Arrangements and Ranking Patterns

Hidehiko Kamiya
Okayama University, Okayama, Japan
Peter Orlik
University of Wisconsin, Madison, WI, USA
Akimichi Takemura
University of Tokyo, Tokyo, Japan
and
Hiroaki Terao
Tokyo Metropolitan University, Tokyo, Japan

March 4, 2005

Abstract

In the unidimensional unfolding model, given m objects in general position on the real line, there arise $1 + m(m-1)/2$ rankings. The set of rankings is called the ranking pattern of the m given objects. Change of the position of these m objects results in change of the ranking pattern. In this paper we use arrangement theory to determine the number of ranking patterns theoretically for all m and numerically for $m \leq 8$. We also consider the probability of the occurrence of each ranking pattern when the objects are randomly chosen.

Key words: unfolding model, ranking pattern, arrangement of hyperplanes, characteristic polynomial, mid-hyperplane arrangement, spherical tetrahedron.

1 Introduction

Various models have been developed for the analysis of ranking data in psychology. These include Thurstonian models, distance-based models, paired and higher-order comparison models, ANOVA-type loglinear models, multistage models and unfolding models, see Marden [11]. The unfolding model was devised by Coombs [3, 4, 5] for the analysis of ranking data based on preferential choice behavior. This model has been widely used in practice in many fields beyond psychology: sociology, marketing science, voting theory, etc. In addition, the same mathematical structure can be found in Voronoi diagrams (Okabe, Boots, Sugihara and Chiu [13]), spatial competition models in urban economics (Hotelling [9], Eaton and Lipsey [7, 8]) and multiple discriminant analysis (Kamiya and Takemura [10]).

According to the unidimensional unfolding model, preferential choice is made in the following manner: all individuals evaluate m objects based on their single common attribute. Each object is represented by a real number x_i, measuring the level of this attribute and viewed as a point on the real line $\mathbb{R}$, the *unidimensional underlying continuum.* At the same time, each individual is represented by a point y on the same line, considered the individual's preference and called his/her *ideal point.* The model assumes that individual y ranks the m objects x_i according to their distances from y, so y prefers x_i to x_j iff $|y-x_i| < |y-x_j|$. We say that the m points representing the objects are in *general position* if they and their midpoints are all distinct. Further, we do not consider partial rankings or ties in this paper, so we treat only those individuals whose ideal points do not coincide with any midpoint of two objects.

Let $\mathbf{x} = (x_1, \dots, x_m)$ be m objects which satisfy these assumptions. By varying the location of the ideal point y throughout $\mathbb{R}$ (except the midpoints), we can account for $\binom{m}{2}+1$ rankings of $\mathbf{x}$. The significance of using this model lies here: there are $m!$ potential rankings, but the psychological structure restricts the variety of rankings that can actually occur. We call the set of $\binom{m}{2} + 1$ rankings of $\mathbf{x}$ the *ranking pattern* of $\mathbf{x}$. By considering different attributes, we can get different sets of m real numbers $x_1, x_2, \dots, x_m$ for the same m objects, and thus obtain different ranking patterns.

Suppose the objects are ordered as $x_1 < \cdots < x_m$. In order to determine the number of ranking patterns, we need to know the number of possible rank orders of the midpoints $x_{ij} = (x_i + x_j)/2,\ 1 \le i < j \le m$. Any rank order of the midpoints $x_{ij},\ 1 \le i < j \le m$, must satisfy the condition that the rank $d(i,j)$ of x_{ij} from left to right on $\mathbb{R}$ be increasing in i for any fixed j as well as increasing in j for any fixed i. Consider the number g_m of functions $d : \{(i,j) \mid 1 \le i < j \le m\} \to \{1, 2, \dots, m(m-1)/2\}$ satisfying this condition. Clearly g_m serves as an upper bound for the number of possible rank orders of the midpoints $x_{ij},\ 1 \le i < j \le m$. Thrall [19] obtained this number by considering a problem similar to that of counting the number of standard Young tableaux. However, g_m is only an upper bound, since the rank order of the midpoints meeting the above-mentioned condition does not necessarily satisfy other restrictions induced by the rank order of the objects.

In this paper, we use the theory of hyperplane arrangements to find the number of possible rank orders of midpoints and thereby obtain the number of ranking patterns generated by the unidimensional unfolding model. For the general theory of hyperplane arrangements, see Orlik and Terao [14]. The organization of this paper is as follows. In Section 2, we define the mid-hyperplane arrangement and show that the number of ranking patterns can be obtained by counting the number of chambers of this arrangement. We give a formula for the number of ranking patterns for all m in Corollary 2.7. Although this provides a theoretical solution of our problem, explicit calculations are difficult. In Section 3, we reduce the calculation to that of counting the number of points in certain finite sets. Based on these results, we obtain the number of ranking patterns for $m \le 7$ in Section 4 and for $m = 8$ in Section 5. We also show in those sections that the characteristic polynomial of the mid-hyperplane arrangement is a product of linear factors in $\mathbb{Z}[t]$ if and only if $m \le 7$. In Section 6, we consider the question of the probabilities of ranking patterns and give the answer for $m \le 5$ objects. For $m = 5$ the problem reduces to that of finding volumes of certain spherical tetrahedra.

Recently Stanley [16] considered a similar problem in the Minkowski space. In this context this note deals with the "classical case."

2 Arrangements and ranking maps

In this section we interpret ranking maps and ranking patterns in terms of arrangements. We define the mid-hyperplane arrangement and prove in Corollary 2.7 that the number of ranking patterns is expressed in terms of the characteristic polynomial of this arrangement.

Let m be an integer with $m \geq 3$. Define two sets of hyperplanes in the m-dimensional Euclidean space $\mathbb{R}^m$.

(I) $H_{ij} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i = x_j\} \quad (1 \leq i < j \leq m)$.

The hyperplane arrangement $\mathcal{B}_m := \{H_{ij} \mid 1 \leq i < j \leq m\}$ is called the *braid arrangement* [14, p.13]. It has $|\mathcal{B}_m| = \binom{m}{2}$ hyperplanes. Let

$$I_4 := \{(p, q, r, s) \mid 1 \leq p < q \leq m,\ p < r < s \leq m,\ p, q, r, s \text{ are distinct}\}.$$

(II) $H_{pqrs} := \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_p + x_q = x_r + x_s\}, \quad (p, q, r, s) \in I_4$.

