Free infinite divisibility of measures with rational function densities

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Abstract

We introduce a map which sends a probability measure into a one with real analytic density. This map is a homomorphism with respect to the free, monotone and Boolean convolutions. The images of some atomic measures by this map are freely infinitely divisible and their densities are rational functions.

Any probability measure μ with rational function density is shown to have a strictly positive free divisibility indicator. That is, there is $t > 0$ such that the Boolean power of μ by t is freely infinitely divisible.

We show that the t-distribution with n degrees of freedom and the F-distribution with $1, n$ degrees of freedom are freely infinitely divisible for odd n. Moreover, the t-distribution with $n \geq 3$ odd and the Gaussian have free divisibility indicators equal to one.

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1 Introduction

The free convolution $\mu \boxplus \nu$ of probability measures μ, ν is defined as the distribution of $X + Y$, where X, Y are self-adjoint, freely independent operators following distributions μ, ν, respectively. The concept of freely infinitely divisible distributions is an analogue of infinitely divisible distributions of probability theory, with the usual convolution $*$ replaced by the free convolution $\boxplus$. Free infinite divisibility is studied in relation to, for example, orthogonal polynomials [1], creation, annihilation and conservation operators on the full Fock space [21], the eigenvalue distributions of large random matrices [10, 14] and matrix-valued Lévy processes [5]. Combinatorial aspects also arise, for example in [8, 19].

Given a probability measure on $\mathbb{R}$, one question is if it is infinitely divisible or not. Many probabilists have worked on this question and now there are many sufficient conditions for a measure to be infinitely divisible, such as the complete monotonicity or log-convexity of the density function [17]. In free probability, by contrast, a useful sufficient condition for free infinite divisibility is not known. Increase of examples will be useful to clarify such sufficient conditions in terms of probability densities. Hence, in this paper, we will construct examples of freely infinitely divisible distributions, in particular those with rational function densities.

In Section 2 we introduce a homomorphism on the set of probability measures with respect to the free, monotone and Boolean convolutions, three well known convolutions in non-commutative probability. It preserves both kinds of infinite divisibility with respect to free and monotone convolutions. This homomorphism is related to Cauchy distributions and has a regularization property, that is, a probability measure is transformed into a one with real analytic density. We note that another regularization related to Cauchy distributions was used in [12]. This homomorphism makes a probability measure freely infinitely divisible if the measure has a variance small enough. As an application, we can construct many probability measures which are both freely and monotonically infinitely divisible.

In Section 3 we prove that the homomorphism increases the free divisibility indicator of a probability measure with finite variance. Moreover, a probability measure μ with rational function density is shown to have a strictly positive free divisibility indicator, that is, there is $t > 0$ such that the Boolean power of μ by t is freely infinitely divisible. This is another general construction of freely infinitely divisible distributions with rational function densities.

In Section 4 we prove that the t-distribution with n degrees of freedom is freely infinitely divisible for any odd n. The proof is quite similar to that of [8] used to prove the free infinite divisibility of the Gaussian. From a result of [4], the F-distribution with $1, n$ degrees of freedom is also freely infinitely divisible for odd n. Moreover, t-distribution has free divisibility indicator one if $n \geq 3$ is odd.

Finally in Section 5 Boolean powers of the Gaussian are studied. Like the t-distribution, the Gaussian is shown to have the free divisibility indicator one. It is shown that a Boolean power of Gaussian is classically infinitely divisible if and only if the power is one or zero.

Preliminaries are summarized below. The reader is respectively referred to [11], [15, 20] and [15, 23] for details on free, monotone and Boolean convolutions. We let $\mathbb{C}_+$ and $\mathbb{C}_-$ respectively denote the upper half-plane and the lower half-plane. Let $G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}$ $(z \in \mathbb{C}_+)$ denote the Cauchy transform of a probability measure μ and let $F_\mu(z) = \frac{1}{G_\mu(z)}$ denote its reciprocal. The map F_μ has a right inverse F_μ^{-1} defined in an open set $\Gamma_{\eta,M} := \{z \in \mathbb{C}_+ : \operatorname{Im} z > M, \ |\operatorname{Im} z| > \eta|\operatorname{Re} z|\}$. The Voiculescu transform of μ is defined by $\phi_\mu(z) := F_\mu^{-1}(z) - z$ for $z \in \Gamma_{\eta,M}$ and the energy transform $K_\mu(z)$ is defined by $K_\mu(z) = z - F_\mu(z)$. Then the free convolution $\boxplus$, monotone one $\triangleright$ and Boolean one $\uplus$ are characterized as follows:

$$\phi_{\mu \boxplus \nu} = \phi_\mu + \phi_\nu \text{ on } \Gamma_{\eta',M'} \text{ for some } \eta', M' > 0,$$
$$F_{\mu \triangleright \nu} = F_\mu \circ F_\nu \text{ on } \mathbb{C}_+, \qquad K_{\mu \uplus \nu} = K_\mu + K_\nu \text{ on } \mathbb{C}_+.$$

The infinite divisibility with respect to a convolution $\star$, any one of the three convolutions, is defined as follows:

Definition 1.1. A probability measure μ on $\mathbb{R}$ is $\star$-infinitely divisible if, for any natural number n, one can find a probability measure μ_n such that $\mu = \mu_n^{\star n} := \mu_n \star \cdots \star \mu_n$, iteration by n times.

