := \mathbb{P}^2_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} H_i,$$

$$I(\mathcal{A}) := \{\{ijk\} \mid H_i \cap H_j \cap H_k \neq \emptyset\}.$$

★ General Principle ★

$I(\mathcal{A})$ has (some) topological info's of $M(\mathcal{A})$.

1 Intro: moduli space $R(I)$

General principle: $I(\mathcal{A})$ has info's of $M(\mathcal{A})$.

$I(\mathcal{A})$ determines e.g.,

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② $H^*(M(\mathcal{A}),\mathcal{L})$ for certain local system $\mathcal{L}$. (Esnault-Schechtman-Viehweg)

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② $H^*(M(\mathcal{A}), \mathcal{L})$ for certain local system $\mathcal{L}$. (Esnault-Schechtman-Viehweg)

③ $\pi_1(M(\mathcal{A}))$ for $n \leq 8$. (K. -M. Fan, Garber-Teicher-Vishne, Falk-Sturmfels, etc.)

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③ $\pi_1(M(\mathcal{A}))$ for $n \leq 8$.

Counter-ex: ④ (Rybnikov) $n = 13$,

$\exists \mathcal{A}_1, \mathcal{A}_2$ s.t.

$$I(\mathcal{A}_1) = I(\mathcal{A}_2) \ \& \ \pi_1(M(\mathcal{A}_1)) \not\simeq \pi_1(M(\mathcal{A}_2)).$$

Challenging Problem: Find other such pairs.

1 Intro: moduli space $R(I)$

$I(\mathcal{A})$ determines: ① $H^*(M(\mathcal{A}), \mathbb{Z})$,

② $H^*(M(\mathcal{A}), \mathcal{L})$ for some $\mathcal{L}$,

③ $\pi_1(M(\mathcal{A}))$ for $n \leq 8$.

But: ④ $\exists \mathcal{A}_1, \mathcal{A}_2$ s.t. $I(\mathcal{A}_1) = I(\mathcal{A}_2)$

& $\pi_1(M(\mathcal{A}_1)) \not\simeq \pi_1(M(\mathcal{A}_2))$.

If the moduli space $\underline{R(I) \text{ is connected } \clubsuit}$

$\Longrightarrow I(\mathcal{A})$ determines $M(\mathcal{A})$

$I(\mathcal{A})$-preserving deformation preserves top. type of $M(\mathcal{A})$ (Randell)

The assumption $\underline{\ \clubsuit\ }$ is not true in general.

1 Intro: moduli space $R(I)$

Focus of this talk:

When is the moduli space $R(I)$
connected/disconnected?

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When is the moduli space $R(I)$
connected/disconnected?

Def. (The moduli space $R(I)$.)
Let $I = I(\mathcal{A}) \subset 2^{\{1,\ldots,n\}}$ be an incidence.

$R(I) :=$

$$\left\{ (H_1, \ldots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \text{ if } i \neq j \\ H_i \cap H_j \cap H_k \neq \emptyset \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \emptyset \text{ if } (ijk) \notin I \end{array} \right\}$$

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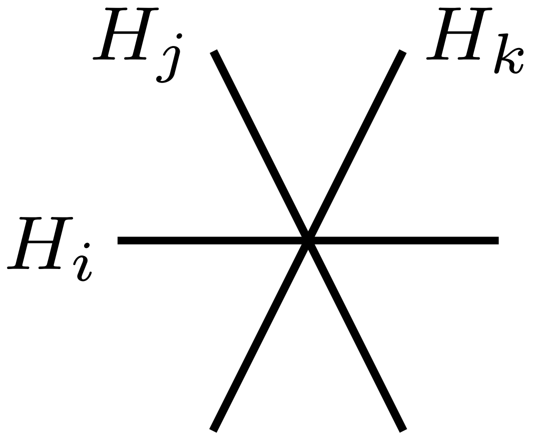
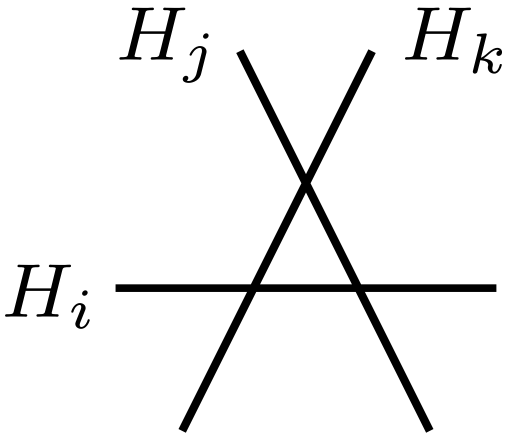


$(ijk) \in I$



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1 Intro: moduli space $R(I)$

$$R(I) = \left\{ (H_1, \ldots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \text{ if } i \neq j \\ H_i \cap H_j \cap H_k \neq \emptyset \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \emptyset \text{ if } (ijk) \notin I \end{array} \right\}$$

$(I \subset 2^{\{1,\ldots,n\}})$.

Remarks:

- $PGL_3(\mathbb{C})$ acts on $R(I)$ diagonally.
- $R(I)$ and $R(I)/PGL_3(\mathbb{C})$ are quasi-projective varieties.

1 Intro: moduli space $R(I)$

Remarks:

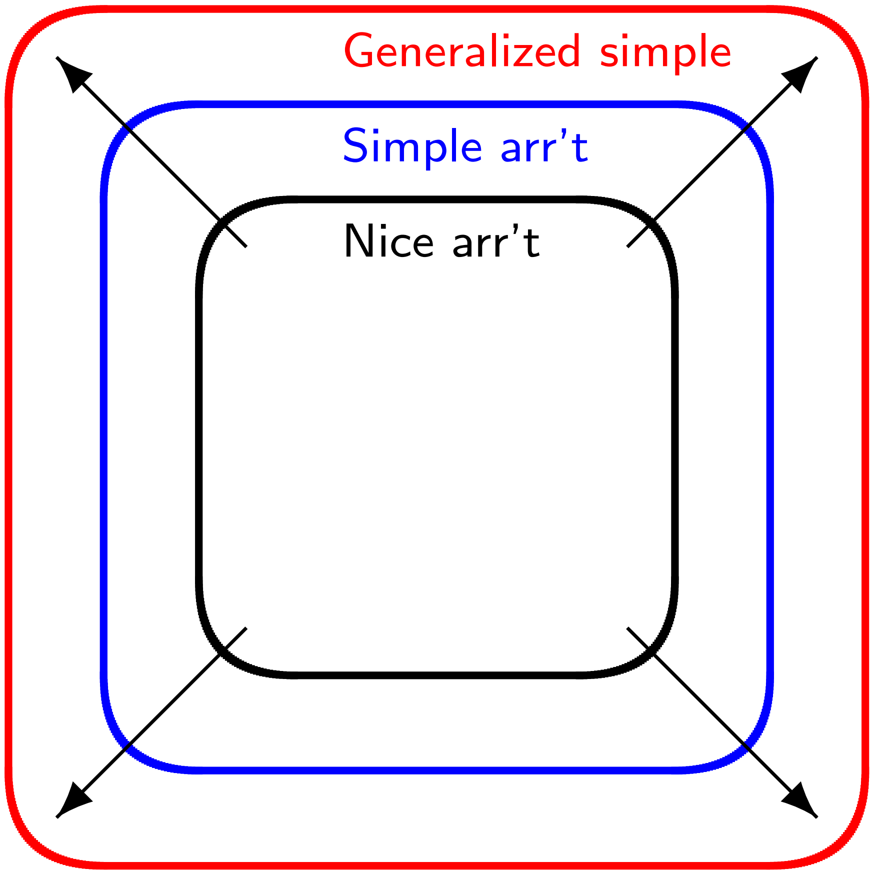
- ◦ Several sufficient conditions for $R(I)$ to be connected is obtained by Jiang-Yau (94) and Wang-Yau (05). The conditions are described in terms of the "graph" associated to $I(\mathcal{A})$ ("nice" and "simple" arrangements.)
- ◦ Our approach is based on the fact:

"An irreducible variety is connected."

1 Intro: moduli space $R(I)$

(Force Dynamics of Jiang-Wang-Yau)

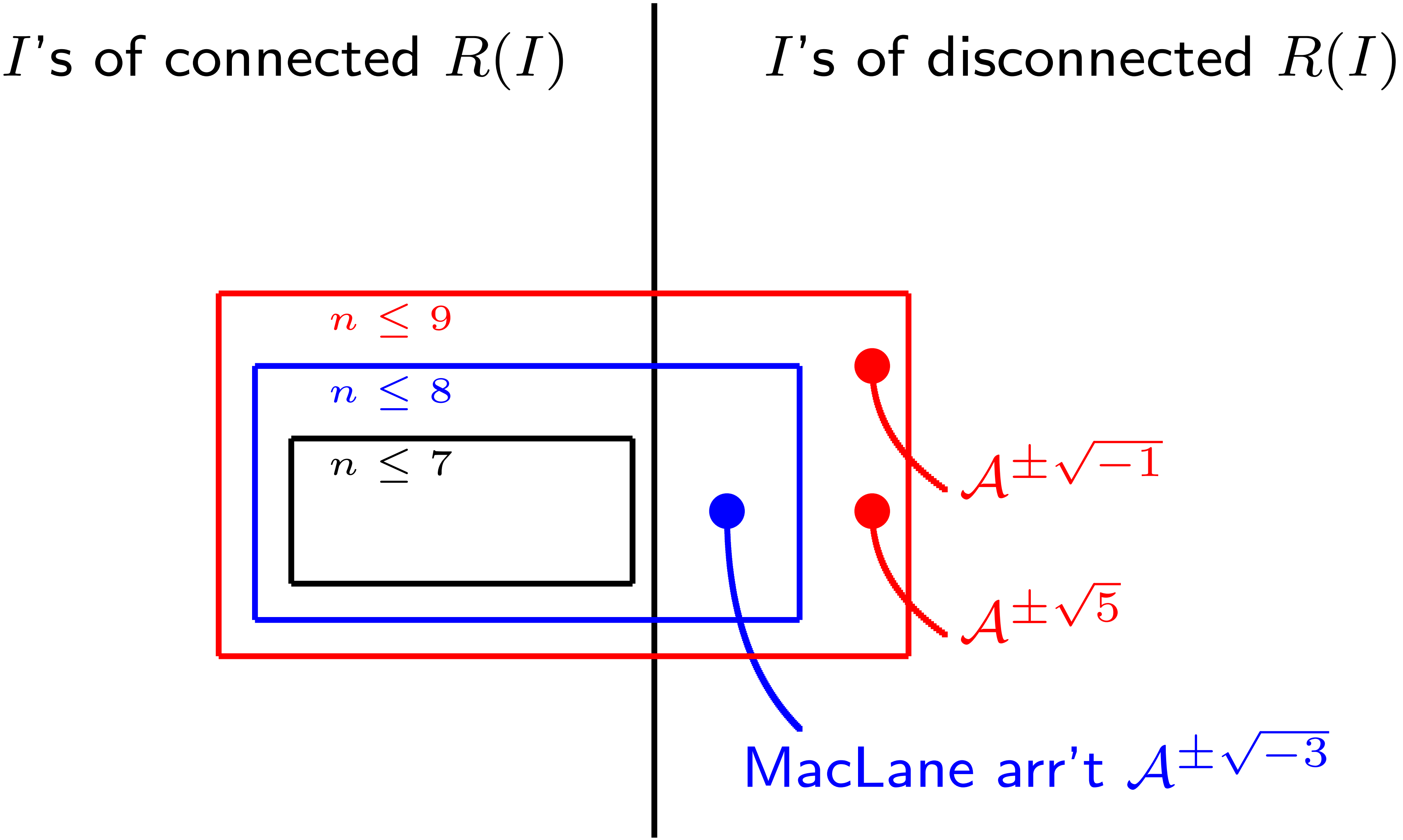
I's of connected $R(I)$



I's of disconnected $R(I)$

1 Intro: moduli space $R(I)$

(Force Dynamics of our work)



I's of connected $R(I)$

I's of disconnected $R(I)$

$n \leq 9$

$n \leq 8$

$n \leq 7$

$\mathcal{A}^{\pm\sqrt{-1}}$

$\mathcal{A}^{\pm\sqrt{5}}$

MacLane arr't $\mathcal{A}^{\pm\sqrt{-3}}$

2 (Dis-)connectivity of $R(I)$

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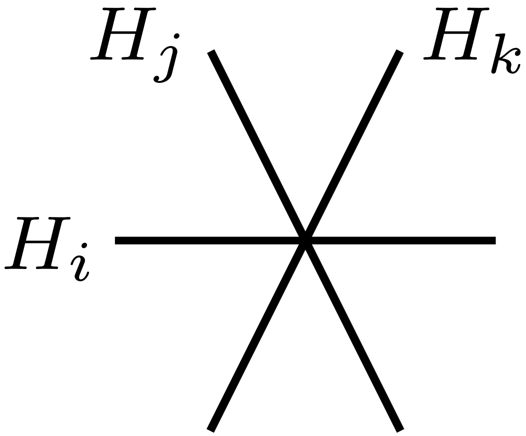

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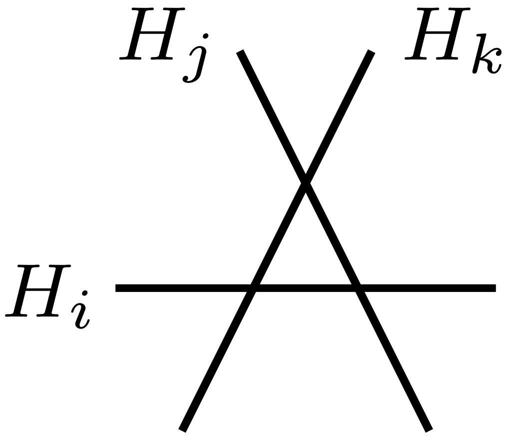

$(ijk) \notin I$

2 (Dis-)connectivity of $R(I)$

$$R(I) = \left\{ (H_1, \ldots, H_n) \in (\mathbb{P}^{2*})^n \;\middle|\; \begin{array}{l} H_i \neq H_j \text{ if } i \neq j \\ H_i \cap H_j \cap H_k \neq \emptyset \text{ if } (ijk) \in I \\ H_i \cap H_j \cap H_k = \emptyset \text{ if } (ijk) \notin I \end{array} \right\}$$

The simplest examples:

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The simplest examples:

◙ $n = 1$

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The simplest examples:

- ⊡ $n = 1 \Longrightarrow (I = \emptyset),\ R(I) = \mathbb{P}^{2*}$.