Define the *mid-hyperplane arrangement*

$$\mathcal{A}_m := \mathcal{B}_m \cup \{H_{pqrs} \mid (p, q, r, s) \in I_4\}.$$

Here $|\mathcal{A}_m| = \binom{m}{2} + 3\binom{m}{4}$. For an arbitrary arrangement $\mathcal{A}$ in $\mathbb{R}^m$, let

$$M(\mathcal{A}) := \mathbb{R}^m \setminus \bigcup_{H \in \mathcal{A}} H$$

be the complement of $\mathcal{A}$. The connected components of $M(\mathcal{A})$ are called *chambers* of $\mathcal{A}$. Let $\mathbf{Ch}(\mathcal{A})$ be the set of all chambers of $\mathcal{A}$.

Let $\mathbb{P}_m$ denote all permutations of $\{1, 2, \ldots, m\}$. For $\pi = (i_1 \ldots i_m) \in \mathbb{P}_m$, let $\hat{\pi}$ denote the corresponding bijection from $\{1, \ldots, m\}$ to itself: $\hat{\pi}(k) = i_k$ $(1 \leq k \leq m)$. In this way we have a one-to-one correspondence between $\mathbb{P}_m$ and the symmetric group $\mathbb{S}_m$, which is defined to be the set of bijections from $\{1, \ldots, m\}$ to itself. The group $\mathbb{S}_m$ acts on the set $\mathbb{P}_m$ by

$$\sigma\pi := (\sigma(i_1) \ldots \sigma(i_m)) \in \mathbb{P}_m$$

for $\sigma \in \mathbb{S}_m$ and $\pi = (i_1 \ldots i_m) \in \mathbb{P}_m$. The action of $\mathbb{S}_m$ on $\mathbb{R}^m$ is defined by

$$\sigma(x_1, \ldots, x_m) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}).$$

Then $\mathbb{S}_m$ acts on $M(\mathcal{A}_m)$ and $M(\mathcal{B}_m)$ and therefore on $\mathbf{Ch}(\mathcal{A}_m)$ and on $\mathbf{Ch}(\mathcal{B}_m)$.

It is well known (e.g., Bourbaki [2, Ch.5, §3, n°2, Th.1]) that the symmetric group $\mathbb{S}_m$ acts on $\mathbf{Ch}(\mathcal{B}_m)$ effectively and transitively. In other words, for any $C, C' \in \mathbf{Ch}(\mathcal{B}_m)$, there exists a unique $\sigma \in \mathbb{S}_m$ with $C' = \sigma C$. In particular, $|\mathbf{Ch}(\mathcal{B}_m)| = m!$. Let

$$C_0 := \{(x_1, x_2, \ldots, x_m) \mid x_1 < x_2 < \cdots < x_m\}$$

be a chamber of the braid arrangement $\mathcal{B}_m$. Then $\mathbf{Ch}(\mathcal{B}_m) = \{\sigma C_0 \mid \sigma \in \mathbb{S}_m\}$.

Fix $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in M(\mathcal{A}_m)$. Plot m points $x_1, x_2, \ldots, x_m$ on the real line $\mathbb{R}$. Let $R(\mathbf{x}) := \mathbb{R} \setminus \{x_{ij} \mid 1 \leq i < j \leq m\}$, where $x_{ij} := (x_i + x_j)/2$ is the midpoint. Define a map

$$\mathcal{R}_{\mathbf{x}} : R(\mathbf{x}) \longrightarrow \mathbb{P}_m$$

as follows:

$$\mathcal{R}_{\mathbf{x}}(y) = (i_1 i_2 \ldots i_m) \Longleftrightarrow |y - x_{i_1}| < |y - x_{i_2}| < \cdots < |y - x_{i_m}|,$$

where $y \in R(\mathbf{x})$ and $(i_1 i_2 \dots i_m) \in \mathbb{P}_m$. The map $\mathcal{R}_{\mathbf{x}}$ is called the *ranking map*. The image of the ranking map $\mathcal{R}_{\mathbf{x}}$ is the *ranking pattern* of $\mathbf{x} \in M(\mathcal{A}_m)$.

Suppose $\mathbf{x} \in C_0 \cap M(\mathcal{A}_m)$. Then $x_1 < x_2 < \cdots < x_m$. For $y \in R(\mathbf{x})$ and $1 \le i < j \le m$, we have

$$y < x_{ij} \iff |y - x_i| < |y - x_j| \iff i \text{ precedes } j \text{ in } \mathcal{R}_{\mathbf{x}}(y),$$

$$y > x_{ij} \iff |y - x_i| > |y - x_j| \iff j \text{ precedes } i \text{ in } \mathcal{R}_{\mathbf{x}}(y).$$

Imagine that the point y moves on the real line $\mathbb{R}$ from left to right. When y is sufficiently small, $\mathcal{R}_{\mathbf{x}}(y) = (12 \dots m)$. Every time y "passes" x_{ij}, the two integers i and j, which are adjacent in $\mathcal{R}_{\mathbf{x}}(y)$, switch their positions. When y is sufficiently large, $\mathcal{R}_{\mathbf{x}}(y) = (m \dots 21)$.

Example 2.1. *Let $m = 3$ and $x_1 < x_2 < x_3$. Then*

$$\mathcal{R}_{\mathbf{x}}(y) = \begin{cases} (123) \text{ if } y < x_{12}, \\ (213) \text{ if } x_{12} < y < x_{13}, \\ (231) \text{ if } x_{13} < y < x_{23}, \\ (321) \text{ if } x_{23} < y. \end{cases}$$

Lemma 2.2. *Let $\sigma \in \mathbb{S}_m$, $\mathbf{x} \in M(\mathcal{A}_m)$ and $y \in R(\mathbf{x})$. Then*

$$\mathcal{R}_{\sigma\mathbf{x}}(y) = \sigma(\mathcal{R}_{\mathbf{x}}(y)).$$

Proof. Suppose $\mathbf{x} = (x_1, \dots, x_m)$. Then

$$\begin{aligned} \mathcal{R}_{\sigma\mathbf{x}}(y) = (i_1 \dots i_m) &\iff |y - x_{\sigma^{-1}(i_1)}| < \cdots < |y - x_{\sigma^{-1}(i_m)}| \\ &\iff \mathcal{R}_{\mathbf{x}}(y) = (\sigma^{-1}(i_1) \dots \sigma^{-1}(i_m)) \iff \sigma(\mathcal{R}_{\mathbf{x}}(y)) = (i_1 \dots i_m). \end{aligned}$$

□

Lemma 2.3. *Let $\sigma \in \mathbb{S}_m$ and $\mathbf{x}, \mathbf{x}' \in \sigma C_0 \cap M(\mathcal{A}_m)$. Then $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$ if and only if the following statement holds true:*

$$x_{pq} > x_{rs} \iff x'_{pq} > x'_{rs}$$

for each $(p, q, r, s) \in I_4$.