Let us denote by $\mathcal{ID}(\boxplus)$ and by $\mathcal{ID}(\triangleright)$ the sets of freely and monotonically infinitely divisible measures, respectively. It is known that any probability measure is $\uplus$-infinitely divisible [23]. Indeed, one can define Boolean powers $\mu^{\uplus t}$ for each μ on $\mathbb{R}$ by the relation $K_{\mu^{\uplus t}} = tK_\mu$. A probability measure is not always $\boxplus$-infinitely divisible, so that $\mu^{\boxplus t}$ may not exist for $t < 1$ with the property $\phi_{\mu^{\boxplus t}} = t\phi_\mu$. However it is known that $\mu^{\boxplus t}$ exists for any $t \geq 1$ and μ [22].

The free infinite divisibility can be characterized in terms of the Voiculescu transform [11]:

Theorem 1.2. *A probability measure μ is in $\mathcal{ID}(\boxplus)$ if and only if ϕ_μ can be analytically extended to $\mathbb{C}_+$ with values in $\mathbb{C}_- \cup \mathbb{R}$. Another equivalent condition is the existence of the Lévy-Khintchine representation [6]*

$$\mathcal{C}_\mu(z) := z\phi_\mu\left(\frac{1}{z}\right) = \eta_\mu z + a_\mu z^2 + \int_{\mathbb{R}} \left(\frac{1}{1-xz} - 1 - xz1_{[-1,1]}(x)\right) d\nu_\mu(x), \quad \operatorname{Im} z < 0, \quad (1.1)$$

where $\eta_\mu \in \mathbb{R}$, $a_\mu \geq 0$ and ν_μ is a Lévy measure, i.e., a non negative measure with the properties $\nu_\mu(\{0\}) = 0$ and $\int_{\mathbb{R}} \min(1, x^2) d\nu(x) < \infty$. The triplet $(\eta_\mu, a_\mu, \nu_\mu)$ is unique.

2 Homomorphisms and free infinite divisibility

We will define a map "$\mu \mapsto \nu_{-a,-b} \triangleright \mu \triangleright \nu_{a,b}$", where $\nu_{a,b}$ is the Cauchy distribution $\frac{b}{\pi[(x-a)^2+b^2]}dx$ with $a \in \mathbb{R}, b \geq 0$. While "$\nu_{-a,-b}$" is not a probability measure, this map makes sense in terms of the reciprocal Cauchy transform. We prove that, if $b > 0$ and μ is not a point measure, then $\nu_{-a,-b} \triangleright \mu \triangleright \nu_{a,b}$ has a real analytic density. We mention that $\mu \triangleright \nu_{a,b}$ $(= \mu \boxplus \nu_{a,b} = \mu * \nu_{a,b} = \mu \uplus \nu_{a,b})$ is another regularization of μ used by Biane and Voiculescu [12]. As we will prove soon, if μ has a finite variance, $\nu_{-a,-b} \triangleright \mu \triangleright \nu_{a,b}$ also has a finite variance while $\mu \triangleright \nu_{a,b}$ does not.

Let $F : \mathbb{C}_+ \to \mathbb{C}_+$ be an analytic map. There is a probability measure μ on $\mathbb{R}$ such that $F = F_\mu$ if and only if the map F satisfies $\lim_{y\to\infty} \frac{F(iy)}{iy} = 1$ [18]. Moreover the probability measure μ is unique. In this case, it is known that $\operatorname{Im} F(z) \geq \operatorname{Im} z$ for any $z \in \mathbb{C}_+$. If $\operatorname{Im} F(z) = \operatorname{Im} z$ for some $z \in \mathbb{C}_+$, then F is of the form $F(z) = z - a$ for some $a \in \mathbb{R}$, that is, $\mu = \delta_a$.

Let $T_{a,b}\mu$ be a probability measure characterized by

$$F_{T_{a,b}\mu}(z) = F_\mu(z - a + ib) + a - ib$$

for $\operatorname{Im} z > 0$, $a \in \mathbb{R}$, $b \geq 0$. It can be shown that $F_{T_{a,b}\mu}$ is analytic in $\mathbb{C}_+$ with values in $\mathbb{C}_+$, and satisfies $\lim_{y\to\infty} \frac{F_{T_{a,b}\mu}(iy)}{iy} = 1$. Hence $T_{a,b}\mu$ is well defined as a probability measure on $\mathbb{R}$.

We denote by $\sigma^2(\mu)$ the variance of a probability measure μ. The notation $f \sim g$, $|x| \to \infty$ means that $\frac{f(x)}{g(x)} \to 1$ as $|x| \to \infty$. We say μ is trivial if μ is a point measure.

Proposition 2.1. *(1) For all probability measures μ and ν,*

$$T_{a,b}(\mu \triangleright \nu) = (T_{a,b}\mu) \triangleright (T_{a,b}\nu), \quad T_{a,b}(\mu \boxplus \nu) = (T_{a,b}\mu) \boxplus (T_{a,b}\nu), \quad T_{a,b}(\mu \uplus \nu) = (T_{a,b}\mu) \uplus (T_{a,b}\nu).$$

(2) $T_{a,b} \circ T_{c,d} = T_{a+c,b+d}$.
(3) For any $b > 0$ and non-trivial μ, $T_{a,b}\mu$ has a real analytic density on $\mathbb{R}$. Moreover, if μ has a finite variance, then the density of $T_{a,b}\mu$ behaves as $\sim \frac{b\sigma^2(\mu)}{\pi x^4}$ as $|x| \to \infty$.
(4) If μ has a finite variance, then $\sigma^2(T_{a,b}\mu) = \sigma^2(\mu)$.

Proof. (1) The first equality is proved directly. The others can be shown from the relations

$$\phi_{T_{a,b}\mu}(z) = \phi_\mu(z - a + ib), \quad K_{T_{a,b}\mu}(z) = K_\mu(z - a + ib).$$

(2) is easy to prove.