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- ⊡ $n = 2$

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- ⧇ $n = 2 \Longrightarrow (I = \emptyset)$

$$\begin{aligned} R(I) &= \{(H_1, H_2) \mid H_1 \neq H_2\} \\ &= (\mathbb{P}^{2*})^2 \setminus \{\text{diagonal}\}. \end{aligned}$$

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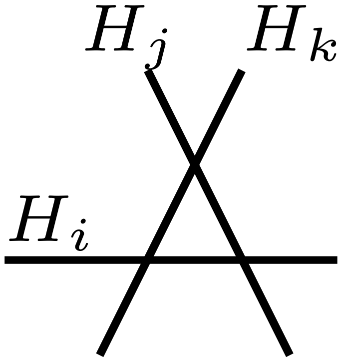

Example: $n \geq 1$ and $I = \emptyset$,
i.e., $H_i \cap H_j \cap H_k = \emptyset \ \forall i, j, k$.

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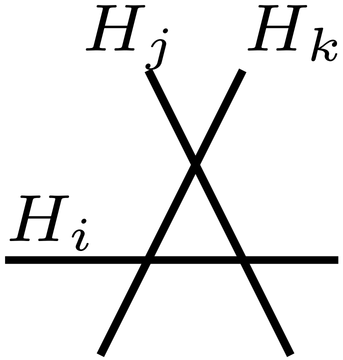

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$\hookrightarrow (\mathbb{P}^{2*})^n$: Zariski open embedding.

$\Rightarrow R(I)$: irreducible $\Rightarrow$ connected!!

2 (Dis-)connectivity of $R(I)$

Example: $n = 8,\ \mathcal{A} = \{H_1, \ldots, H_8\}$.
$I = \{123, 456, 147, 257, 367, 248, 358, 168\}$.

$$R(I)/PGL_3(\mathbb{C}) =???$$

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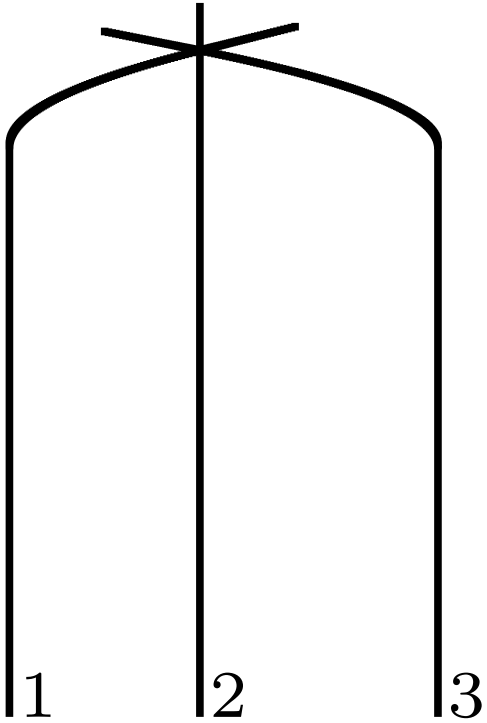

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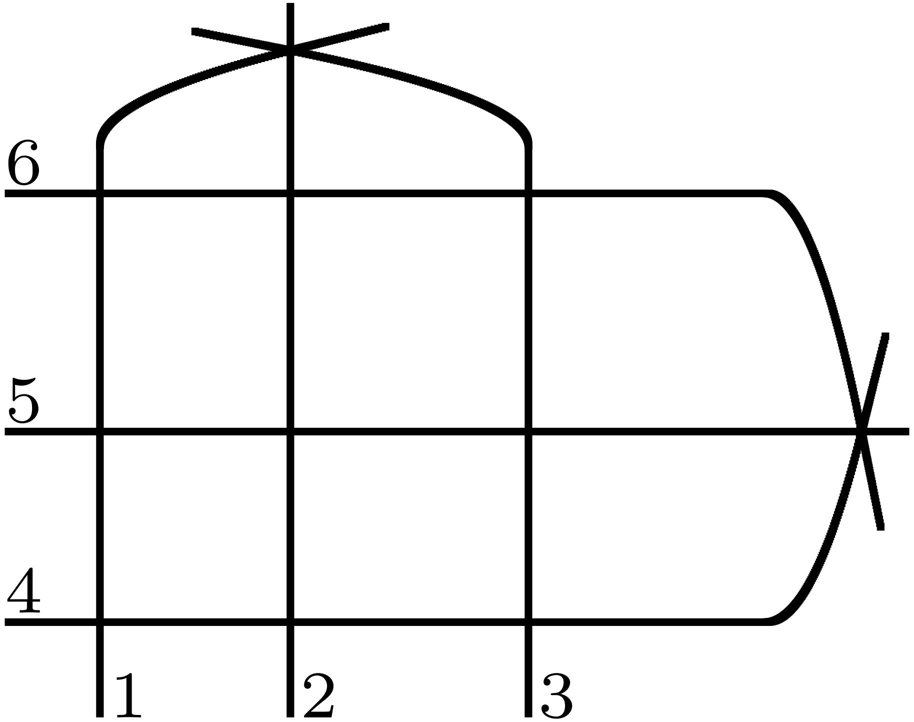

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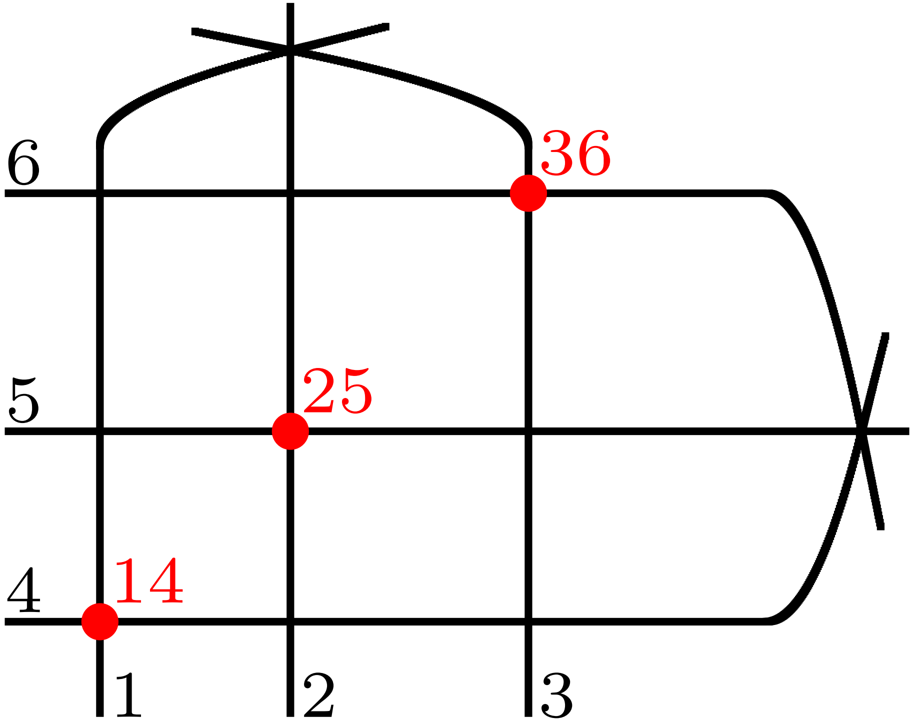

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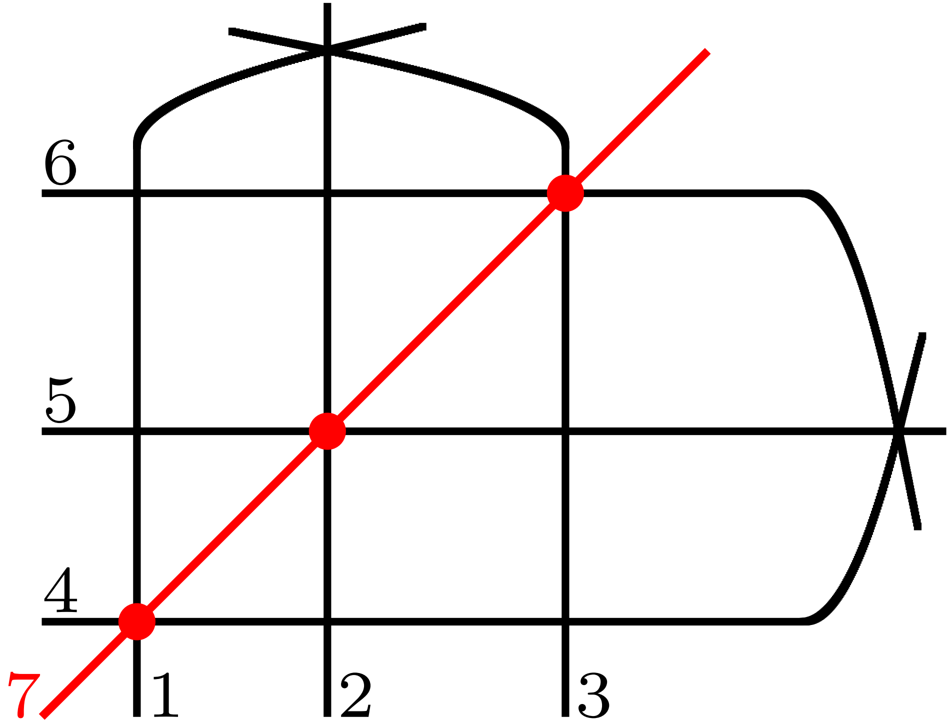

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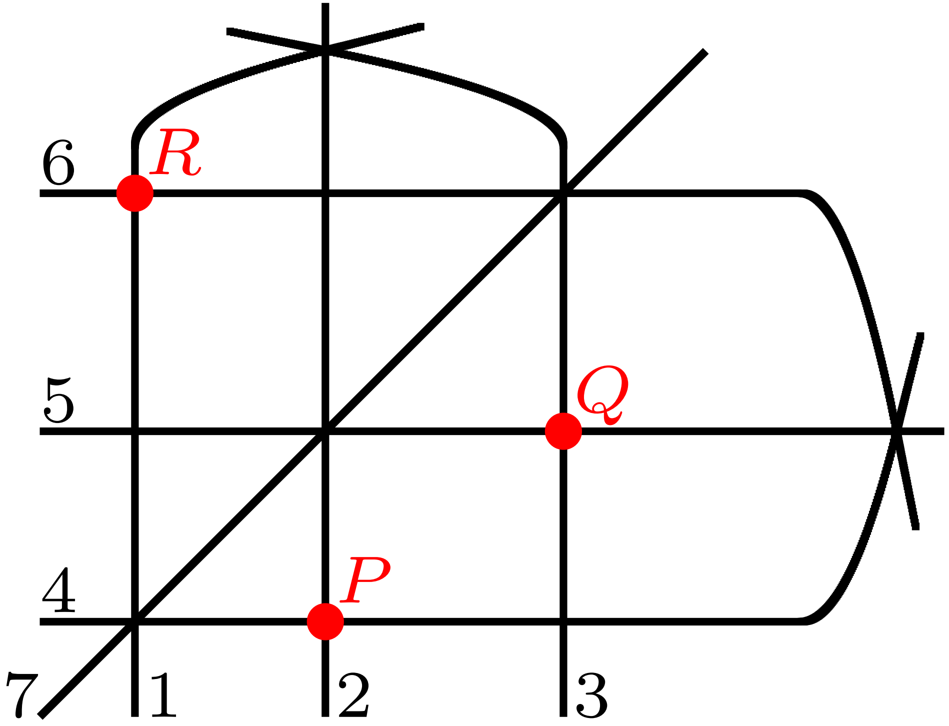
Example: $n = 8$, $\mathcal{A} = \{H_1, \ldots, H_8\}$.

$I = \{123, 456, 147, 257, 367, \underset{P}{248}, \underset{Q}{358}, \underset{R}{168}\}$.



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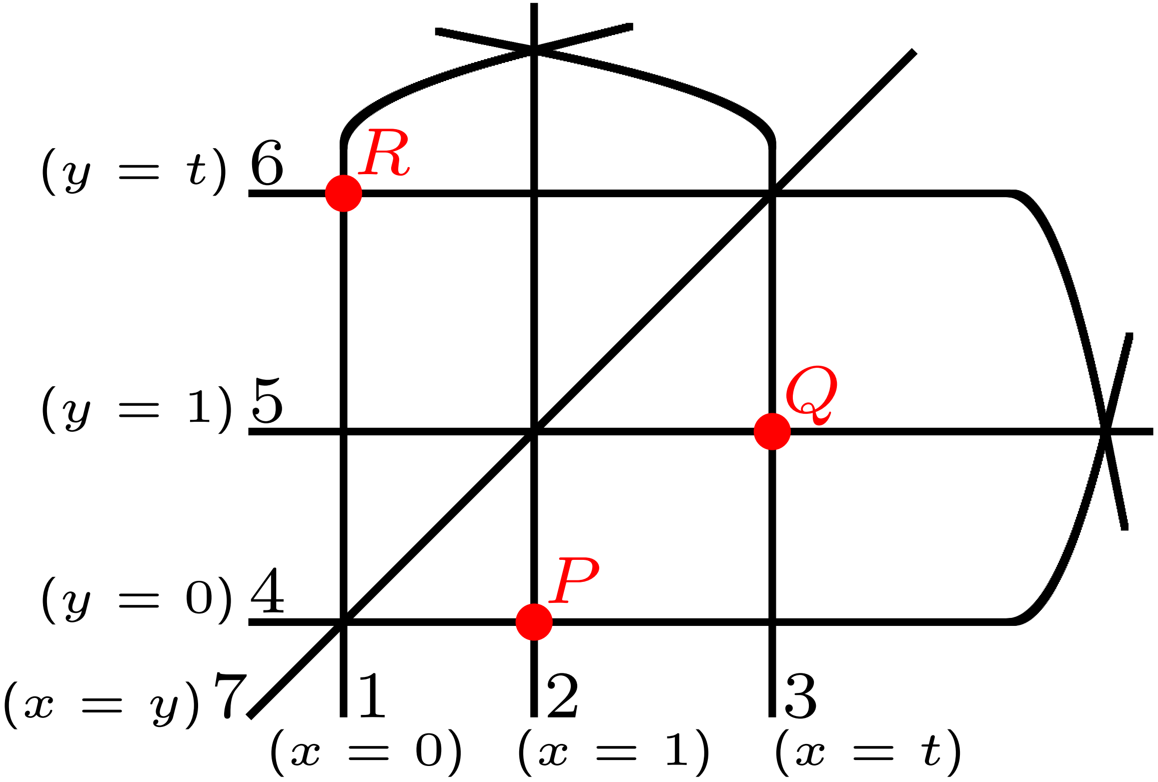