Proof. Each chamber of $\mathcal{A}_m$ inside σC_0 is equal to the intersection of σC_0 and half-spaces defined by either $2(x_{pq} - x_{rs}) = x_p + x_q - x_r - x_s > 0$ or $2(x_{pq} - x_{rs}) = x_p + x_q - x_r - x_s < 0$ for $(p, q, r, s) \in I_4$. □

Theorem 2.4. *Let $\sigma \in \mathbb{S}_m$ and $\mathbf{x}, \mathbf{x}' \in \sigma C_0 \cap M(\mathcal{A}_m)$. Then $\mathbf{x}$ and $\mathbf{x}'$ have the same ranking pattern if and only if $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$.*

Proof. Assume first that $\sigma = 1$, so $\mathbf{x}, \mathbf{x}' \in C_0 \cap M(\mathcal{A}_m)$. Suppose that $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$. Write $\mathbf{x}' = (x'_1, x'_2, \dots, x'_m)$ and $x'_{ij} := (x'_i + x'_j)/2$ $(1 \le i < j \le m)$. By Lemma 2.3, we have

$$x_{i_1 j_1} < x_{i_2 j_2} < \cdots < x_{i_t j_t}, \quad x'_{i_1 j_1} < x'_{i_2 j_2} < \cdots < x'_{i_t j_t},$$

where $t = \binom{m}{2}$. This shows

$$\operatorname{im} \mathcal{R}_{\mathbf{x}} = \{\pi_0, \pi_1, \dots, \pi_t\} = \operatorname{im} \mathcal{R}_{\mathbf{x}'},$$

where $\pi_0, \pi_1, \ldots \pi_t \in \mathbb{P}_m$ are defined inductively by

$$\begin{aligned}\pi_0 &= (12\ldots m),\\ \pi_s &= [i_s j_s]\pi_{s-1} \ \ (1 \le s \le t).\end{aligned}$$

Here $[ij] \in \mathbb{S}_m$ $(1 \le i < j \le m)$ denotes the transposition of i and j.

Conversely, assume $\operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'}$. For $\pi = (i_1 i_2 \ldots i_m) \in \mathbb{P}_m$, let $\iota(\pi)$ denote the number of inversions in π:

$$\iota(\pi) := |\{(k_1, k_2) \mid k_1 < k_2, i_{k_1} > i_{k_2}\}|.$$

As the point y moves on the real line from left to right, $\iota(\mathcal{R}_{\mathbf{x}}(y))$ increases one by one. So we may write

$$\operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'} = \{\pi_0, \pi_1, \ldots, \pi_t\}$$

such that $\iota(\pi_s) = s$ $(0 \le s \le t)$. Also there exists a unique transposition $[i_s j_s]$ such that $\pi_s = [i_s j_s]\pi_{s-1}$ $(1 \le s \le t)$. Thus $x_{i_1 j_1} < x_{i_2 j_2} < \cdots < x_{i_t j_t}$ and $x'_{i_1 j_1} < x'_{i_2 j_2} < \cdots < x'_{i_t j_t}$. It follows from Lemma 2.3 that $\mathbf{x}$ and $\mathbf{x}'$ lie in the same chamber of $\mathcal{A}_m$.

For a general $\sigma \in \mathbb{S}_m$, let $\mathbf{y} := \sigma^{-1}\mathbf{x} \in C_0 \cap M(\mathcal{A}_m)$ and $\mathbf{y}' := \sigma^{-1}\mathbf{x}' \in C_0 \cap M(\mathcal{A}_m)$. By Lemma 2.2,

$$\begin{aligned}\operatorname{im}\mathcal{R}_{\mathbf{x}} = \operatorname{im}\mathcal{R}_{\mathbf{x}'} &\Leftrightarrow \sigma^{-1}(\operatorname{im}\mathcal{R}_{\mathbf{x}}) = \sigma^{-1}(\operatorname{im}\mathcal{R}_{\mathbf{x}'}) \Leftrightarrow \operatorname{im}\mathcal{R}_{\sigma^{-1}\mathbf{x}} = \operatorname{im}\mathcal{R}_{\sigma^{-1}\mathbf{x}'}\\ &\Leftrightarrow \operatorname{im}\mathcal{R}_{\mathbf{y}} = \operatorname{im}\mathcal{R}_{\mathbf{y}'} \Leftrightarrow \mathbf{y} \text{ and } \mathbf{y}' \text{ lie in the same chamber of } \mathcal{A}_m\\ &\Leftrightarrow \mathbf{x} \text{ and } \mathbf{x}' \text{ lie in the same chamber of } \mathcal{A}_m.\end{aligned}$$

□

Let $r(m)$ denote the number of ranking patterns when $\mathbf{x}$ runs over the set $C_0 \cap M(\mathcal{A}_m)$:

$$r(m) := |\{\operatorname{im}\mathcal{R}_{\mathbf{x}} \mid \mathbf{x} \in C_0 \cap M(\mathcal{A}_m)\}|.$$

Note that for each $\sigma \in \mathbb{S}_m$, $|\{\operatorname{im}\mathcal{R}_{\mathbf{x}} \mid \mathbf{x} \in \sigma C_0 \cap M(\mathcal{A}_m)\}|$ is equal to $r(m)$ by Lemma 2.2.

Theorem 2.5. $r(m) = |\mathbf{Ch}(\mathcal{A}_m)|/(m!)$.

Proof. By Theorem 2.4, $r(m)$ is equal to the number of chambers of $\mathcal{A}_m$ which lie inside C_0. Thus we have $|\mathbf{Ch}(\mathcal{A}_m)| = r(m)|\mathbb{S}_m| = r(m)(m!)$. □

Now recall some general results about the number of chambers and the characteristic polynomial [14]. Let $\mathbb{K}$ be a field and V an ℓ-dimensional vector space over $\mathbb{K}$. Assume that $\mathcal{A}$ is an arbitrary arrangement of hyperplanes in V. Let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. An element $X \in L$ is called an *edge* of $\mathcal{A}$. Define a *partial order* on L by $X \le Y \Longleftrightarrow Y \subseteq X$. Note that this is reverse inclusion. Thus V is the unique minimal element of L.

Let $\mu : L \to \mathbb{Z}$ be the Möbius function of L defined by $\mu(V) = 1$, and for $X > V$ by the recursion

$$\sum_{Y \le X} \mu(Y) = 0.$$

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(X) t^{\dim X}.$$

Theorem 2.6 (Zaslavsky [20]). *If* $\mathbb{K} = \mathbb{R}$, *then* $|\chi(\mathcal{A}, -1)| = |\mathbf{Ch}(\mathcal{A})|$.

We combine this result with Theorem 2.5:

Corollary 2.7. $r(m) = |\chi(\mathcal{A}_m, -1)|/(m!)$.