(3) If μ is not a point measure, $\operatorname{Im} F_\mu(z) > \operatorname{Im} z$ for $z \in \mathbb{C}_+$. Using the Stieltjes inversion formula, we have

$$T_{a,b}\mu(dx) = \frac{\operatorname{Im} F_\mu(x-a+ib) - b}{\pi |F_\mu(x-a+ib)+a-ib|^2} dx.$$

The density function is real analytic and strictly positive on $\mathbb{R}$. The reciprocal Cauchy transform F_μ has the Nevanlinna representation as $F_\mu(z) = z - m_1(\mu) + \int_{\mathbb{R}} \frac{\rho(du)}{u-z}$ with $m_1(\mu) := \int_{\mathbb{R}} u\, \mu(du)$ and a finite non-negative measure ρ. The measure ρ satisfies $\rho(\mathbb{R}) = \sigma^2$ [18]. Then, $F_\mu(x-a+ib)+a-ib \sim x$ as $|x| \to \infty$ and

$$\operatorname{Im}\left(F_\mu(x-a+ib)+a-ib\right) = \int_{\mathbb{R}} \frac{b}{(u-x+a)^2+b^2}\rho(du) \sim \frac{b\sigma^2}{x^2}.$$

Therefore, the density $p(x)$ of $T_{a,b}\mu$ behaves as $p(x) \sim \frac{b\sigma^2}{\pi x^4}$ as $|x| \to \infty$.

(4) $\sigma^2(\mu)$ is the same as $\rho(\mathbb{R})$, which is also equal to $\lim_{y\to\infty} |F_\mu(iy) - iy + m_1(\mu)| y$ [18]. Applying this formula to $T_{a,b}\mu$, we conclude that $\sigma^2(\mu) = \sigma^2(T_{a,b}\mu)$. □

Corollary 2.2. *$T_{a,b}(\mathcal{ID}(\triangleright)) \subset \mathcal{ID}(\triangleright)$, $T_{a,b}(\mathcal{ID}(\boxplus)) \subset \mathcal{ID}(\boxplus)$ for $a \in \mathbb{R}$, $b \geq 0$.*

Proof. These inclusions follow from the homomorphism properties and the very definition of the infinite divisibility. □

We prove a probability measure with small variance becomes freely infinitely divisible after the transformation by $T_{a,b}$. The following class is useful to prove this.

Definition 2.3. A probability measure μ is said to be in class $\mathcal{UI}$ if F_μ is univalent in $\mathbb{C}_+$ and moreover, F_μ^{-1} has an analytic continuation from $F_\mu(\mathbb{C}_+)$ to $\mathbb{C}_+$ as a univalent function.

The class $\mathcal{UI}$ is a subclass of $\mathcal{ID}(\boxplus)$ as proved in [2]. This fact was essentially used in [8] to prove the $\boxplus$-infinite divisibility of a normal law. Using the relation $F_{T_{a,b}\mu}^{-1}(z) = F_\mu^{-1}(z-a+ib)+a-ib$, we can prove that $T_{a,b}(\mathcal{UI}) \subset \mathcal{UI}$.

Theorem 2.4. *Let μ be a probability measure with finite variance σ^2. Then $T_{a,b}\mu \in \mathcal{UI}$ for $b \geq 2\sigma$.*

Remark 2.5. Assume moreover that μ is not $\boxplus$-infinitely divisible. If we use the free divisibility indicator $\phi(\mu)$, the inequality $b \geq 2\sigma$ can be weakened to $b \geq 2\sigma\sqrt{1-\phi(\mu)}$. See Proposition 3.1.

Proof. Let $\mathbb{C}_s := \{z \in \mathbb{C} : \operatorname{Im} z > s\}$. Maassen proved that F_μ is univalent in $\mathbb{C}_\sigma$ and $F_\mu(\mathbb{C}_\sigma) \supset \mathbb{C}_{2\sigma}$ ([18], Lemma 2.4). Therefore, $F_{T_{a,b}\mu}(\mathbb{C}_{-b+\sigma}) = F_\mu(\mathbb{C}_\sigma) + a - ib \supset \mathbb{C}_{2\sigma-b} \supset \mathbb{C}_+$, so that $F_{T_{a,b}\mu}^{-1}$ can be defined in $\mathbb{C}_+$ as a univalent function. □

Starting from atomic measures, we obtain many freely infinitely divisible measures whose densities are rational functions.

Example 2.6. Let $b > 0, n > 1, \lambda_j > 0, \sum_{j=1}^n \lambda_j = 1$ and $a_j \in \mathbb{R}$. The measure $T_{a,b}(\sum_{j=1}^n \lambda_j \delta_{a_j})$ has a rational function density. If $b > 0$ is large enough, it is freely infinitely divisible.

For instance, let $\mu_p := \frac{p}{2}(\delta_{-1}+\delta_1)+(1-p)\delta_0$, $0 \le p \le 1$. Then $G_{\mu_p}(z) = \frac{z^2-1+p}{z(z^2-1)}$, so that $G_{T_{0,b}\mu_p}(z) = \frac{z^2-b^2-1+p+2ibz}{z^3-(1+b^2)z+2ib(z^2-p/2)}$. The probability measure is given by

$$T_{0,b}\mu_p(dx) = \frac{bp(x^2+b^2+1-p)}{\pi[x^6+2(b^2-1)x^4+(b^4+2b^2(1-2p)+1)x^2+p^2b^2]}dx.$$

From Theorem 2.4, $T_{0,b}\mu_p \in \mathcal{UI}$ for $b \ge 2\sqrt{p}$.