Points P, Q, R are collinear

$\Updownarrow$

$t^2 - t + 1 = 0, t = \frac{-1 \pm \sqrt{-3}}{2}$

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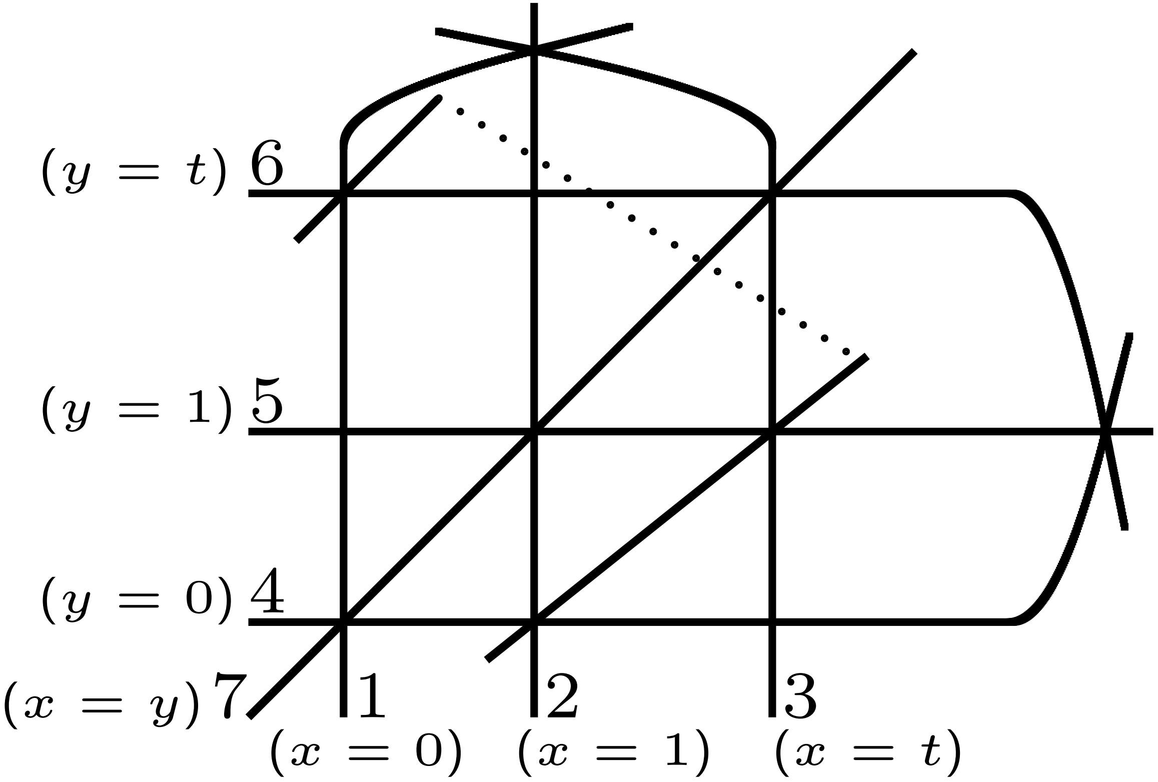
$P \qquad Q \qquad R$



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$\boxed{\text{MacLane arrangement}}$

$R(I)/PGL_3 = \{\mathcal{A}^{+\sqrt{-3}}, \mathcal{A}^{-\sqrt{-3}}\}$: 2-points

2 (Dis-)connectivity of $R(I)$

Def.

$$\mathrm{mult}(\mathcal{A}) = \left\{ p \in \mathbb{P}^2 \;\middle|\; \begin{array}{l} \exists i,j,k\text{: distinct} \\ \text{s.t. } p = H_i \cap H_j \cap H_k \end{array} \right\}$$

Theorem 1 (Nazir, –)

$\mathcal{A} = \{H_1, \ldots, H_n\}, I = I(\mathcal{A}), (R(I) \neq \emptyset)$,

$\mathcal{A}' = \{H_1, \ldots, H_{n-1}\}, I' = I(\mathcal{A}')$.

If $\mu := |\,\mathrm{mult}(\mathcal{A}) \cap H_n| \leq 2$ and $R(I')$: irred.,

$\Longrightarrow R(I)$ is also irreducible.

2 (Dis-)connectivity of $R(I)$

Thm. 1 $\mathcal{A} = \{H_1, \dots, H_n\}, \mathcal{A}' = \mathcal{A} \setminus \{H_n\}$,
$\mu = |\,\mathrm{mult}(\mathcal{A}) \cap H_n| \leq 2$ & $R(I')$ irred.
$\Longrightarrow R(I)$: irred.

Sketch of proof for $\mu = 1$

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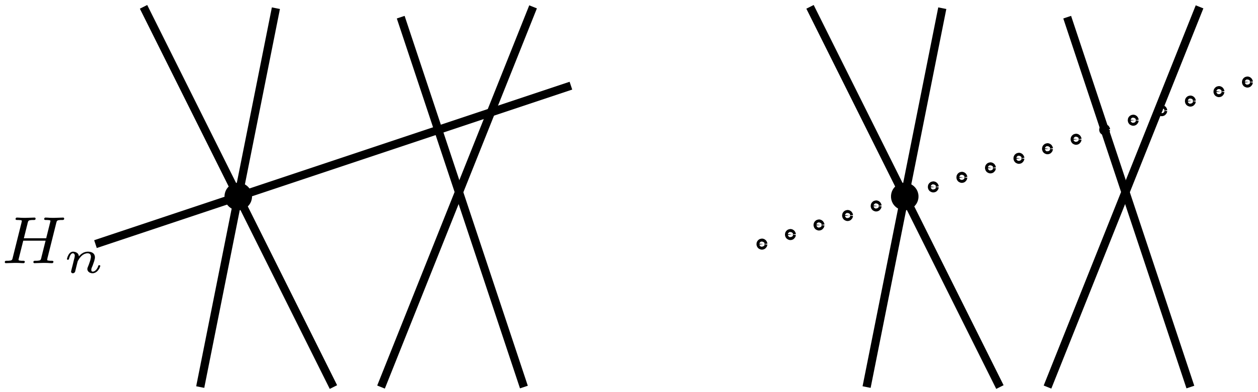
$$\pi : R(I) \longrightarrow R(I')$$

$$(H_1, \ldots, H_n) \longmapsto (H_1, \ldots, H_{n-1})$$



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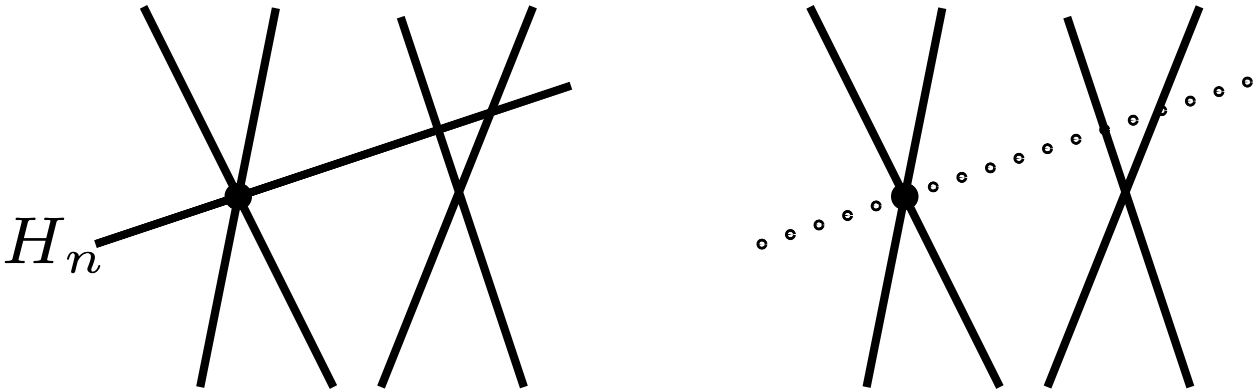
$$\pi : R(I) \longrightarrow R(I')$$

$$\pi^{-1}(\mathcal{A}') = \mathbb{P}^1 \setminus \{pts\}$$

$$(H_1, \dots, H_n) \longmapsto (H_1, \dots, H_{n-1})$$



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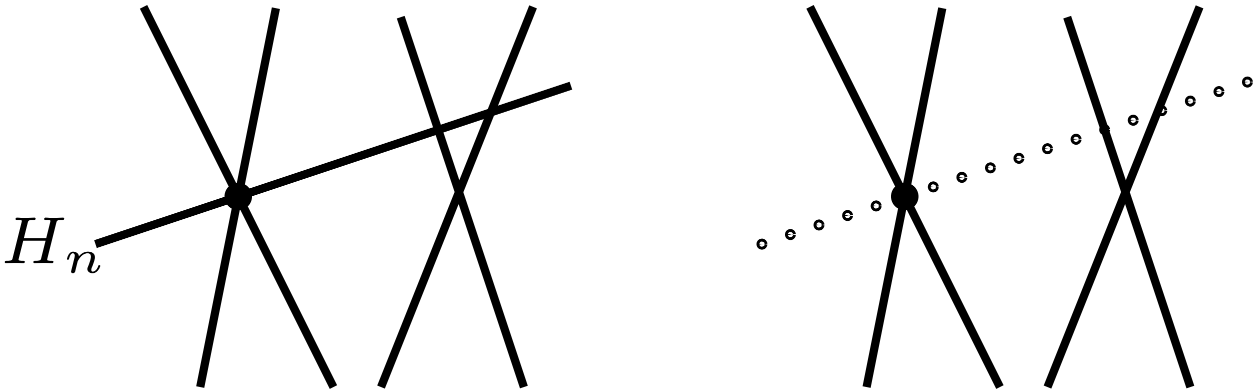


$\pi^{-1}(\mathcal{A}') = \mathbb{P}^1 \setminus \{pts\}$

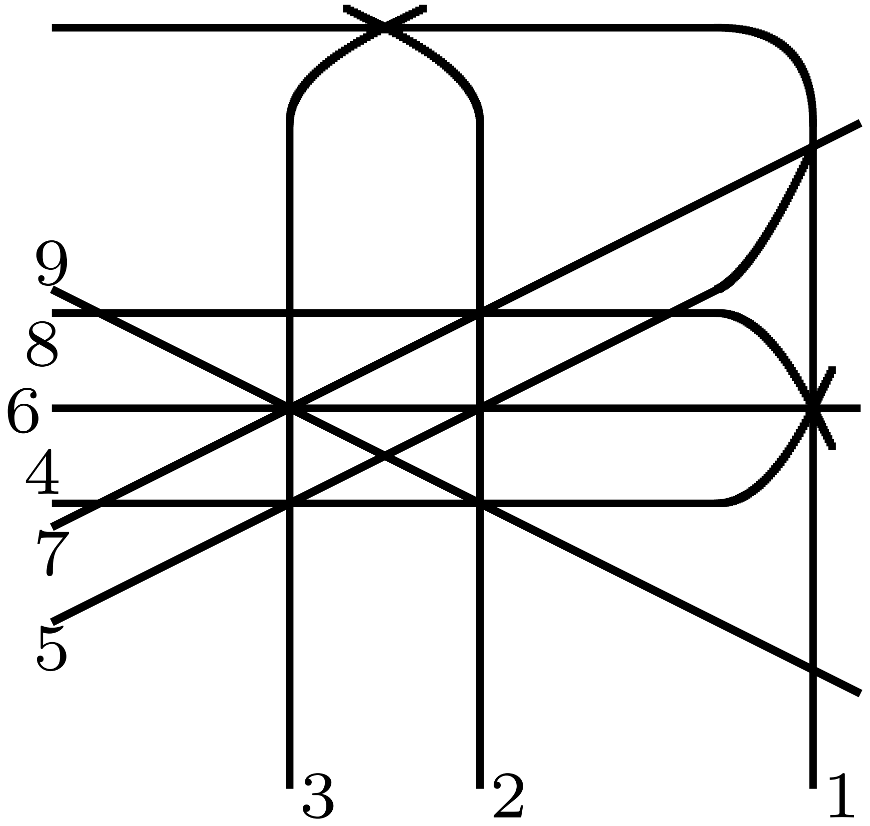
$R(I)$ is a Zariski open set of a $\mathbb{P}^1$-fibration over $R(I')$.

2 (Dis-)connectivity of $R(I)$

Thm. 1 $\mathcal{A} = \{H_1, \ldots, H_n\}, \mathcal{A}' = \mathcal{A} \setminus \{H_n\}$,

$\mu = |\,\mathrm{mult}(\mathcal{A}) \cap H_n| \leq 2$ & $R(I')$ irred. $\Rightarrow$ $R(I)$: irred.