3 The characteristic polynomial of $\mathcal{A}_m$

It follows from Corollary 2.7 that the number of ranking patterns is determined by the characteristic polynomial $\chi(\mathcal{A}_m, t)$. Although this polynomial has a simple definition and it is not hard to write a computer program to determine it from the linear forms of the hyperplanes of $\mathcal{A}_m$, implementing the calculation is a different matter. In this section, we use arrangements over finite fields to simplify the calculation of $\chi(\mathcal{A}_m, t)$.

An arrangement $\mathcal{A}$ is called *essential* if the dimension of a maximal element of $L(\mathcal{A})$ is zero. The mid-hyperplane arrangement $\mathcal{A}_m$ is not essential because the line $l = \mathrm{span}\{\mathbf{1}\} = \{\lambda\mathbf{1} \mid \lambda \in \mathbb{R}\} \subset \mathbb{R}^m$, where $\mathbf{1} \in \mathbb{R}^m$ is the vector of 1's is a maximal element. This implies that $\chi(\mathcal{A}_m, t)$ is divisible by t. The fact that l is contained in every hyperplane of $\mathcal{A}_m$ implies that $\chi(\mathcal{A}_m, t)$ is also divisible by $(t-1)$. Thus $\chi(\mathcal{A}_m, t)/t(t-1)$ is a monic polynomial of degree $m-2$.

Let H_0 be the hyperplane defined by $x_1 = 0$. Define $\mathcal{A}_m^* := \mathcal{A}_m \cup \{H_0\}$. Then $\mathcal{A}_m^*$ is essential and the lattice $L(\mathcal{A}_m)$ is isomorphic to the sublattice defined by $L(\mathcal{A}_m^*)_{\geq H_0} := \{X \in L(\mathcal{A}_m^*) \mid X \geq H_0\}$.

The following theorem was essentially proved by Rota and Crapo in [6]. It is found in this form in [17] (4.10) and [14] (Theorem 2.69) and was effectively used by Athanasiadis (e.g., [1]).

Theorem 3.1. *Let $\mathbb{F}_q$ be a finite field of q elements. If $\mathbb{K} = \mathbb{F}_q$, then $\chi(\mathcal{A}, q) = |M(\mathcal{A})|$.*

When $\mathbb{K} = \mathbb{F}_q$ and V is a finite set of q^ℓ elements, $\chi(\mathcal{A}, q)$ can be evaluated by counting the number of points not on any hyperplane $H \in \mathcal{A}$ in V. Let q be a prime number greater than m. Let $\mathcal{A}_{m,q}^*$ be the modulo q reduction of $\mathcal{A}_m^*$ in $(\mathbb{Z}_q)^m$. In other words, the hyperplanes belonging to $\mathcal{A}_{m,q}^*$ are:

(0) $H_0 := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_1 = 0\}$,

(I_q) $H_{ij} := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_i = x_j\}$ $\quad (1 \leq i < j \leq m)$, and

(II_q) $H_{pqrs} := \{(x_1, \ldots, x_m) \in (\mathbb{Z}_q)^m \mid x_p + x_q = x_r + x_s\}$, $\quad (p, q, r, s) \in I_4$.

The arrangement $\mathcal{A}_{m,q}^*$ is essential. The modulo q reduction $\mathcal{A}_{m,q}$ of $\mathcal{A}_m$ is composed of the hyperplanes of type (I_q) and (II_q) above. Note that the lattice $L(\mathcal{A}_{m,q})$ is isomorphic to the sublattice defined by $L(\mathcal{A}_{m,q}^*)_{\geq H_0} := \{X \in L(\mathcal{A}_{m,q}^*) \mid X \geq H_0\}$. Therefore, the intersection lattices $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic if $L(\mathcal{A}_m^*)_{\geq H_0}$ and $L(\mathcal{A}_{m,q}^*)_{\geq H_0}$ are isomorphic.

Let C be the coefficient matrix of $\mathcal{A}_m^*$. For example when $m = 4$,

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & -1 & -1 \end{pmatrix}.$$

Consider the m-minors of C. Each m-minor is parametrized by the set of m columns used for the minor. It is known that the intersection lattice of an essential arrangement is completely determined by the information which m-minors vanish and which do not [18, Proposition 3]. Thus we have

Theorem 3.2. *Define $f(m) := \max\{|\det T| \mid T$ is an m-minor of C and T contains the column $(1, 0, \ldots, 0)^T$ as its first column$\}$. Let q be a prime number greater than $f(m)$. Then $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic.*

Next we will find an upper bound for $f(m)$. Let $m \geq 3$ as always. Consider the following three conditions concerning a matrix:

(i) every entry of the matrix is either -1, 0 or 1,

(ii) in every column 1 appears at most twice,

(iii) in every column -1 appears at most twice.

Define

$$g(m) := \max\{|\det A| \mid A \text{ is an } (m-1)\times(m-1)\text{-matrix satisfying (i,ii,iii)}\}.$$

It is clear that $f(m) \le g(m)$.

Lemma 3.3.

$$g(m) = 2^{m-2} \text{ for } m \le 5, \quad g(m) \le 8\cdot 3^{m-5} \text{ for } m \ge 6.$$

Proof. We argue by induction on m. For $m = 3, 4$ direct computation shows the result. The values are attained by

$$g(3) = 2 = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad g(4) = 4 = \det\begin{pmatrix} -1 & 1 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Suppose $m = 5$. We show that $g(5) = 8$. Note that

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix} = 8.$$

Thus $g(5) \ge 8$. We must show $g(5) = \det A \le 8$. Let $\mathbf{a}_i$ be the ith column of A. Denote the number of nonzero elements in $\mathbf{a}_i$ by $val(\mathbf{a}_i)$. If A has a column $\mathbf{a}_i$ with $val(\mathbf{a}_i) \le 2$, then $g(5) \le 2g(4) = 8$. So we may assume that $3 \le val(\mathbf{a}_i) \le 4$ for every i. Define $\mathbf{a}_i^*$ to be the uniquely determined four-dimensional column vector with two 1's and two -1's which satisfies the following property:

$$\begin{aligned} \text{if } val(\mathbf{a}_i) &= 4, \text{ then } \mathbf{a}_i^* = \mathbf{a}_i, \\ \text{if } val(\mathbf{a}_i) &= 3, \text{ then } \mathbf{a}_i^* \text{ is obtained from } \mathbf{a}_i \text{ by replacing} \\ &\qquad \text{the unique zero in } \mathbf{a}_i \text{ by either 1 or } -1. \end{aligned}$$