In the particular case $p=1$ and $b=2$, $T_{0,2}\mu_1$ is a scaled t-distribution with degree three:

$$T_{0,2}\mu_1(dx) = \frac{2}{\pi(1+x^2)^2}dx, \quad x \in \mathbb{R}\,.$$

The Lévy measure ν of (1.1) is calculated as follows:

$$\nu(dx) = \frac{1}{\pi x^2}\left(1-\left(\frac{|x|\sqrt{x^2+16}-x^2}{8}\right)^{1/2}\right)dx, \quad x \in \mathbb{R}\,.$$

We can see $T_{0,b}\mu_1 \notin \mathcal{ID}(\boxplus)$ for $b<2$ as follows. The Voiculescu transform of μ_1 is $\phi_{\mu_1}(z) = \frac{-z+\sqrt{z^2+4}}{2}$. This function cannot be analytic in $\mathbb{C}_b$ if $b<2$. Therefore, $T_{0,b}\mu_1 \notin \mathcal{ID}(\boxplus)$ for $b<2$.

We can construct probability measures which are both monotonically and freely infinitely divisible:

Corollary 2.7. *Let $\mu \in \mathcal{ID}(\triangleright)$. If μ has a finite variance σ^2, then $T_{a,b}\mu \in \mathcal{ID}(\triangleright) \cap \mathcal{UI} \subset \mathcal{ID}(\triangleright) \cap \mathcal{ID}(\boxplus)$ for $b \ge 2\sigma$.*

Example 2.8. Let μ be the centered arcsine law with variance t: $\mu(dx) = \frac{dx}{\pi\sqrt{2t-x^2}}$, $F_\mu(z) = \sqrt{z^2-2t}$. Then

$$T_{0,b}\mu(dx) = \frac{(r(x)-b^2)\sqrt{r(x)-x^2+2t+b^2}+\sqrt{2}b(2t-x^2)}{\sqrt{2}\pi(x^4+4(b^2-t)x^2+4t^2)}dx,$$

where $r(x) = \sqrt{x^4+2(b^2-2t)x^2+(2t+b^2)^2}$. Corollary 2.7 says that $T_{0,b}\mu \in \mathcal{ID}(\boxplus) \cap \mathcal{ID}(\triangleright)$ for $b \ge 2\sqrt{t}$. More strongly, in this case $T_{0,b}\mu \in \mathcal{ID}(\boxplus) \cap \mathcal{ID}(\triangleright)$ if and only if $b \ge \sqrt{2t}$. This can be proved directly from the Voiculescu transform $\phi_\mu(z) = \sqrt{z^2+2t}-z$.

3 Free divisibility indicator

We know from Theorem 2.4 that the map $T_{a,b}$ makes a probability measure $\boxplus$-infinitely divisible for large $b>0$. This property can be understood quantitatively in terms of the so-called free divisibility indicator [9].

A family of maps $\mathbb{B}_t$ on the set of probability measures is defined by

$$\mathbb{B}_t(\mu) := (\mu^{\boxplus(1+t)})^{\uplus\frac{1}{1+t}}.$$

The free divisibility indicator $\phi(\mu)$ of a measure μ is defined by

$$\phi(\mu) := \sup\{t \ge 0 : \mu \text{ is in the image of } \mathbb{B}_t\}.$$

The following properties are known:

(1) $\mathbb{B}_t \circ \mathbb{B}_s = \mathbb{B}_{t+s}$ for $s, t \geq 0$ [9].

(2) A probability measure μ belongs to $\mathcal{ID}(\boxplus)$ if and only if μ is in the image of $\mathbb{B}_1$ [9].

(3) If $\phi(\mu) < \infty$, there is a probability measure ν such that $\mu = \mathbb{B}_{\phi(\mu)}(\nu)$ [9].

(4) $\phi(\mu) = \sup\{t \geq 0 : \mu^{\uplus t} \in \mathcal{ID}(\boxplus)\}$ [3].

We can sharpen Theorem 2.4 as follows.

Proposition 3.1. *If μ is a probability measure with a finite variance $\sigma^2(\mu)$, then*

$$\phi(T_{a,b}\mu) \geq \phi(\mu) + \frac{b^2}{4\sigma^2(\mu)}, \quad a \in \mathbb{R}, b \geq 0.$$

Proof. If $\phi(\mu) = \infty$, then $\phi(T_{a,b}\mu) = \infty$ since, for any $t > 0$, we can find a ν such that $\mu = \mathbb{B}_t(\nu)$ by definition, so that $T_{a,b}\mu = \mathbb{B}_t(T_{a,b}\nu)$.

If $\phi(\mu) < \infty$, let ν be a probability measure such that $\mu = \mathbb{B}_{\phi(\mu)}(\nu)$ in the property (3) above. Then, $T_{a,b}\mu = \mathbb{B}_{\phi(\mu)}(T_{a,b}\nu)$. Since the variance of $\nu^{\uplus t}$ is $t\sigma^2(\mu)$, $(T_{a,b}\nu)^{\uplus t} = T_{a,b}(\nu^{\uplus t})$ is $\boxplus$-infinitely divisible if $b \geq 2\sqrt{t\sigma^2(\mu)}$ from Theorem 2.4. Using the property (4) above, we have the inequality $\phi(T_{a,b}\nu) \geq \frac{b^2}{4\sigma^2(\mu)}$, which yields the conclusion. □

The following example will be generalized in Section 4.

Example 3.2. The Student t-distribution $\nu(dx) := \frac{2}{\pi(1+x^2)^2}dx$ is written as $\nu = T_{0,2}(\mu)$, where μ is the Bernoulli law $\frac{1}{2}(\delta_{-1} + \delta_1)$. Example 2.6 or Theorem 3.1 means that $\phi(\nu) \geq 1$. We can prove $\phi(\nu) = 1$ as follows. By direct calculation, we have $\phi_{\nu^{\uplus t}}(z) = \frac{-(z+2i)+\sqrt{(z+2i)^2+4t}}{2}$. If $t > 1$, the measure $\nu^{\uplus t}$ is not $\boxplus$-infinitely divisible since the Voiculescu transform cannot be analytic in the upper half-plane. The property (4) then implies that $\phi(\nu) = 1$.