Example $(n = 9)$



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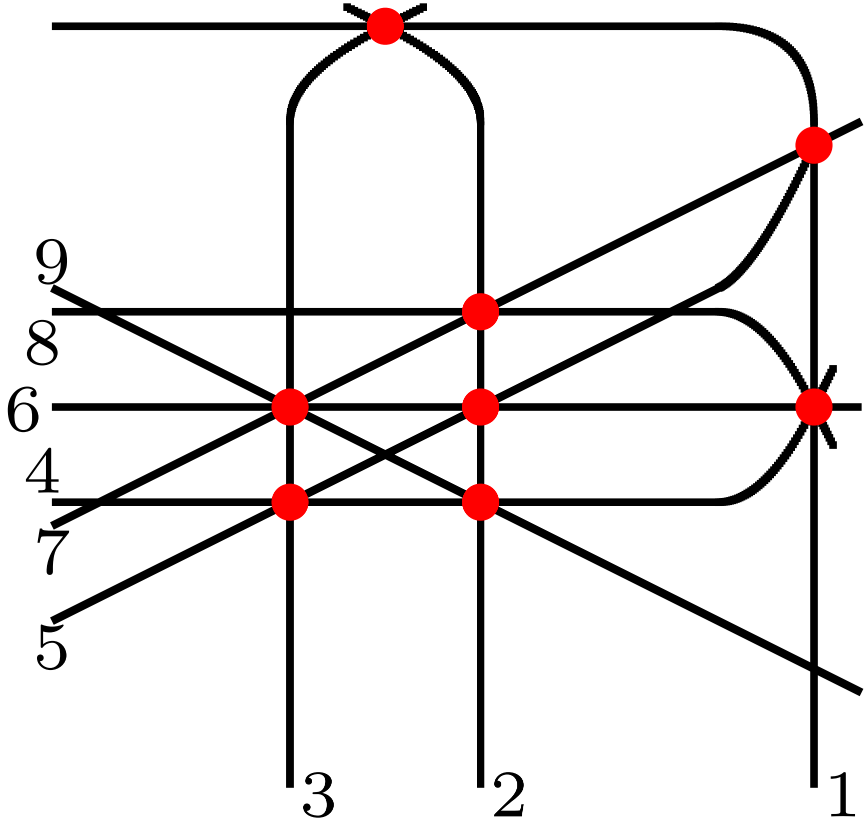

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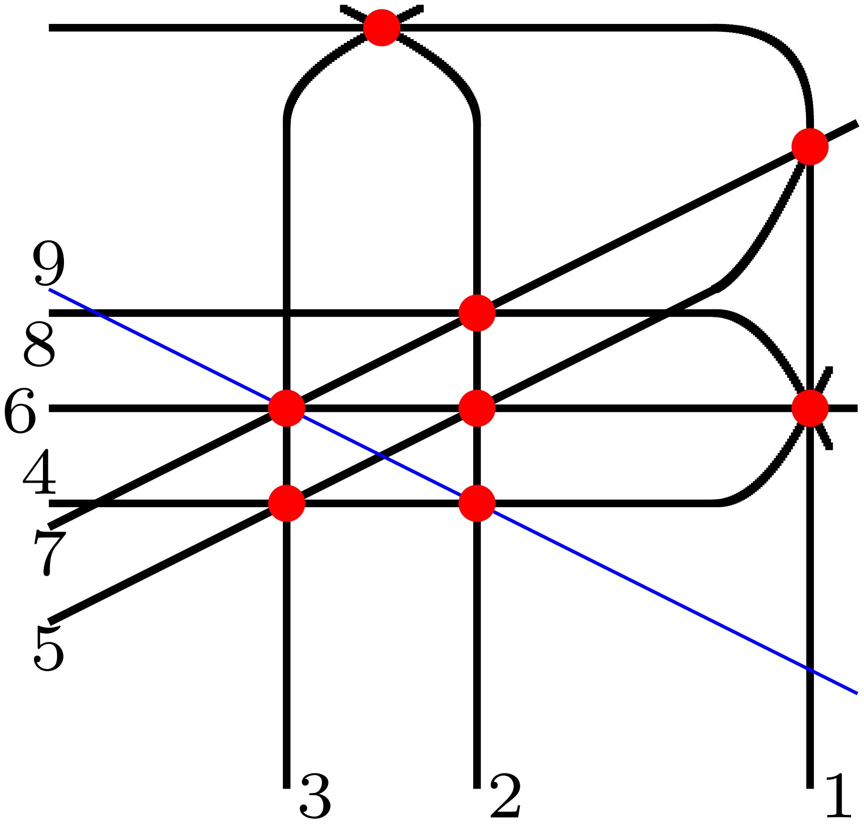

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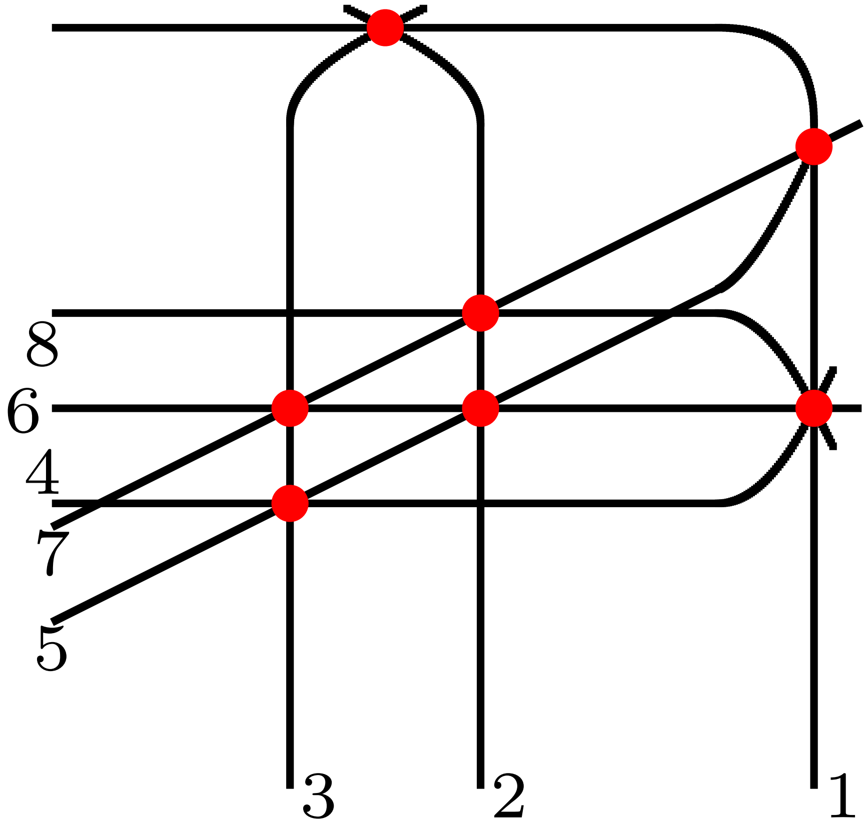

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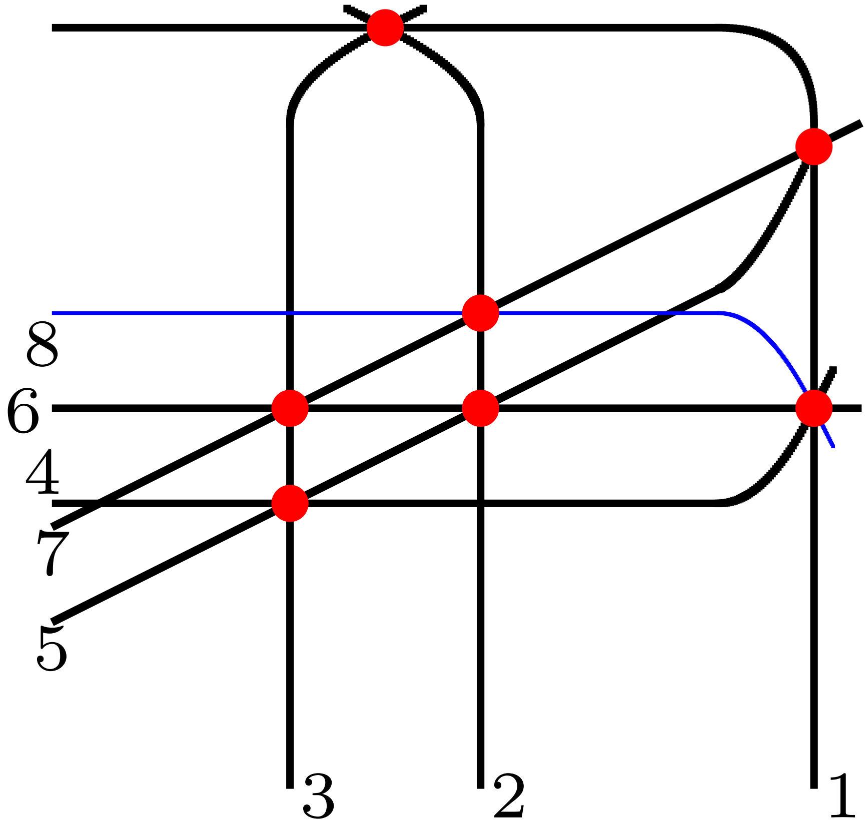

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Induction + Thm. 1

⇓

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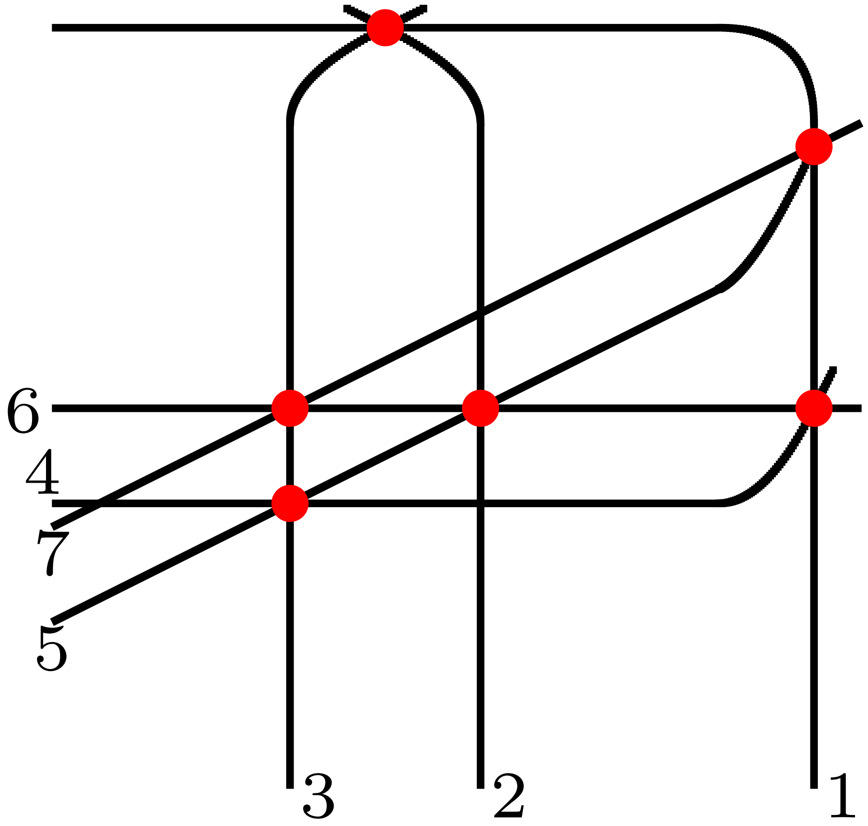

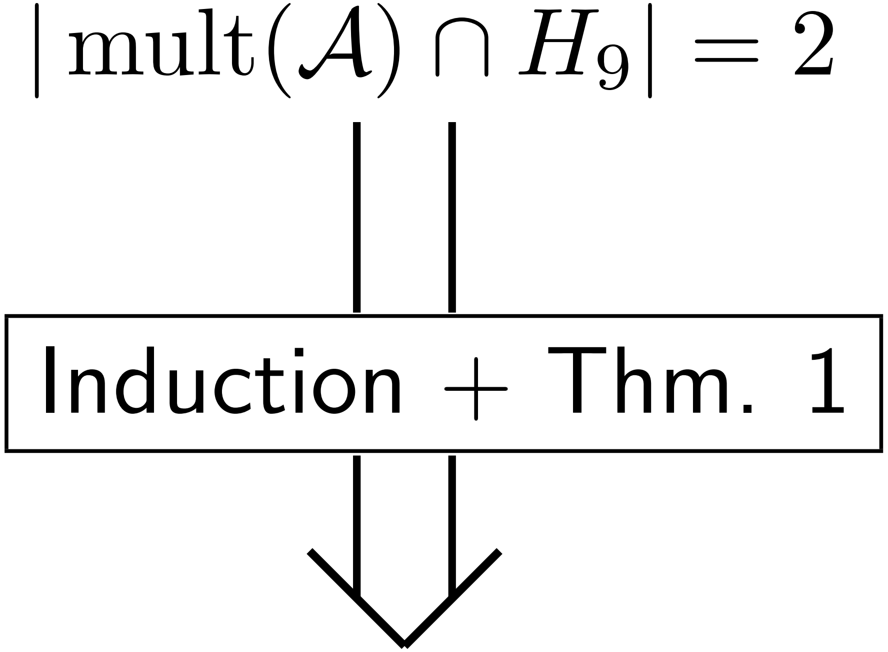

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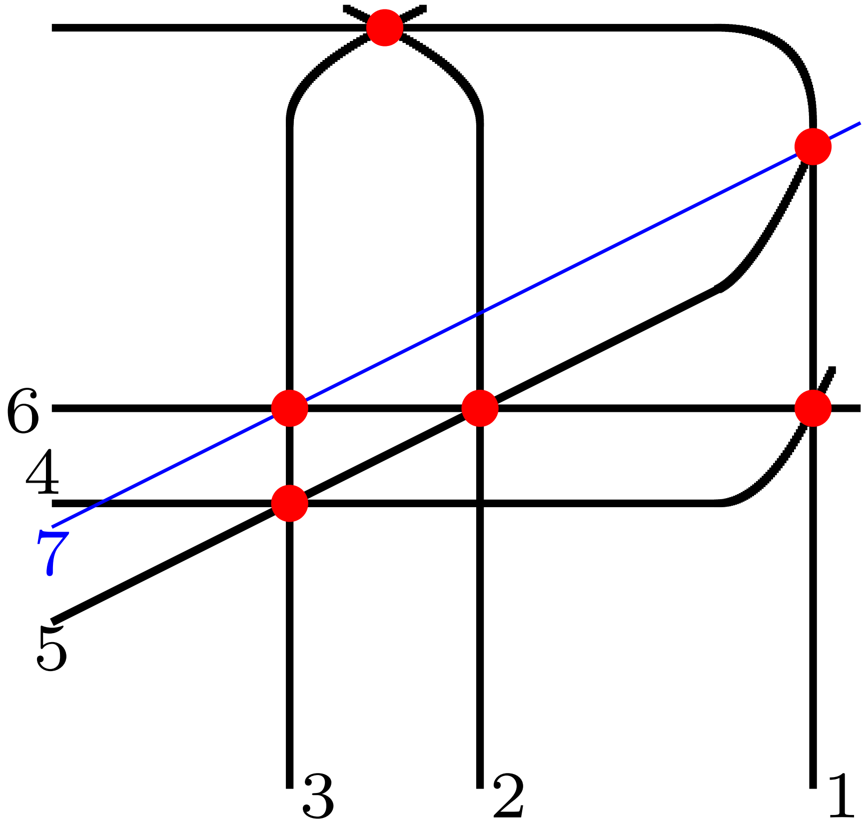