For example,

$$\text{if } \mathbf{a}_i = \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} \text{ then } \mathbf{a}_i^* = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Since $\binom{4}{2}/2 = 3 < 4$, among the four vectors $\mathbf{a}_i^*$ $(1 \le i \le 4)$ at least two are either equal to or the negative of each other. Without loss of generality, we may assume $\mathbf{a}_1^* = \mathbf{a}_2^*$. Then $\mathbf{a}_{12} := \mathbf{a}_1 - \mathbf{a}_2$ is composed only of $0, -1$ and 1 with $val(\mathbf{a}_{12}) = 2$. We get

$$\begin{aligned} g(5) = \det A = \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) &= \det(\mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \\ &= \det(\mathbf{a}_{12}, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4) \le val(\mathbf{a}_{12}) g(4) = 2g(4) = 8. \end{aligned}$$

For the induction step, assume $m \ge 6$. Choose an $(m-1)\times(m-1)$-matrix A satisfying (i), (ii) and (iii) with $\det A = g(m)$. Use the Laplace expansion formula along $\mathbf{a}_i$ to get

$$g(m) \le val(\mathbf{a}_i) g(m-1)$$

for each i. If $val(\mathbf{a}_i) = 4$ for every column $\mathbf{a}_i$ of A, then all the rows sum up to the zero vector and thus $\det A = 0$. This is a contradiction. Therefore we may assume that there exists a column $\mathbf{a}_i$ with $val(\mathbf{a}_i) \le 3$; so we obtain $g(m) \le 3g(m-1)$. ∎

Theorem 3.2 and Lemma 3.3 imply

Theorem 3.4. *If a prime number q satisfies*

$$q > \begin{cases} 2^{m-2} & \text{if } m \leq 5, \\ 8 \cdot 3^{m-5} & \text{if } m \geq 6, \end{cases}$$

then the intersection lattices $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic and

$$\chi(\mathcal{A}_m, q) = |M(\mathcal{A}_{m,q})|.$$

The following theorem shows that we can fix $x_1 = 0$, $x_2 = 1$ in counting $|M(\mathcal{A}_{m,q})|$.

Theorem 3.5. *Define*

$$M_1(m, q) := \{(0, 1, x_3, \ldots, x_m) \in M(\mathcal{A}_{m,q})\}.$$

Under the assumption of Theorem 3.4, we have

$$\frac{\chi(\mathcal{A}_m, q)}{q(q-1)} = |M_1(m, q)|.$$

Proof. Consider the action of the additive group $\mathbb{F}_q$ on $M(\mathcal{A}_{m,q})$ by

$$(x_1, x_2, \ldots, x_m) \mapsto (x_1 + \alpha, x_2 + \alpha, \ldots, x_m + \alpha) \ \ (\alpha \in \mathbb{F}_q).$$

The set of orbits under this action is represented by the set

$$M_0 := \{(0, x_2, x_3, \ldots, x_m) \in M(\mathcal{A}_{m,q})\}.$$

Thus $|M_0| = \chi(\mathcal{A}_m, q)/q$. Next consider the action of the multiplicative group $\mathbb{F}_q^\times := \mathbb{F} \setminus \{0\}$ on M_0 by

$$(0, x_2, \ldots, x_m) \mapsto (0, x_2\beta, \ldots, x_m\beta) \ \ (\beta \in \mathbb{F}_q^\times).$$

The set of orbits under this action is represented by the set $M_1(m, q)$. Thus $|M_1(m, q)| = |M_0|/(q-1) = \chi(\mathcal{A}_m, q)/q(q-1)$. □

We calculate $\chi(\mathcal{A}_m, t)$ as follows. Let q_i $(i = 1, \ldots, m-2)$ be primes satisfying the conditions of Theorem 3.4. Count the number of points in the set $M_1(m, q_i)$ for each i. By Theorem 3.5, we have $\chi(\mathcal{A}_m, q_i)/q_i(q_i - 1) = |M_1(m, q_i)|$. Since $\chi(\mathcal{A}_m, t)/t(t-1)$ is a monic polynomial of degree $m-2$, the values $|M_1(m, q_i)|$ $(i = 1, \ldots, m-2)$ determine the characteristic polynomial $\chi(\mathcal{A}_m, t)$.

4 The number of ranking patterns for $m \leq 7$

We use the results of the last section to determine $\chi(\mathcal{A}_m, t)$, $|\mathbf{Ch}(\mathcal{A}_m)|$ and $r(m)$ for $m \leq 7$. The case $m = 3$ is known because $\mathcal{A}_3 = \mathcal{B}_3$. Let $m = 4$. If $q > 4$ is a prime, then Theorem 3.5 gives

$$\frac{\chi(\mathcal{A}_4, q)}{q(q-1)} = |M_1(4, q)|.$$

Let

$$p(t) := \frac{\chi(\mathcal{A}_4, t)}{t(t-1)}.$$

Then $p(t)$ is a monic quadratic polynomial. We let $q_1 = 5$, $q_2 = 7$ and find

$$p(5) = |M_1(4,5)| = 0 \text{ and } p(7) = |M_1(4,7)| = 8.$$

Theorem 2.6 and Corollary 2.7 give

$$p(t) = t^2 - 8t + 15 = (t-3)(t-5),\ \chi(\mathcal{A}_4, t) = t(t-1)(t-3)(t-5),$$
$$|\mathbf{Ch}(\mathcal{A}_4)| = 48, \text{ and } r(4) = 2.$$

Using the same method, computer calculations provide the following table:

Theorem 4.1.

m	$\chi(\mathcal{A}_m, t)$	$\lvert\mathbf{Ch}(\mathcal{A}_m)\rvert$	$r(m)$
3	$t(t-1)(t-2)$	6	1
4	$t(t-1)(t-3)(t-5)$	48	2
5	$t(t-1)(t-7)(t-8)(t-9)$	1440	12
6	$t(t-1)(t-13)(t-14)(t-15)(t-17)$	120960	168
7	$t(t-1)(t-23)(t-24)(t-25)(t-26)(t-27)$	23587200	4680

Corollary 4.2. *If $m \leq 7$, then the characteristic polynomial $\chi(\mathcal{A}_m, t)$ is a product of linear factors in $\mathbb{Z}[t]$.*

Remark. Define $a_n = n(n^{n-1}-1)((n-2)!)/(n-1)$. We note that $r(m) = a_{m-2}$ for $m = 3, 4, 5, 6, 7$ but we do not have any reasonable interpretation for the coincidence at this writing.

5 The number of ranking patterns for $m \geq 8$

In this section we determine $r(8)$ and prove a theorem about the characteristic polynomial $\chi(\mathcal{A}_m, t)$ for $m \geq 8$.

For $m = 8$ we used a computer to count $|M_1(8, q)|$ with the primes $q = 223$, 227, 229, 233, 239, 241, all greater than $8 \cdot 3^{8-5} = 216$. Theorem 3.5 implies:

Theorem 5.1.

$$\chi(\mathcal{A}_8, t) = t(t-1)(t-35)(t-37)(t-39)(t-41)(t^2 - 85t + 1926),$$
$$|\mathbf{Ch}(\mathcal{A}_8)| = 9248117760,$$
$$r(8) = 229386.$$

Remark. The coincidence of $r(m)$ and a_{m-2} does not hold for $m = 8$. Here $r(8) = 229386 > a_6 = 223920$.