There are sufficient conditions for a free divisibility indicator to be zero. For instance, if $\phi(\mu) > 0$, then the singular continuous part is zero and the Lebesgue absolutely continuous part is real analytic wherever it is positive and finite [9].

We show a sufficient condition for a divisibility indicator to be strictly positive: rational function densities are sufficient. This leads to another construction of freely infinitely divisible distributions with rational function densities.

Proposition 3.3. *Let $f(x)$ be a rational function which is strictly positive on $\mathbb{R}$ and satisfies $\int_{\mathbb{R}} f(x)dx = 1$. Then the probability measure $\mu(dx) := f(x)dx$ has a strictly positive free divisibility indicator. Equivalently, there is $t > 0$, depending on f, such that $\mu^{\uplus t} \in \mathcal{ID}(\boxplus)$.*

Proof. From the residue theorem, G_μ and hence F_μ is a rational function without zeros or poles in $\mathbb{C}_+$. $F_\mu(z)$ can be written as $z + g(z)$, where g is a rational function. We can find a $\delta > 0$ such that $g(z)$ does not have a pole in $\mathbb{C}_{-2\delta} := \{z \in \mathbb{C} : \operatorname{Im} z > -2\delta\}$. Define $F_t(z) := F_{\mu^{\uplus t}}(z) = z + tg(z)$ and $M_1 := \sup_{x\in\mathbb{R}} \operatorname{Im} g(x - i\delta) < \infty$. It then holds, for any $0 < t < \frac{\delta}{M_1}$, that $\operatorname{Im} F_t(x - i\delta) = -\delta + t\operatorname{Im} g(x - i\delta) \leq 0$, $x \in \mathbb{R}$. For any $w \in L_{-\delta} := \partial\mathbb{C}_{-\delta}$ and $z \in L_{-\frac{3}{2}\delta} \cup L_{-\frac{1}{2}\delta}$, we have $|F_t(z) - F_t(w) - (z - w)| = t|g(z) - g(w)| \leq M_2 t$, where $M_2 := \sup\{|g(z) - g(w)| : z \in L_{-\frac{3}{2}\delta} \cup L_{-\frac{1}{2}\delta}, w \in L_{-\delta}\}$. If $t < \min\{\frac{\delta}{M_1}, \frac{\delta}{2M_2}\}$, then $|F_t(z) - F_t(w) - (z - w)| < |z - w|$ on $L_{-\frac{3}{2}\delta} \cup L_{-\frac{1}{2}\delta}$ for any fixed $w \in L_{-\delta}$, implying that F_t

is injective in $\{z \in \mathbb{C} : \operatorname{Im} z \in (-\frac{3}{2}\delta, -\frac{1}{2}\delta)\}$ and $F_t(L_{-\delta})$ is a simple curve. This means F_t is injective in $\mathbb{C}_{-\delta}$ and $F_t(\mathbb{C}_{-\delta}) \supset \mathbb{C}_+$ from the Darboux-Picard theorem (see [13], pp. 310). Hence $\mu^{\uplus t} \in \mathcal{UI}$. □

If $\mu(dx) = f(x)dx$ as in Proposition 3.3, then $\mu^{\uplus t}$ also has a rational function density for any $t > 0$. Thus we obtain many freely infinitely divisible distributions with rational function densities.

4 Student t-distributions and F-distributions

The probability measure with the density proportional to $\frac{1}{(1+x^2)^{\frac{n+1}{2}}}$ is called the t-distribution with n degrees of freedom (the scaling is different from the convention). More generally, we set

$$\mathbf{St}_r(dx) := \frac{c_r}{(1+x^2)^{\frac{r+1}{2}}}dx, \quad x \in \mathbb{R}, \quad r > 0,$$

where c_r is a normalizing constant given by $c_r = \frac{1}{B(\frac{1}{2}, \frac{r}{2})}$ in terms of the beta function. For any integer n, $\mathbf{St}_n$ is known to be classically infinitely divisible [16].

It is known that $\mathbf{St}_1$ is in $\mathcal{ID}(\boxplus)$ and moreover $\phi(\mathbf{St}_1) = \infty$ [9]. We have met $\mathbf{St}_3$ in Section 2 and observed $\phi(\mathbf{St}_3) = 1$ in Section 3. In this section, we prove the following:

Theorem 4.1. *If n is an odd integer, the measure $\mathbf{St}_n$ is freely infinitely divisible. Moreover, the free divisibility indicator of $\mathbf{St}_n$ is one for any odd integer $n > 1$.*

The proof is quite similar to that of the Gaussian [8]. As we prove soon, when r is an odd integer, the Cauchy transform of $\mathbf{St}_r$ is a rational function. This fact is important in the proof. If r is not an odd integer, we have to look at the Riemannian sheet associated to the Cauchy transform, or have to avoid it somehow, and the proof of this paper becomes invalid at least in the current form.

We collect several properties for general $r > 0$:

Lemma 4.2. *(1) The Cauchy transform $G_r := G_{\mathbf{St}_r}$ continues analytically from $\mathbb{C}_+$ to $\mathbb{C}\backslash\{iy : y < -1\}$.*

(2) $G_{r+2}(z) = \frac{1}{1+z^2}\left(\frac{c_{r+2}}{c_r}G_r(z) + z\right)$.