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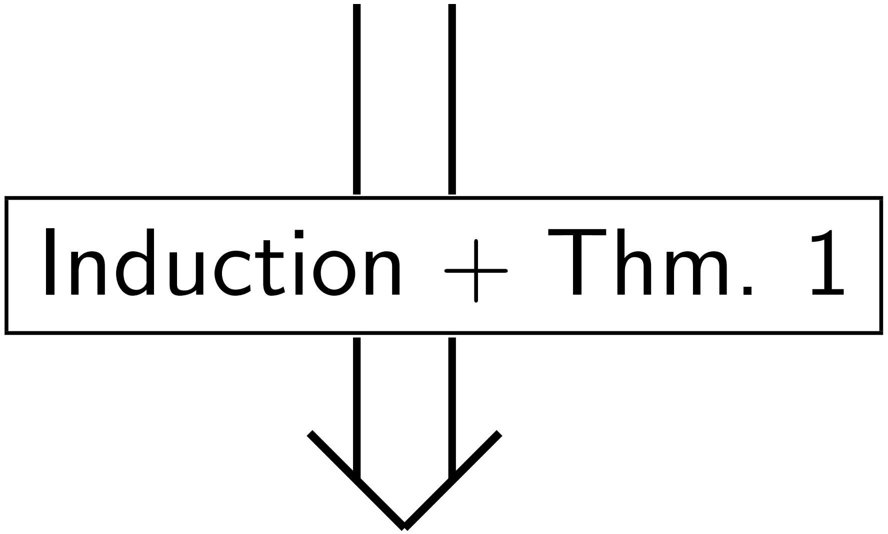

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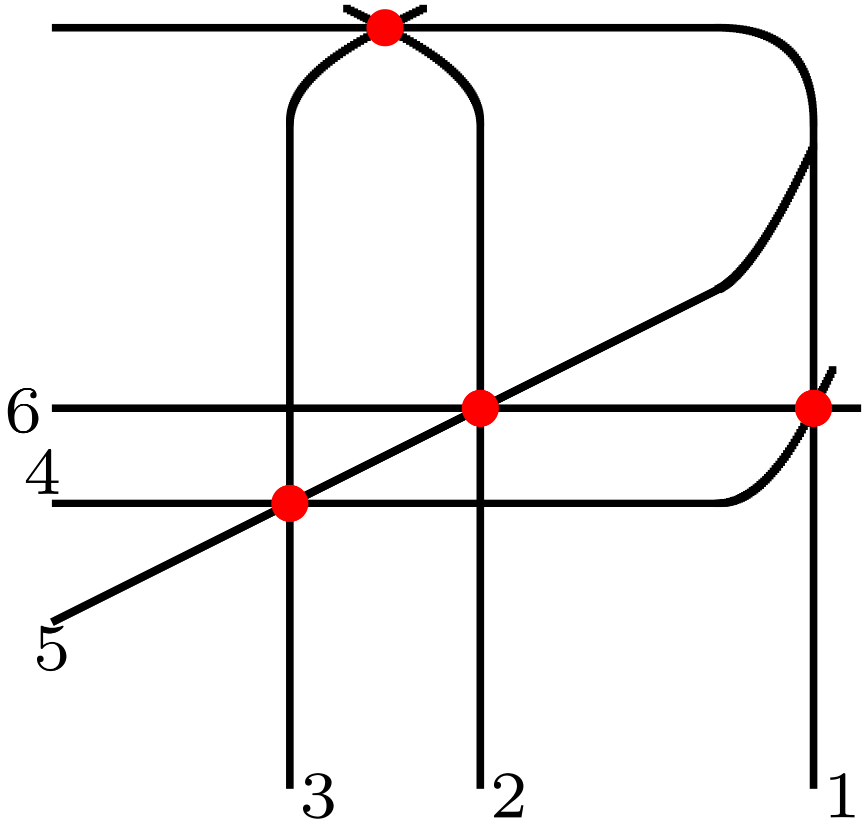

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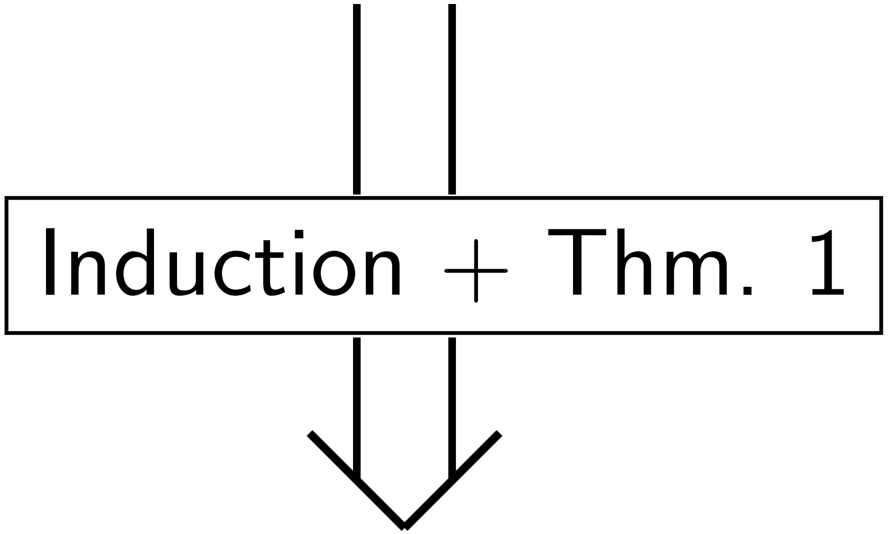
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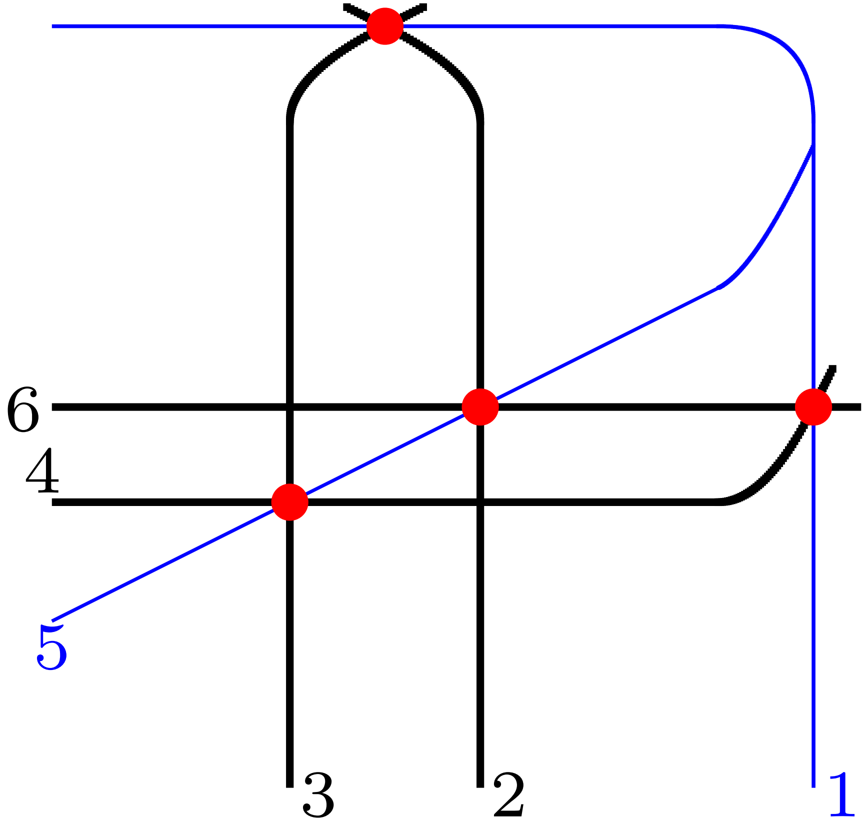

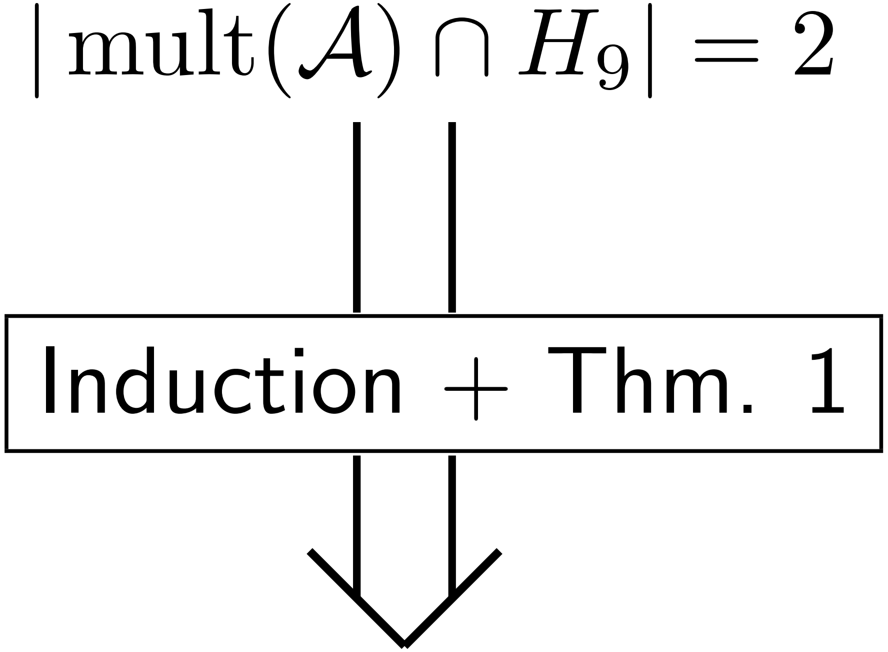

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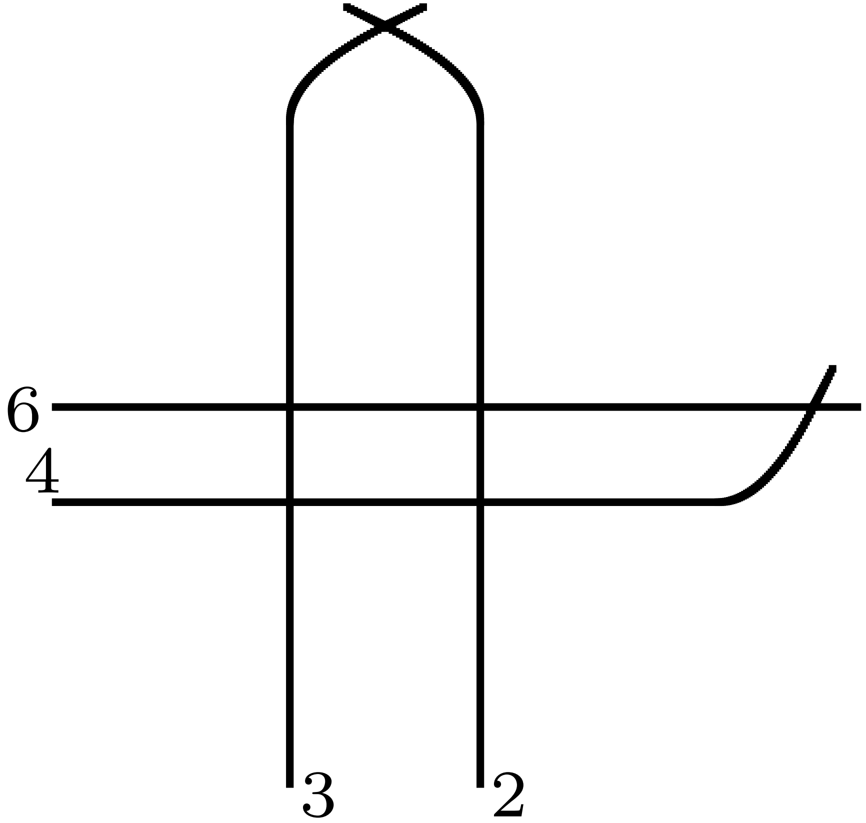

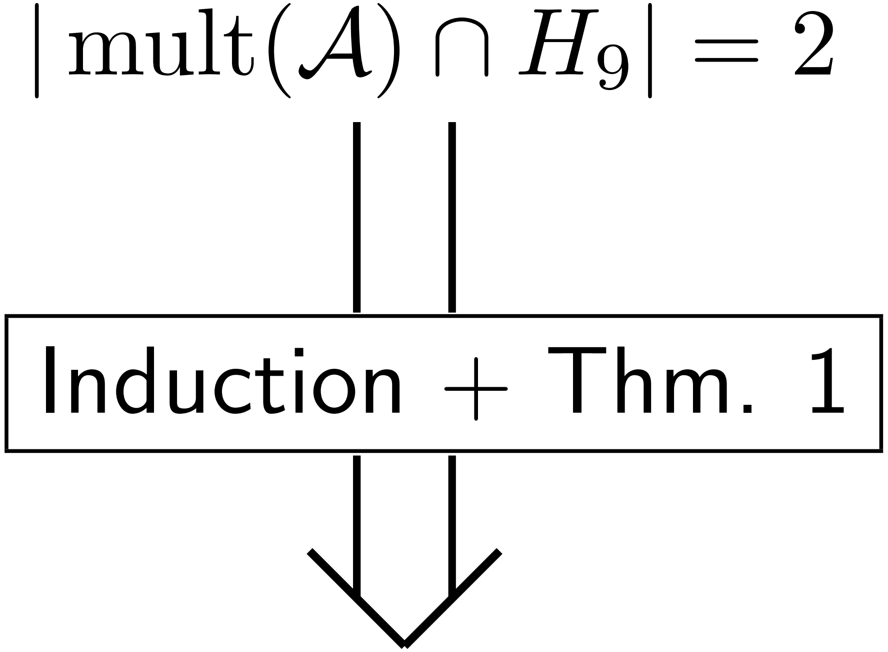

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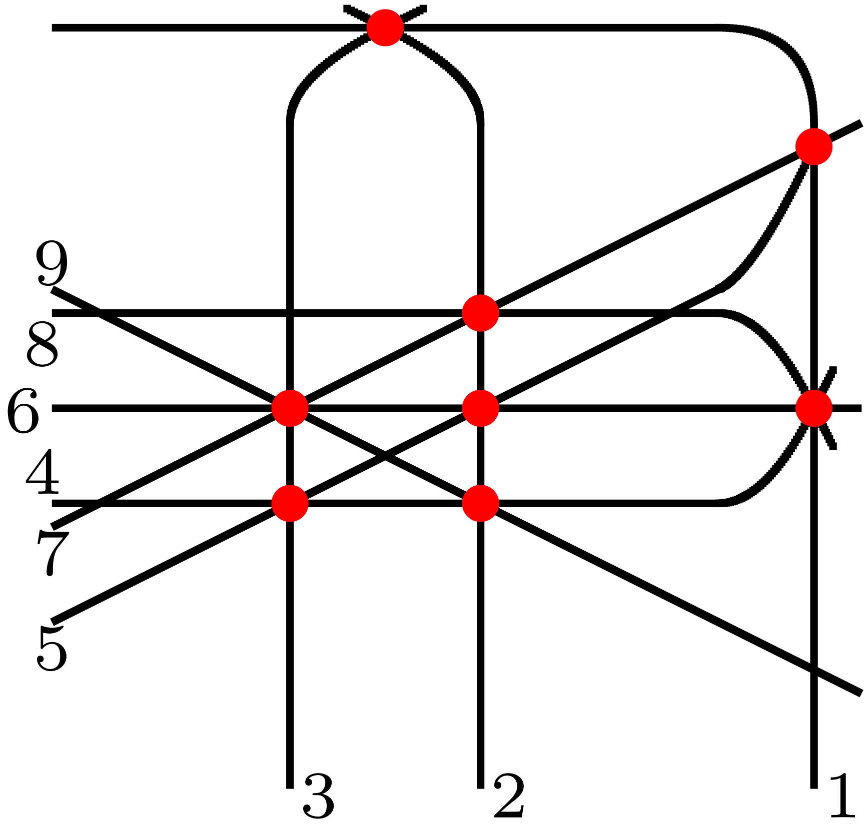

$$|\,\mathrm{mult}(\mathcal{A}) \cap H_9| = 2$$

$$\Downarrow$$

Induction + Thm. 1

$R(I)$: irreducible.