Evaluating $r(m)$ for $m \geq 9$ is not feasible at present with our brute-force counting method. Improving the bound in Lemma 3.3 might reduce the computational time. Computer experiments indicate that $L(\mathcal{A}_m)$ and $L(\mathcal{A}_{m,q})$ are isomorphic for much smaller q than the value guaranteed by Lemma 3.3. It may be interesting to investigate the growth of $r(m)$ even if its values remain unknown.

Next we consider the factorization problem. Write

$$\chi(\mathcal{A}_m, t) = \sum_{k=0}^{m} \mu_k t^{m-k}.$$

It is known that

$$\mu_0 = 1,\ \mu_1 = -|\mathcal{A}_m| = -\binom{m}{2} - 3\binom{m}{4},\ \mu_m = 0.$$

Although we do not have a general formula for μ_k, routine calculations yield a formula for μ_2:

Theorem 5.2.

$$\mu_2 = 2\binom{m}{3} + 15\binom{m}{4} + 120\binom{m}{5} + 375\binom{m}{6} + 630\binom{m}{7} + 315\binom{m}{8}.$$

Theorem 5.3. *The characteristic polynomial $\chi(\mathcal{A}_m, t)$ is a product of linear factors in $\mathbb{Z}[t]$ if and only if $m \le 7$.*

Proof. This follows from Corollary 4.2 when $m \le 7$. Let $m \ge 8$. Suppose that the characteristic polynomial is a product of linear factors in $\mathbb{Z}[t]$:

$$\chi(\mathcal{A}_m, t) = \sum_{k=0}^{m} \mu_k t^{m-k} = t(t-1)(t-b_2)\dots(t-b_{m-1})$$

for $b_2, \dots, b_{m-1} \in \mathbb{Z}$.

Applying Theorem 5.2, we have

$$\sum_{i=2}^{m-1} b_i = -\mu_1 - 1 = |\mathcal{A}_m| - 1 = -1 + \binom{m}{2} + 3\binom{m}{4},$$

$$\begin{aligned}\sum_{2\le i<j\le m-1} b_i b_j &= \mu_2 - \sum_{i=2}^{m-1} b_i \\ &= 1 - \binom{m}{2} + 2\binom{m}{3} + 12\binom{m}{4} \\ &\quad + 120\binom{m}{5} + 375\binom{m}{6} + 630\binom{m}{7} + 315\binom{m}{8}.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=2}^{m-1}\left(b_i - \frac{\sum_{i=2}^{m-1} b_i}{m-2}\right)^2 &= \sum_{i=2}^{m-1} b_i^2 - \frac{\left(\sum_{i=2}^{m-1} b_i\right)^2}{m-2} \\ &= \left(\sum_{i=2}^{m-1} b_i\right)^2 - 2\sum_{2\le i<j\le m-1} b_i b_j - \frac{\left(\sum_{i=2}^{m-1} b_i\right)^2}{m-2} \\ &= \frac{(m-3)\left(\sum_{i=2}^{m-1} b_i\right)^2}{m-2} - 2\sum_{2\le i<j\le m-1} b_i b_j.\end{aligned}$$

Compute

$$\begin{aligned}h(m) &:= (m-2)\sum_{i=2}^{m-1}\left(b_i - \frac{\sum_{i=2}^{m-1} b_i}{m-2}\right)^2 \\ &= (m-3)\left\{-1 + \binom{m}{2} + 3\binom{m}{4}\right\}^2 - 2(m-2)\left\{1 - \binom{m}{2} + 2\binom{m}{3}\right. \\ &\quad \left. +12\binom{m}{4} + 120\binom{m}{5} + 375\binom{m}{6} + 630\binom{m}{7} + 315\binom{m}{8}\right\} \\ &= 1 + \frac{98m}{3} - \frac{1573m^2}{16} + \frac{5423m^3}{48} - \frac{12787m^4}{192} + \frac{527m^5}{24} - \frac{391m^6}{96} \\ &\quad + \frac{19m^7}{48} - \frac{m^8}{64}.\end{aligned}$$

Thus $h(m) \ge 0$ for $m > 2$. On the other hand, we may check by standard calculus techniques that $h(m) < 0$ whenever $m \ge 8$. This is a contradiction. □

6 Probabilities of ranking patterns

We counted the number of possible ranking patterns in the preceding sections. Here we investigate the probabilities of ranking patterns when the objects $x_1, \dots, x_m$ are randomly determined. For $m = 4$, the problem is trivial by symmetry considerations as long as the four objects are independently and identically distributed.

We consider the case $m = 5$ and assume that $\mathbf{x} = (x_1, \dots, x_5) \in \mathbb{R}^5$ is distributed according to an arbitrary spherical distribution. Note that $\mathbf{x} \in M(\mathcal{A}_5)$ with probability one. For $m = 5$, there are 1440 possible ranking patterns in all. By relabelling the indices it suffices to consider the case

$$x_1 < \cdots < x_5. \tag{1}$$

Furthermore, by replacing x_i by $-x_i$, it suffices to consider the case

$$x_1 < \cdots < x_5, \quad x_{24} < x_{15}. \tag{2}$$

Under restriction (2), we have $1440/(5! \cdot 2) = r(5)/2 = 6$ possible ranking patterns, which are characterized by the following midpoint orders (Lemma 2.3, Theorem 2.4):

$$\begin{array}{cl}
\text{(I)} & x_{14} < x_{23} < x_{24} < x_{15} < x_{25} < x_{34}, \\
\text{(II)} & x_{14} < x_{23} < x_{24} < x_{15} < x_{34} < x_{25}, \\
\text{(III)} & x_{14} < x_{23} < x_{24} < x_{34} < x_{15} < x_{25}, \\
\text{(IV)} & x_{23} < x_{14} < x_{24} < x_{15} < x_{25} < x_{34}, \\
\text{(V)} & x_{23} < x_{14} < x_{24} < x_{15} < x_{34} < x_{25}, \\
\text{(VI)} & x_{23} < x_{14} < x_{24} < x_{34} < x_{15} < x_{25}.
\end{array}$$

We are interested in the conditional probabilities of the six midpoint orders above assuming (2). Recall that these midpoint orders represent chambers of $\mathcal{A}_5$ (Lemma 2.3). We argue next that our problem reduces to computing the spherical volumes of the restrictions of some chambers of $\mathcal{A}_5$ to the three-dimensional unit sphere.