(3) $\frac{d}{dz}G_r(z) = \frac{(r+1)c_r}{c_{r+2}}(1 - zG_{r+2}(z)) = \frac{r+1}{1+z^2}\left(\frac{c_r}{c_{r+2}} - zG_r(z)\right)$.

Proof. (1) This can be seen by changing the contour of the integral, that is, the representation

$$G_r(z) = \int_{\partial D_{a,\delta}} \frac{c_r}{(1+x^2)^{\frac{r+1}{2}}} \frac{1}{z-x}dx,$$

where $D_{a,\delta} := \{z \in \mathbb{C} : \arg(z + ia) \in (-\frac{1}{2}\pi + \delta, \frac{3}{2}\pi - \delta)\}$, gives an extension to $D_{a,\delta}$ for any $a < 1$, $\delta > 0$.

(2) From simple calculations,

$$\begin{aligned}G_r(z) &= c_r \int_{\mathbb{R}} \frac{1+z^2+2z(x-z)+(x-z)^2}{(1+x^2)^{\frac{r+1}{2}+1}} \frac{1}{z-x} dx \\ &= \frac{c_r}{c_{r+2}} G_{r+2}(z)(1+z^2) - 2z\frac{c_r}{c_{r+2}} + z\frac{c_r}{c_{r+2}} \\ &= \frac{c_r}{c_{r+2}} \left((z^2+1)G_{r+2}(z) - z\right).\end{aligned}$$

(3) By integration by parts,

$$\begin{aligned}G_r'(z) &= -c_r \int_{\mathbb{R}} \frac{1}{(1+x^2)^{\frac{r+1}{2}}} \frac{1}{(z-x)^2} dx \\ &= -c_r \int_{\mathbb{R}} \frac{(r+1)x}{(1+x^2)^{\frac{r+3}{2}}} \frac{1}{z-x} dx \\ &= (r+1)\frac{c_r}{c_{r+2}} \left(1 - zG_{r+2}(z)\right).\end{aligned}$$

The last equality of (3) follows from (2). □

The following is crucial to prove the main theorem:

Proposition 4.3. *Let $r > 0$ be real and $n > 0$ be an odd integer.*
(1) If $F_r(z) := \frac{1}{G_r(z)} \in \mathbb{C}_+$, then $F_r'(z) \neq 0$.
(2) G_n is a rational function in $\mathbb{C}$ with a unique pole at $-i$. If $n > 1$, then the degree of the pole is larger than one.
(3) F_n is an analytic bijection from a neighborhood of $i(-1,\infty)$ onto a neighborhood of $i(0,\infty)$.

Proof. (1) Assume that $F_r(z) \in \mathbb{C}_+$. If $F_r'(z)$ were equal to zero and $z \in \mathbb{C}_+$, then $F_{r+2}(z) = z$ from Lemma 4.2(3), a contradiction. If $F_r'(z)$ were equal to zero and $z \in \mathbb{C}_- \cup \mathbb{R}$, then $F_r(z) = \frac{c_{r+2}}{c_r} z$ from Lemma 4.2(3), again a contradiction.
(2) From the residue theorem, $G_1(z) = \frac{1}{z+i}$. The inductive application of Lemma 4.2(2) implies G_n is a rational function and has a pole at $-i$ and possibly at i. However, G_n is analytic in $\mathbb{C}_+$, so that i is not a pole. It also follows that the degree of the pole $-i$ is larger than one.
(3) Note that $G_n(iy) \in i\mathbb{R}$ for $y > -1$ from the contour integral representation of Lemma 4.2(1). If F_n had poles in $i(-1,0)$, let is be the one with the largest imaginary part. Now two cases are possible: $\frac{1}{i}F_n(is+i0) = -\infty$ and $\frac{1}{i}F_n(is+i0) = \infty$. In the former case, there is a zero in $i(s,\infty)$, but G_n has no pole in $i(-1,\infty)$, a contradiction. In the latter case, there is $t > s$ such that $F_n'(it) = 0$. Since F_n does not have a zero in $i(s,\infty)$, we have $\frac{1}{i}F_n(it) > 0$. This contradicts (1). Hence $F_n(z)$ does not have a pole in $i(-1,0)$. Since $F_n(-i+i0) = 0$, $\frac{1}{i}F_n(iy) > 0$ for $y > -1$ and hence $F_n'(iy) > 0$ for $y > -1$ from (1). □

Now we prove the main theorem of this section.

Proof of Theorem 4.1. Let $n > 1$ be an odd integer. We do not consider $\mathbf{St}_1$, the Cauchy distribution, which is known to be freely infinitely divisible. Let $\sigma_n^2 < \infty$ be the variance of $\mathbf{St}_n$. As shown in Lemma 2.4 of [18], the map F_n is injective in $\mathbb{C}_{\sigma_n} := \{z \in \mathbb{C} : \operatorname{Im} z > \sigma_n\}$ and takes each point of $\mathbb{C}_{2\sigma_n}$ precisely once there.

From Proposition 4.3(3) there is an open set $U \supset i(-1,\infty)$ such that F_n is injective in U. In particular, for each $y > 0$, there is an $\varepsilon > 0$, dependent on y, such that the equation $F_n(z) = x + iy$ for each $|x| < \varepsilon$ has a unique solution $z \in U$. Let us denote by C_y^ε the curve $F_n^{-1}(\{x + iy : |x| < \varepsilon\})$ defined as above. The property (1) of Proposition 4.3 enables us to extend the end points of C_y^ε until they tend to infinity or encounter poles of F_n and then we obtain a curve $C_y \supset C_y^\varepsilon$ satisfying the property $F_n(C_y) \subset \mathbb{R} + iy$. Since F_n is a rational function and $\lim_{y\to\infty} F_n(iy) = \infty$, it holds that $F_n(z) \to \infty$ as $|z| \to \infty$ or as z tends to a pole, and this means $F_n(C_y) = iy + \mathbb{R}$.