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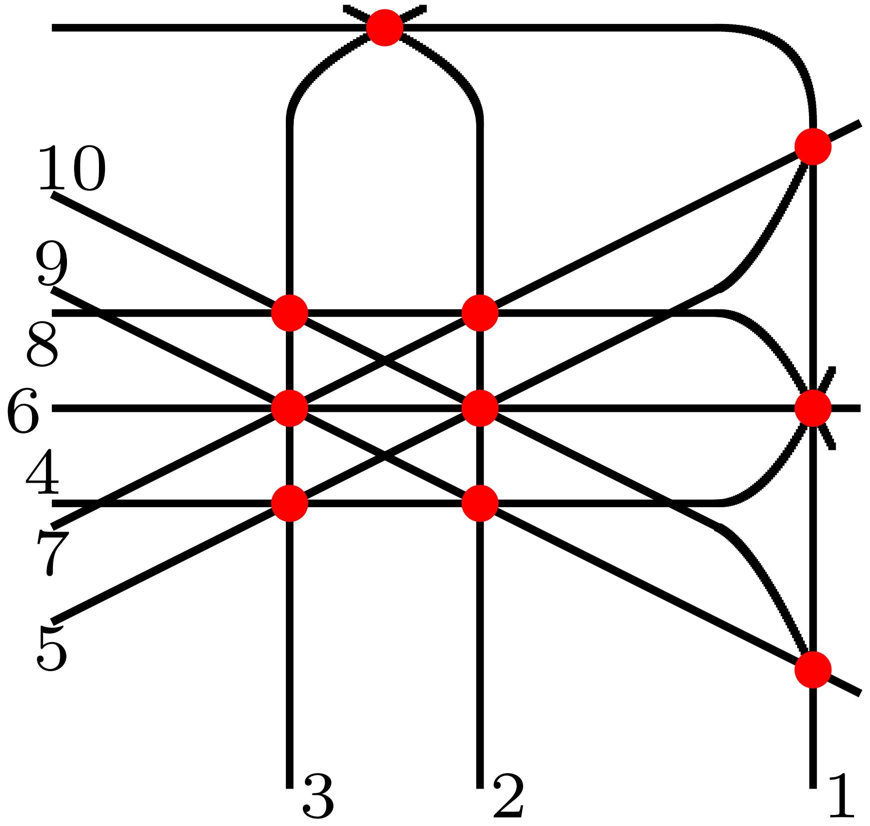
Example $(n = 10)$



$|\,\mathrm{mult}(\mathcal{A}) \cap H_i| \geq 3$, $\forall i$.

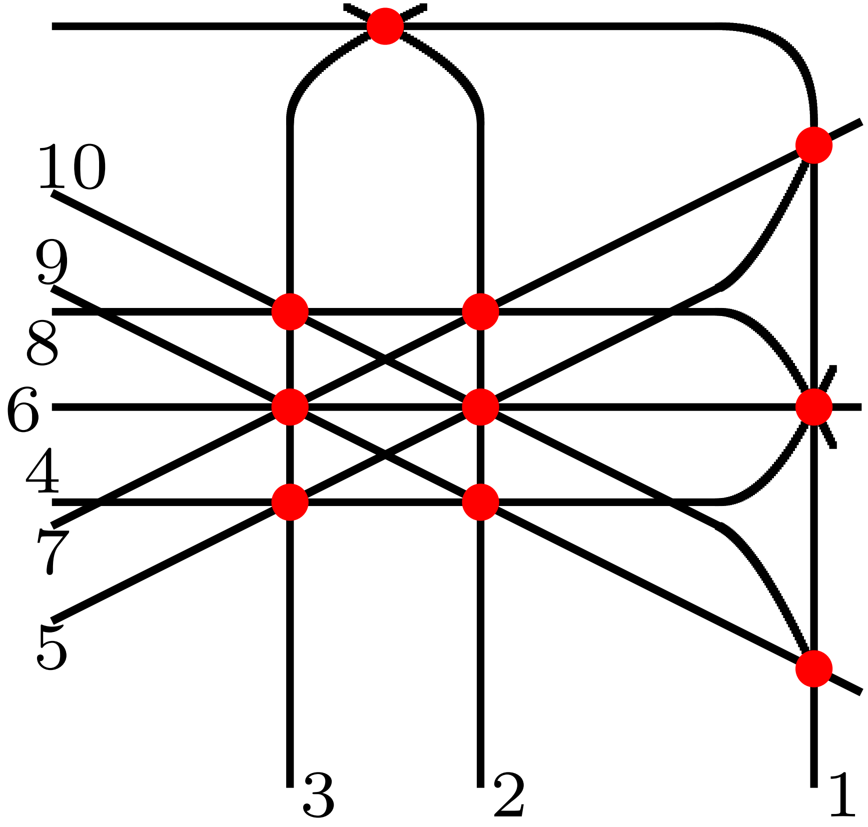
We can not apply Thm.1. But still $R(I)$ irred. by the next result.

2 (Dis-)connectivity of $R(I)$

Theorem 2 (Nazir, –) $\mathcal{A} = \{H_1, \dots, H_n\}$.

$\exists (ijk) \in I(\mathcal{A})$ s.t. $\mathrm{mult}(\mathcal{A}) \subset H_i \cup H_j \cup H_k$,

$\Longrightarrow R(I(\mathcal{A}))$ is irreducible.



Example:

- $\circ$ $(123) \in I(\mathcal{A})$.
- $\circ$ $\mathrm{mult}(\mathcal{A}) \subset H_1 \cup H_2 \cup H_3$

$\Downarrow$ (Thm. 2)

$R(I)$: irreducible.

2 (Dis-)connectivity of $R(I)$

Thm. 2: $\exists (ijk) \in I(\mathcal{A})$ s.t. $\mathrm{mult}(\mathcal{A}) \subset H_i \cup H_j \cup H_k$,

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Sketch of proof.

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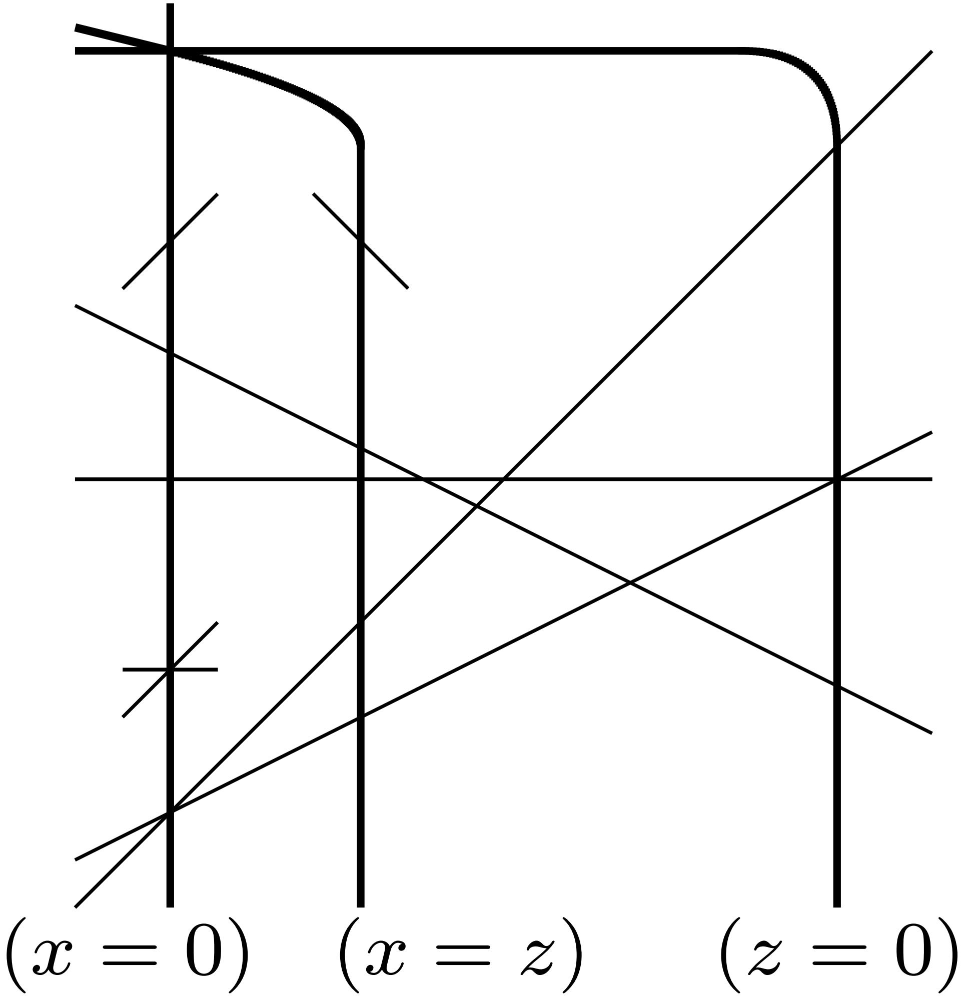

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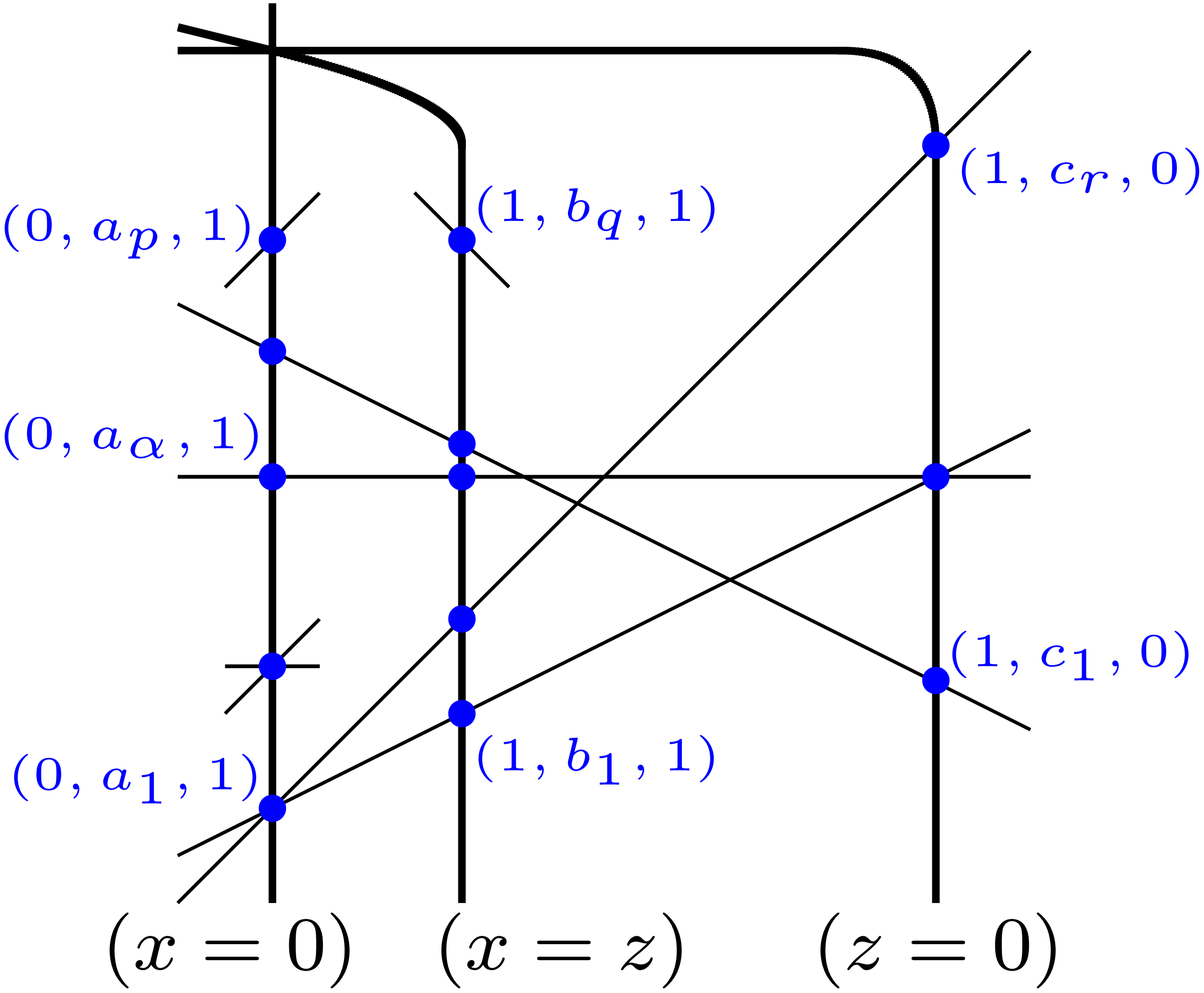