We begin by recalling that all hyperplanes in $\mathcal{A}_5$ contain the line $l = \operatorname{span}\{\mathbf{1}\} = \{\lambda \mathbf{1} \mid \lambda \in \mathbb{R}\} \subset \mathbb{R}^5$, where $\mathbf{1} \in \mathbb{R}^5$ is the vector of 1's. The orthogonal projection of $\mathbf{x} = (x_1, \dots, x_5) \in \mathbb{R}^5$ onto $H_0' = l^{\perp} = \{(x_1, \dots, x_5) \in \mathbb{R}^5 \mid x_1 + \cdots + x_5 = 0\}$ will be denoted by $\mathbf{z} := (x_1 - \bar{x}, \dots, x_5 - \bar{x})$, where $\bar{x} = (x_1 + \cdots + x_5)/5$. Since $\mathbf{x}$ is assumed to be distributed as a spherical distribution, the marginal distribution of the orthogonal projection $\mathbf{z}$ is a spherical distribution of one less dimension (Muirhead [12, p.34]). Now, any $\mathbf{x} \in M(\mathcal{A}_5)$ and its orthogonal projection $\mathbf{z}$ are on the same side of each hyperplane in $\mathcal{A}_5$, so for any chamber $C \in \mathbf{Ch}(\mathcal{A}_5)$, we have $\operatorname{Prob}(\mathbf{x} \in C) = \operatorname{Prob}(\mathbf{z} \in C_{H_0'})$ with $C_{H_0'} := C \cap H_0'$. This $C_{H_0'}$ can be regarded as a chamber of the arrangement $\mathcal{A}_5' := \{H \cap H_0' \mid H \in \mathcal{A}_5\}$ in H_0'.

Each hyperplane in $\mathcal{A}_5'$ contains the origin. Thus its chambers are the interiors of polyhedral cones in H_0'. As a result, for each $C_{H_0'} \in \mathbf{Ch}(\mathcal{A}_5')$, we have that $\mathbf{z} \in C_{H_0'}$ is equivalent to $\mathbf{z}/\|\mathbf{z}\| \in C_{\mathbb{S}^3} := C_{H_0'} \cap \mathbb{S}^3$, where $\mathbb{S}^3 := \{(x_1, \dots, x_5) \in H_0' \mid x_1^2 + \cdots + x_5^2 = 1\}$ is the unit sphere in H_0'. Together with the uniformity of the distribution of $\mathbf{z}/\|\mathbf{z}\|$ on $\mathbb{S}^3$, this yields $\operatorname{Prob}(\mathbf{z} \in C_{H_0'}) = \operatorname{Prob}(\mathbf{z}/\|\mathbf{z}\| \in C_{\mathbb{S}^3}) = \operatorname{Vol}(C_{\mathbb{S}^3})/\operatorname{Vol}(\mathbb{S}^3)$.

We conclude that for any chamber C of $\mathcal{A}_5$,

$$\operatorname{Prob}(\mathbf{x} \in C) = \frac{\operatorname{Vol}(C_{\mathbb{S}^3})}{\operatorname{Vol}(\mathbb{S}^3)}$$

with $C_{\mathbb{S}^3} = C \cap \mathbb{S}^3$. Thus the probability of $\mathbf{x}$ being in chamber $C \in \mathbf{Ch}(\mathcal{A}_5)$ is proportional to the volume of $C_{\mathbb{S}^3} = C \cap \mathbb{S}^3$. Therefore, the desired conditional probabilities under (2) are given by the ratios of the volumes of the chambers $C_{\mathbb{S}^3}$ corresponding to the six midpoint orders to the volume of the union $T := \{(x_1, \dots, x_5) \mid x_1 \leq \cdots \leq x_5,\ x_{24} \leq x_{15}\} \cap \mathbb{S}^3$ of their closures.

The binding inequalities of the spherical chambers associated with the six midpoint orders are

$$\begin{array}{cl} \text{(I)} & x_{14} < x_{23},\ x_{25} < x_{34},\ x_3 < x_4,\ x_{24} < x_{15}, \\ \text{(II)} & x_{15} < x_{34},\ x_{14} < x_{23},\ x_{24} < x_{15},\ x_3 < x_4,\ x_{34} < x_{25}, \\ \text{(III)} & x_{14} < x_{23},\ x_2 < x_3,\ x_3 < x_4,\ x_{34} < x_{15}, \\ \text{(IV)} & x_1 < x_2,\ x_{25} < x_{34},\ x_{23} < x_{14},\ x_{24} < x_{15}, \\ \text{(V)} & x_{15} < x_{34},\ x_{23} < x_{14},\ x_{24} < x_{15},\ x_{34} < x_{25}, \\ \text{(VI)} & x_1 < x_2,\ x_2 < x_3,\ x_{23} < x_{14},\ x_{34} < x_{15}. \end{array}$$

With the exception of (II), the closures of these chambers are spherical tetrahedra

$$\begin{array}{cl} \text{(I)} & FBGH, \\ \text{(III)} & AFED, \\ \text{(IV)} & FBGC, \\ \text{(V)} & CGFE, \\ \text{(VI)} & AFCE \end{array}$$

where

$$\begin{aligned} &A = (-1,-1,-1,-1,4)/\sqrt{20}, \quad B = (-3,-3,2,2,2)/\sqrt{30}, \\ &C = (-2,-2,-2,3,3)/\sqrt{30}, \quad D = (-1,0,0,0,1)/\sqrt{2}, \\ &E = (-7,-2,-2,3,8)/\sqrt{130}, \quad F = (-4,-4,1,1,6)/\sqrt{70}, \\ &G = (-2,-1,0,1,2)/\sqrt{10}, \quad H = (-8,-3,2,2,7)/\sqrt{130}; \end{aligned}$$

Chamber (II) is a quadrilateral pyramid $FEDHG$, which can be divided into two tetrahedra, say, $FEDG$ and $FDGH$. Note that this observation implies that the closures of the chambers of the mid-hyperplane arrangement $\mathcal{A}_m$ are not necessarily simplices. See Figures 1 and 2.