From the construction, the set $\bigcup_{0<y<2\sigma_n} C_y$ is open and $F_n : \mathbb{C}_+ \cup (\bigcup_{0<y<2\sigma_n} C_y) \to \mathbb{C}_+$ is an analytic bijection. Thus we can extend F_n^{-1} from the domain $\mathbb{C}_{2\sigma_n}$ to $\mathbb{C}_+$ as a univalent map. Therefore, $\mathbf{St}_n \in \mathcal{UI}$.

Next we prove that $\phi(\mathbf{St}_n) = 1$. Let $t > 1$ be sufficiently close to 1. Then the function $F(z) := F_{\mathbf{St}_n^{\uplus t}}(z) = (1-t)z + tF_n(z)$ has critical points in $i(-1,0)$, since $F'(-i) = 1-t < 0$ and $F'(0) > 0$ (the latter inequality holds if $t > 1$ is close to 1). Let $iy_0 \in i(-1,0)$ be the critical point with the largest imaginary part. We find that $y_1 := \frac{1}{i}F(iy_0) = (1-t)y_0 + \frac{t}{i}F_n(iy_0) > (1-t)y_0 > 0$. If $t > 1$ is close to 1, there is no critical point in $i(y_0,\infty)$ and then the inverse map F^{-1} is well defined around $i(y_1,\infty)$. However, $(F^{-1})'(iy_1 + i\varepsilon) \to \infty$ as $\varepsilon \searrow 0$, so that F^{-1} does not extend analytically to $\mathbb{C}_+$. In view of Theorem 1.2, this means $\mathbf{St}_n^{\uplus t} \notin \mathcal{ID}(\boxplus)$ for $t > 1$ sufficiently close to 1. From the property (4) of Section 3, we conclude $\phi(\mathbf{St}_n) = 1$. □

Remark 4.4. The fact that the Cauchy transform is a rational function is used to prove $F_\mu(C_y) = L_y$. In the case of the Gaussian, the Cauchy transform is not a rational function, but the use of a differential equation helps us to prove this [8].

If a random variable X has a symmetric distribution in $\mathcal{ID}(\boxplus)$, then the square X^2 also has a $\boxplus$-infinitely divisible distribution [4]. If X follows the t-distribution with $2n-1$ degrees of freedom, then X^2 follows the F-distribution $\mathbf{F}_{1,2n-1}$ with degrees $1, 2n-1$ whose density is:

$$\mathbf{F}_{1,2n-1}(dx) = c_n x^{-\frac{1}{2}}(1+x)^{-n}dx, \quad x > 0.$$

This measure is $\boxplus$-infinitely divisible, and moreover classically infinitely divisible since the density is completely monotone (see [17], Theorem 10.7).

Thus we have obtained two families of probability measures in $\mathcal{ID}(*) \cap \mathcal{ID}(\boxplus)$: $\mathbf{St}_{2n-1}$ and $\mathbf{F}_{1,2n-1}$, $n = 1, 2, 3, \cdots$. We note that, after taking an appropriate scaling and the limit $n \to \infty$, $\mathbf{St}_{2n-1}$ converges weakly to the Gaussian and $\mathbf{F}_{1,2n-1}$ converges to the chi-square distribution with one degree of freedom.

A general F-distribution with m, n degrees of freedom has the density, up to a scaling,

$$\mathbf{F}_{m,n}(dx) = c_{m,n} x^{\frac{m}{2}-1}(1+x)^{-\frac{m+n}{2}}dx, \quad x > 0,$$

where $c_{m,n} = \frac{1}{B(\frac{m}{2},\frac{n}{2})}$. Here we mention what happens for $m > 1$.

If X is a symmetric random variable following the law

$$\mathbf{S}_{m,n}(dx) = c_{m,n}|x|^{m-1}(1+x^2)^{-\frac{m+n}{2}}dx, \quad x \in \mathbb{R},$$

then X^2 follows $\mathbf{F}_{m,n}$. If $m > 1$, the density of $\mathbf{S}_{m,n}$ vanishes at 0, and this means $F_{\mathbf{S}_{m,n}}(0) = \infty$. From a property of a subordination function (see Theorem 4.6 of [7]), if μ is in $\mathcal{ID}(\boxplus)$, F_μ extends to a continuous function on $\mathbb{C} \cup \mathbb{R}$ with values in $\mathbb{C} \cup \mathbb{R}$. Hence $\mathbf{S}_{m,n}$ is not freely infinitely divisible if $m > 1$ and we cannot apply the result of [4], but this does not imply that $\mathbf{F}_{m,n} \notin \mathcal{ID}(\boxplus)$.

5 Note on infinite divisibility of Gaussian

The free divisibility indicator is not continuous with respect to the weak convergence, as one can observe from Wigner's semicircle law $\mathbf{W}_t$ with mean 0 and variance t. Indeed, $\phi(\mathbf{W}_t) = 1$ for any $t > 0$, while $\phi(\mathbf{W}_0) = \infty$. Hence, Theorem 4.1 together with an approximation argument is not sufficient to calculate the exact value of the free divisibility indicator of Gaussian. In this section we will show that the value is still equal to one as the t-distribution case. The classical infinite divisibility of the Boolean power of Gaussian is also studied here.

The following properties are similar to those proved for Askey-Wimp-Kerov distributions [8], but the Gaussian case was excluded in [8]. We therefore prove them.