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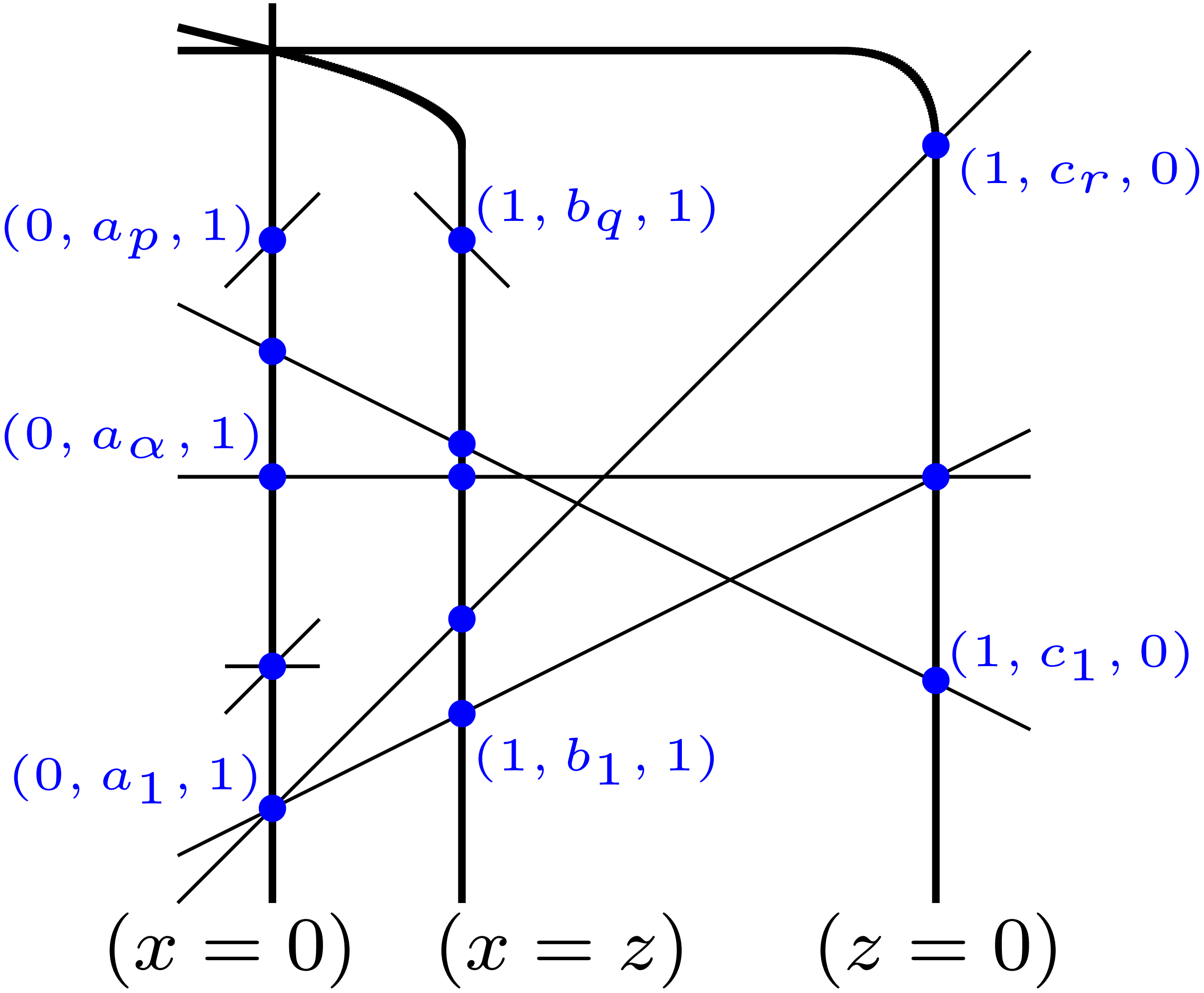

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$$R(I) \hookrightarrow \mathbb{C}^{p+q+r}$$

$$\mathcal{A} \mapsto (a_\alpha, b_\beta, c_\gamma).$$

Closed conditions:

2 (Dis-)connectivity of $R(I)$

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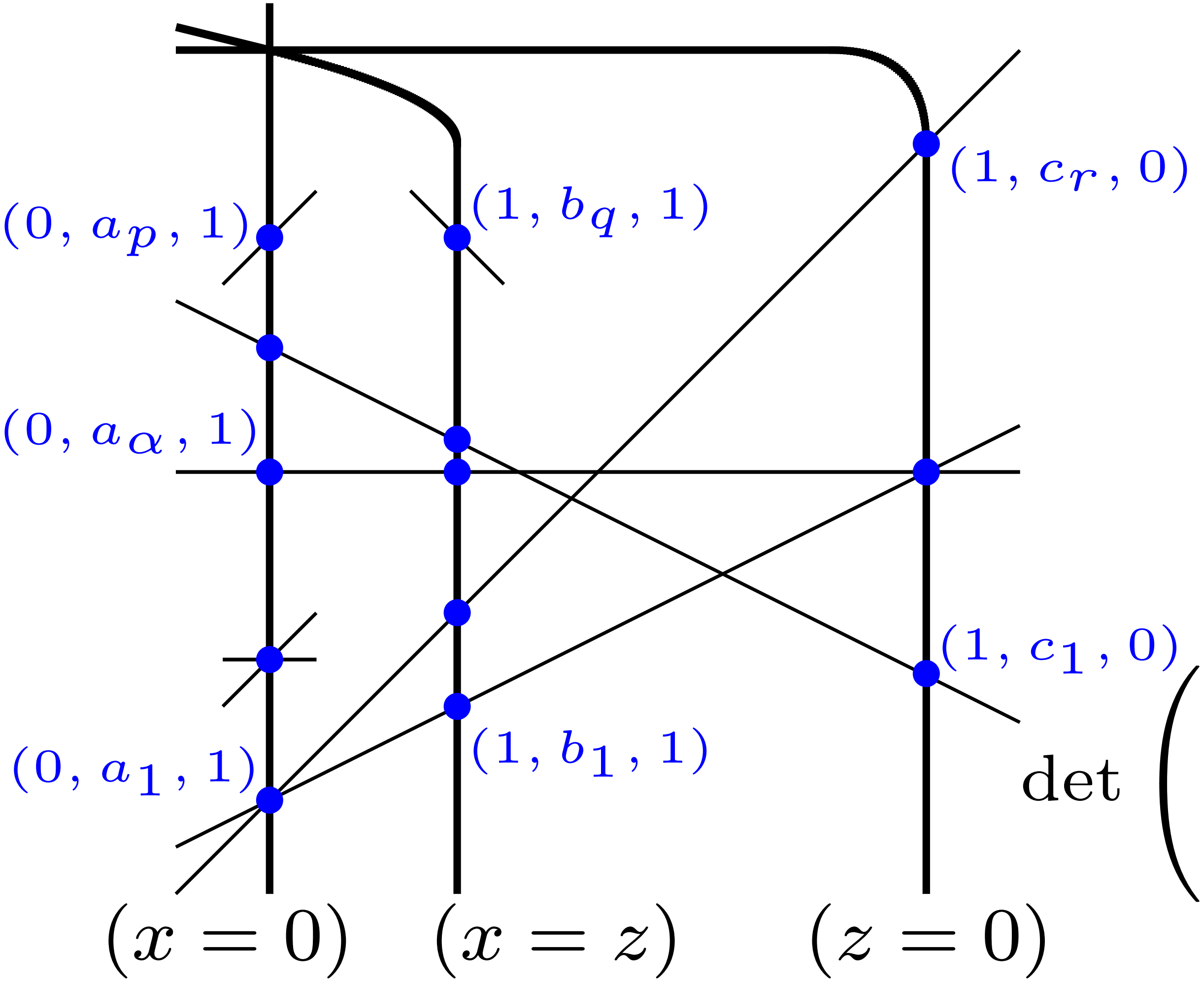

Sketch of proof.

$$R(I) \hookrightarrow \mathbb{C}^{p+q+r}$$

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Closed conditions:

$$\det \begin{pmatrix} 0 & a_\alpha & 1 \\ 1 & b_\beta & 1 \\ 1 & c_\gamma & 0 \end{pmatrix} = a_\alpha - b_\beta + c_\gamma = 0.$$

Linear relations!

2 (Dis-)connectivity of $R(I)$

Thm. 2: $\exists (ijk) \in I(\mathcal{A})$ s.t. $\mathrm{mult}(\mathcal{A}) \subset H_i \cup H_j \cup H_k$,

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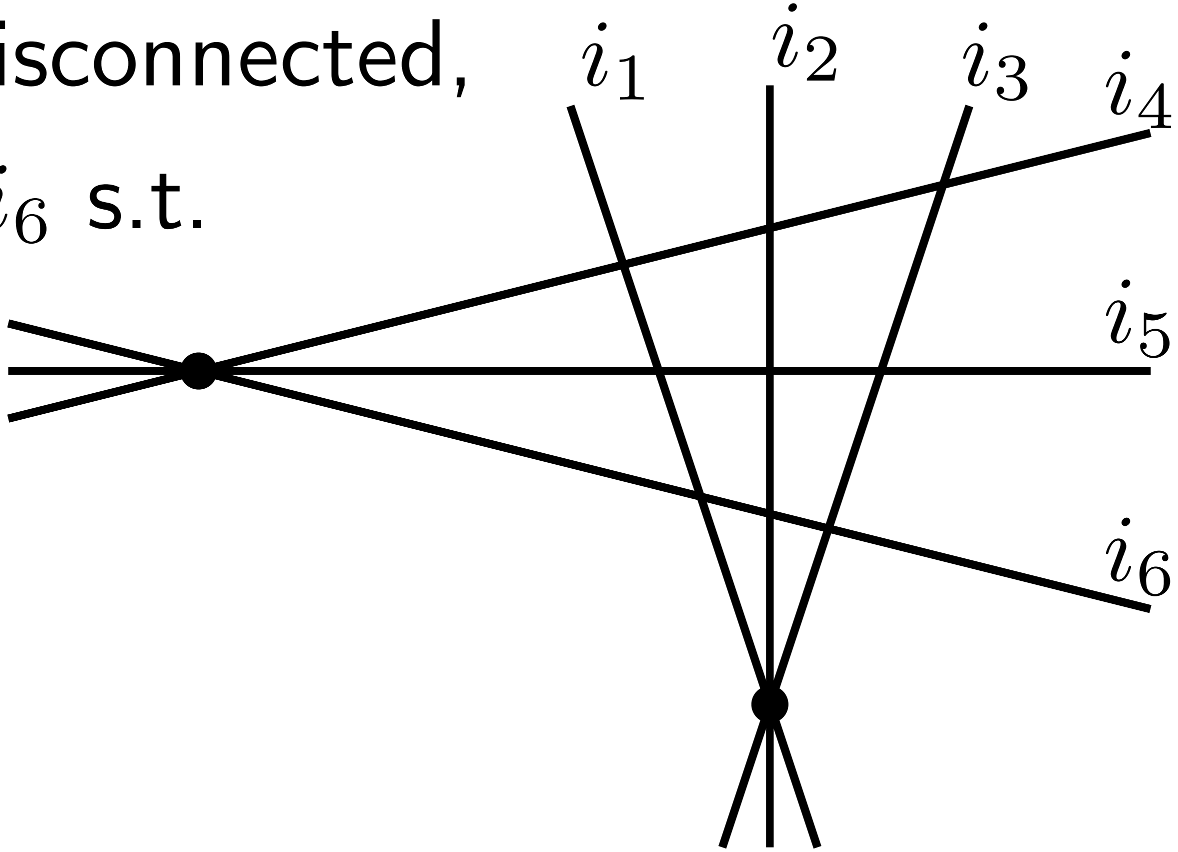
Corollary If $R(I)$ is disconnected,

$\Longrightarrow \exists i_1, i_2, \ldots, i_6$ s.t.



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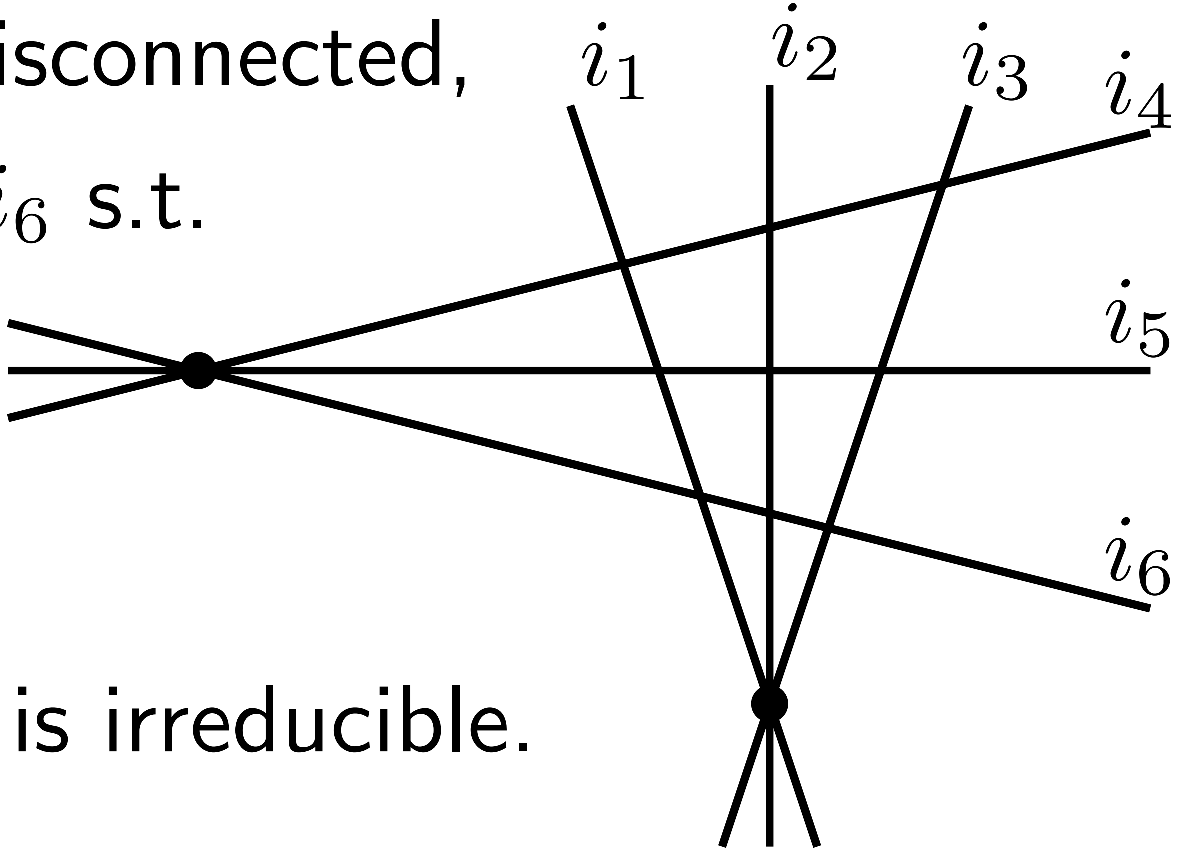


Corollary

If $n \leq 7$, then $R(I)$ is irreducible.