Lemma 5.1. *Let F be the reciprocal Cauchy transform of the standard Gaussian on $\mathbb{R}$.*
(1) F continues analytically to $i(-\infty, \infty)$ and $\frac{1}{i}F(iy) > 0$, $y \in \mathbb{R}$.
(2) $F'(iy) > 0$, $y \in \mathbb{R}$.
(3) $\lim_{y\to-\infty} F(iy) = 0$.

Proof. Let G be the Cauchy transform of the Gaussian and $f(y)$ denote the function $\frac{1}{i}F(iy)$.

(1) One can replace the contour $\mathbb{R}$ for G by $\mathbb{R} - ic$ for any $c > 0$, which extends G to an entire analytic function in $\mathbb{C}$, and hence F is a meromorphic function in $\mathbb{C}$ without zeros. By symmetry, $f(y)$ takes real numbers wherever it is defined (it soon turns out that f is defined in $\mathbb{R}$). Assume that F has a pole in $i\,\mathbb{R}$ and let iy_0 be the pole in $i(-\infty, 0)$ with the largest imaginary part. Then the limit $\lim_{y\searrow y_0} f(y)$ is either $-\infty$ or ∞. In the former case, one can find a point $y_1 \in (y_0, 0)$ such that $f(y_1) = 0$, a contradiction to the fact that G is entire analytic. In the latter case, there is a point $y_2 \in (y_0, \infty)$ such that $f'(y_2) = 0$ since $f(\infty) = \infty$. The map f cannot have a zero in (y_0, ∞), so that $f(y_2) > 0$ which contradicts Remark 3.4 of [8]. Therefore F does not have a pole or a zero in $i\,\mathbb{R}$.

(2) The function f satisfies a differential equation

$$f'(y) = f(y)^2 - yf(y) \tag{5.1}$$

as proved in [8], Eq. (3.6). If $y > 0$, $f(y) > y$ from a basic property of a reciprocal Cauchy transform, and hence $f'(y) = f(y)(f(y) - y) > 0$. If $y < 0$, $f'(y) > 0$ from (1) and (5.1).

(3) From (2), the limit $a := \lim_{y\to-\infty} f(y)$ exists in $[0, \infty)$. If a were strictly positive, then $f'(-\infty) = \infty$ from (5.1). However $f(y) = f(0) - \int_y^0 f'(x)dx$, implying $f(-\infty) = -\infty$, a contradiction. Hence $a = 0$. □

Proposition 5.2. *Let $\mathbf{G}$ be the Gaussian on $\mathbb{R}$ with mean 0 and variance 1.*
(1) $\phi(\mathbf{G}) = 1$, or equivalently, $\mathbf{G}^{\uplus t} \in \mathcal{ID}(\boxplus)$ if and only if $0 \le t \le 1$.
(2) $\mathbf{G}^{\uplus t} \in \mathcal{ID}(*)$ if and only if $t \in \{0, 1\}$.

Proof. We follow the notation of Lemma 5.1.

(1) For $t > 1$ let $F_t(z) := F_{\mathbf{G}^{\uplus t}}(z) = (1-t)z + tF(z)$ and $f_t(y) := \frac{1}{i}F_t(iy)$. A consequence of Lemma 5.1 is that $f_t(-\infty) = \infty$. Since $f_t(\infty) = \infty$ and $f_t'(\infty) = 1$, we can find a point $y_0 \in \mathbb{R}$ such that $f_t'(y_0) = 0$ and $f_t'(y) > 0$ for $y \in (y_0, \infty)$. Let $y_1 := f_t(y_0) = (1-t)y_0 + tf(y_0)$. If $y_0 \le 0$, then $y_1 > 0$ from Lemma 5.1(1). If $y_0 > 0$, then $y_1 > (1-t)y_0 + ty_0 = y_0 > 0$ from a basic property of a reciprocal Cauchy transform. In both cases $y_1 > 0$. The inverse map F_t^{-1} analytically extends to a neighborhood of $i(y_1, \infty)$, but $(F_t^{-1})'(iy_1 + i0) = \infty$. From Theorem 1.2, $\mathbf{G}^{\uplus t}$ is not $\boxplus$-infinitely divisible.

(2) By changing the contour, we can write

$$xG(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{x}{x-y+i} e^{-\frac{1}{2}(y-i)^2} dy.$$

We divide the integral into two parts. First we find

$$\left| \frac{1}{\sqrt{2\pi}} \int_{\sqrt{x}}^{\infty} \frac{x}{x-y+i} e^{-\frac{1}{2}(y-i)^2} dy \right| \leq \frac{1}{\sqrt{2\pi}} \int_{\sqrt{x}}^{\infty} y^2 e^{-\frac{1}{2}y^2+\frac{1}{2}} dy \to 0 \text{ as } x \to \infty.$$

Next, we have $\sup_{y \in (-\infty, \sqrt{x}]} |\frac{x}{x-y+i} - 1| \to 0$ as $x \to \infty$ and hence

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{x}} \frac{x}{x-y+i} e^{-\frac{1}{2}(y-i)^2} dy \to 1 \text{ as } x \to \infty$$

from the dominated convergence theorem. By symmetry, we conclude $xG(x) \to 1$ as $|x| \to \infty$.

From the Stieltjes inversion formula, the density of $\mathbf{G}^{\uplus t}$ can be written as

$$\frac{t}{|(1-t)xG(x)+t|^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

For each $t > 0$, the above density behaves as $\sim \frac{t}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ for large $|x| > 0$ since $xG(x) \to 1$. If $\mathbf{G}^{\uplus t}$ were in $\mathcal{ID}(*)$, its Gaussian-like tail behavior implies that $\mathbf{G}^{\uplus t}$ is precisely a Gaussian (see Corollary 9.9 of [17]). This contradicts the fact that $\mathbf{G}^{\uplus t}$ is not a Gaussian for $t \neq 1$. □

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