3 Classification up to 9 lines

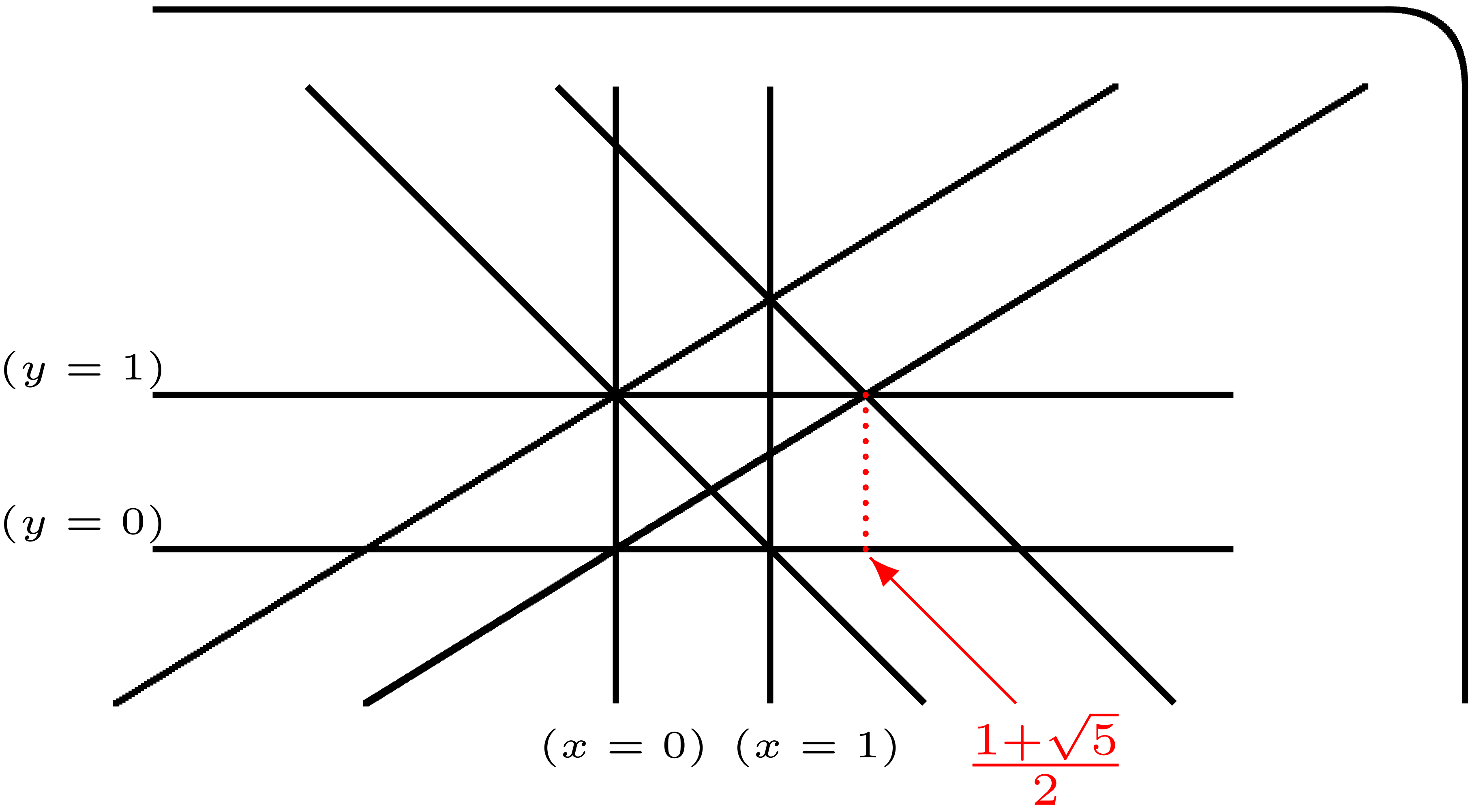
3 Classification up to 9 lines

n	disconnected $R(I)$	minimal field
≤ 7	NO	$\mathbb{Q}$
8	$\mathcal{A}^{\pm\sqrt{-3}}$ (MacLane)	$\mathbb{Q}(\sqrt{-3})$
9	$\mathcal{A}^{\pm\sqrt{-3}} \cup \{H_9\}$ (up to numbering)	$\mathbb{Q}(\sqrt{-3})$
	$\mathcal{A}^{\pm\sqrt{5}}$(Falk-Sturmfels)	$\mathbb{Q}(\sqrt{5})$
	$\mathcal{A}^{\pm\sqrt{-1}}$	$\mathbb{Q}(\sqrt{-1})$
10	$\mathcal{A}^{\pm\sqrt{2}}$	$\mathbb{Q}(\sqrt{2})$
	and many others	

Cor. For $n \leq 9$, $I(\mathcal{A})$ determines $\pi_1(M(\mathcal{A}))$.

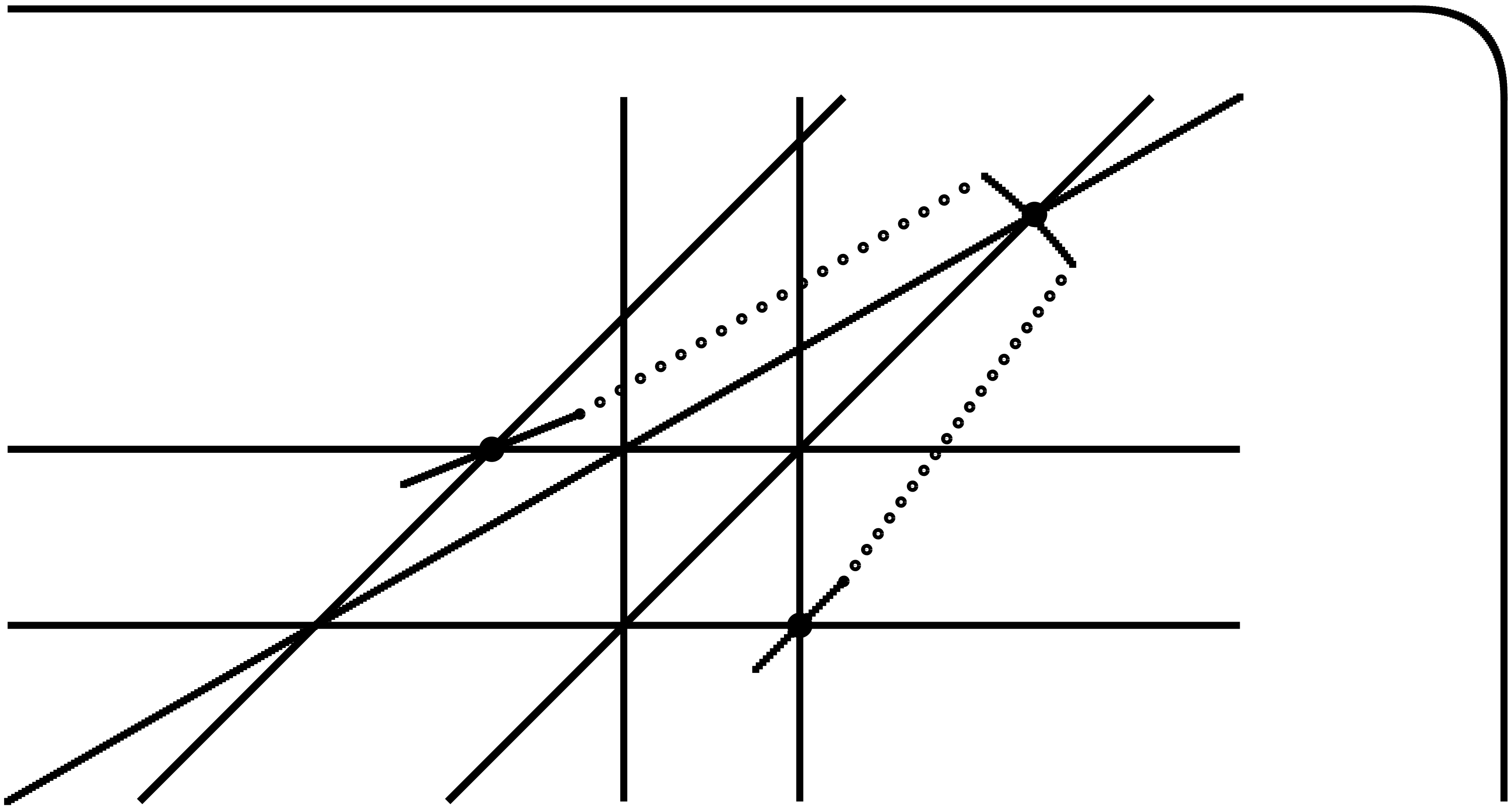
3 Classification up to 9 lines

Falk-Sturmfels Arrangement $\mathcal{A}^{+\sqrt{5}}$ $(n=9)$



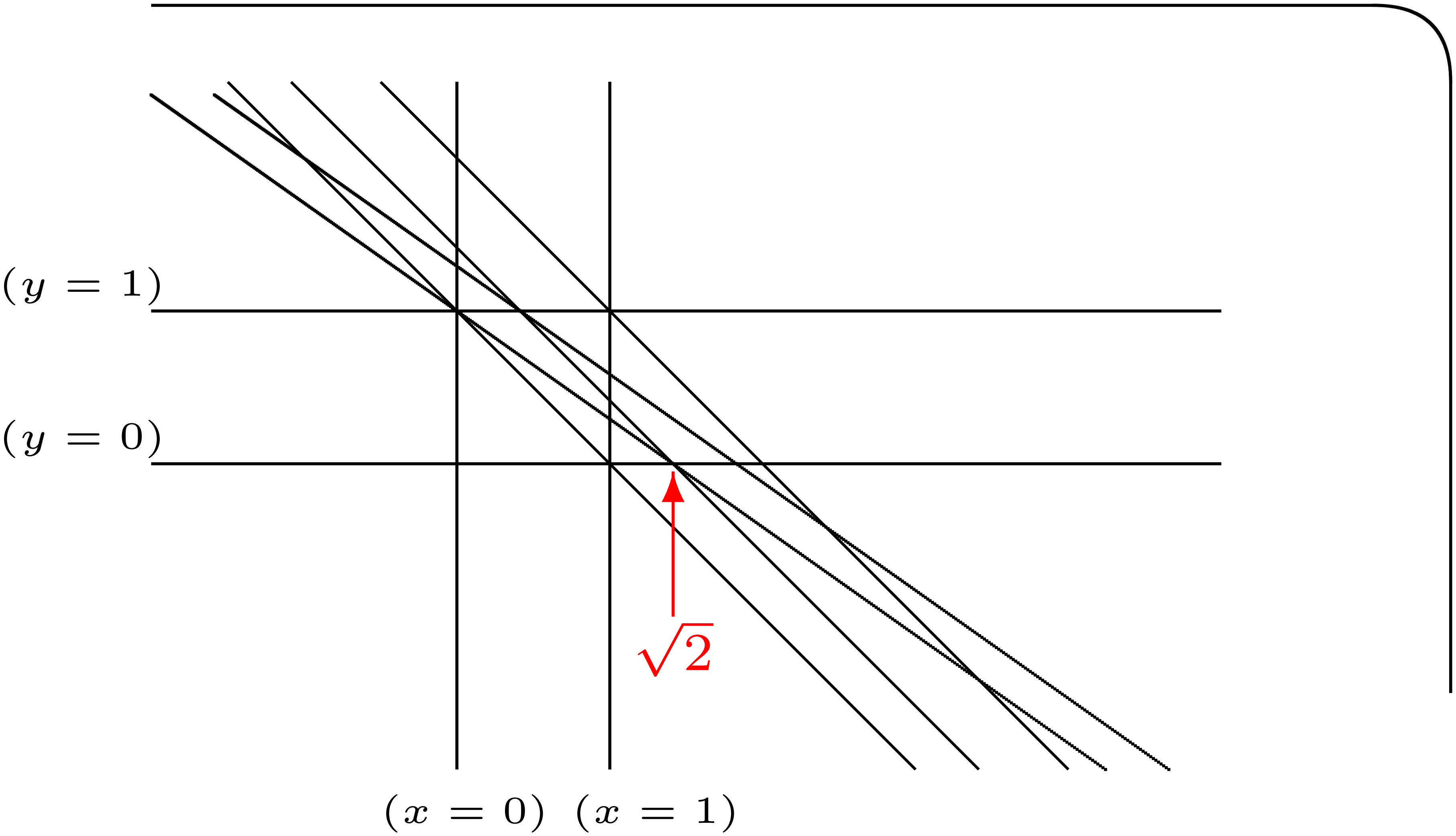
3 Classification up to 9 lines

$\underline{\mathcal{A}^{\pm\sqrt{-1}}\ (n=9)}$



3 Classification up to 9 lines

$\mathcal{A}^{\pm\sqrt{2}}$ $(n = 10)$



Reference:

S. Nazir, M. Yoshinaga: On the connectivity of the realization spaces of line arrangements.
(arXiv:1009.0202)

Thank you for your attention.