The volumes of the seven spherical tetrahedra mentioned above can be computed as

$$\begin{array}{cl} \text{(I)} & \mathrm{Vol}(FBGH) = 0.00628091, \\ \text{(II)} & \mathrm{Vol}(FEDG) = 0.00486715, \quad \mathrm{Vol}(FDGH) = 0.00481365, \\ \text{(III)} & \mathrm{Vol}(AFED) = 0.0189182, \\ \text{(IV)} & \mathrm{Vol}(FBGC) = 0.0146084, \\ \text{(V)} & \mathrm{Vol}(CGFE) = 0.00650684, \\ \text{(VI)} & \mathrm{Vol}(AFCE) = 0.0262516. \end{array}$$

These values can be obtained by using Schläfli's [15] result concerning partial derivatives of the volume of a spherical tetrahedron with respect to its dihedral angles. Note that these values add up to the volume of the spherical tetrahedron $T = ABCD = \{(x_1, \dots, x_5) \in \mathbb{S}^3 \mid x_1 \leq \cdots \leq x_5,\ x_{24} \leq x_{15}\}$:

$$\mathrm{Vol}(T) = \frac{\mathrm{Vol}(\mathbb{S}^3)}{5! \cdot 2} = \frac{2\pi^2}{5! \cdot 2} = 0.0822467.$$

Let $S = \{(x_1, \dots, x_5) \in \mathbb{S}^3 \mid x_1 \le \cdots \le x_5\}$: $\mathrm{Vol}(S) = 2\mathrm{Vol}(T)$. We use the values above to arrive at

$$\begin{aligned}
\mathrm{Prob}((\mathrm{I}) \mid S) &:= \mathrm{Prob}(x_{14} < x_{23} < x_{24} < x_{15} < x_{25} < x_{34} \mid \\
&\qquad\qquad x_1 < \cdots < x_5) \\
&= \frac{\mathrm{Vol}(FBGH)}{\mathrm{Vol}(S)} \\
&= \frac{0.00628091}{2 \times 0.0822467} = 0.0381834.
\end{aligned}$$

By replacing x_i by $-x_i$, we also consider the following cases:

$$\begin{array}{cl}
(\mathrm{I}') & x_{23} < x_{14} < x_{15} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{II}') & x_{14} < x_{23} < x_{15} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{III}') & x_{14} < x_{15} < x_{23} < x_{24} < x_{34} < x_{25}, \\
(\mathrm{IV}') & x_{23} < x_{14} < x_{15} < x_{24} < x_{25} < x_{34}, \\
(\mathrm{V}') & x_{14} < x_{23} < x_{15} < x_{24} < x_{25} < x_{34}, \\
(\mathrm{VI}') & x_{14} < x_{15} < x_{23} < x_{24} < x_{25} < x_{34}.
\end{array}$$

Using the symmetry we get

$$\begin{aligned}
\mathrm{Prob}((\mathrm{I}) \mid S) = \mathrm{Prob}((\mathrm{I}') \mid S) &= 0.0381834, \\
\mathrm{Prob}((\mathrm{II}) \mid S) = \mathrm{Prob}((\mathrm{II}') \mid S) &= 0.0588522, \\
\mathrm{Prob}((\mathrm{III}) \mid S) = \mathrm{Prob}((\mathrm{III}') \mid S) &= 0.1150086, \\
\mathrm{Prob}((\mathrm{IV}) \mid S) = \mathrm{Prob}((\mathrm{IV}') \mid S) &= 0.0888085, \\
\mathrm{Prob}((\mathrm{V}) \mid S) = \mathrm{Prob}((\mathrm{V}') \mid S) &= 0.0395569, \\
\mathrm{Prob}((\mathrm{VI}) \mid S) = \mathrm{Prob}((\mathrm{VI}') \mid S) &= 0.1595905.
\end{aligned}$$

We have confirmed that these values coincide with the result of our simulation study with $\mathbf{x} \sim N_5(\mathbf{0}, I_5)$, where I_5 denotes the 5×5-identity matrix.

References

[1] Ch. Athanasiadis, Generalized Catalan numbers, Weyl groups and arrangements of hyperplanes, *Bull. London Math. Soc.* **36** (2004), 294–302.

[2] N. Bourbaki, "Groupes et Algèbres de Lie," Chapitres 4, 5 et 6, Hermann, Paris, 1968.

[3] C. H. Coombs, Psychological scaling without a unit of measurement, *Psychol. Rev.* **57** (1950), 145–158.

[4] C. H. Coombs, A theory of psychological scaling, *Engng. Res. Inst. Bull.* No.34, University of Michigan Press, Ann Arbor, 1952.

[5] C. H. Coombs, "A Theory of Data," John Wiley & Sons, New York, 1964.

[6] H. Crapo and G.-C. Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries," preliminary edition, MIT Press, Cambridge, MA, 1970.

[7] B. C. Eaton and R. G. Lipsey, The principle of minimum differentiation reconsidered: Some new developments in the theory of spatial competition, *Rev. Econ. Stud.* **42** (1975), 27–49.

[8] B. C. Eaton and R. G. Lipsey, The non-uniqueness of equilibrium in the Löschan location model, *Amer. Econ. Rev.* **66** (1976), 77–93.

[9] H. Hotelling, Stability in competition, *Economic Journal* **39** (1929), 41–57.

[10] H. Kamiya and A. Takemura, On rankings generated by pairwise linear discriminant analysis of m populations, *J. Multivariate Anal.* **61** (1997), 1–28.

[11] J. I. Marden, "Analyzing and Modeling Rank Data," Chapman & Hall, London, 1995.

[12] R. J. Muirhead, "Aspects of Multivariate Statistical Theory," John Wiley & Sons, New York, 1982.

[13] A. Okabe, B. Boots, K. Sugihara and S. N. Chiu, "Spatial Tessellations: Concepts and Applications of Voronoi Diagrams," 2nd ed., John Wiley & Sons, Chichester, 2000.

[14] P. Orlik and H. Terao, "Arrangements of Hyperplanes," Grundlehren der math. Wiss. **300**, Springer-Verlag, Berlin, 1992.

[15] L. Schläfli, On the multiple integral $\int^n dxdy \cdots dz$, whose limits are $p_1 = a_1x + b_1y + \cdots + h_1z > 0,\ p_2 > 0, \ldots, p_n > 0$, and $x^2 + y^2 + \cdots + z^2 < 1$, *Quart. J. Math.* **2** (1858), 269–300. Continued in **3** (1860), 54–68, 97–108.

[16] R. Stanley, Ordering events in Minkowski space, preprint; math.CO/0501256.

[17] H. Terao, The Jacobians and the discriminants of finite reflection groups, *Tohoku Math. J.* **41** (1989), 237-247.

[18] H. Terao, Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss-Manin connections, *Topology and Its Appl.* **118** (2002), 255–274.

[19] R. M. Thrall, A combinatorial problem, *Michigan Math. J.* **1** (1952), 81–88.

[20] T. Zaslavsky, Facing up to arrangements: Face-count formulas for partitions of space by hyperplanes, *Mem. Amer. Math. Soc.* vol. 1, no. 154 (1975).

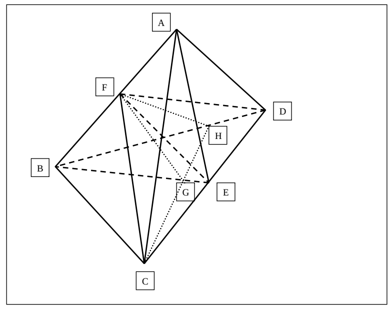

Figure 1

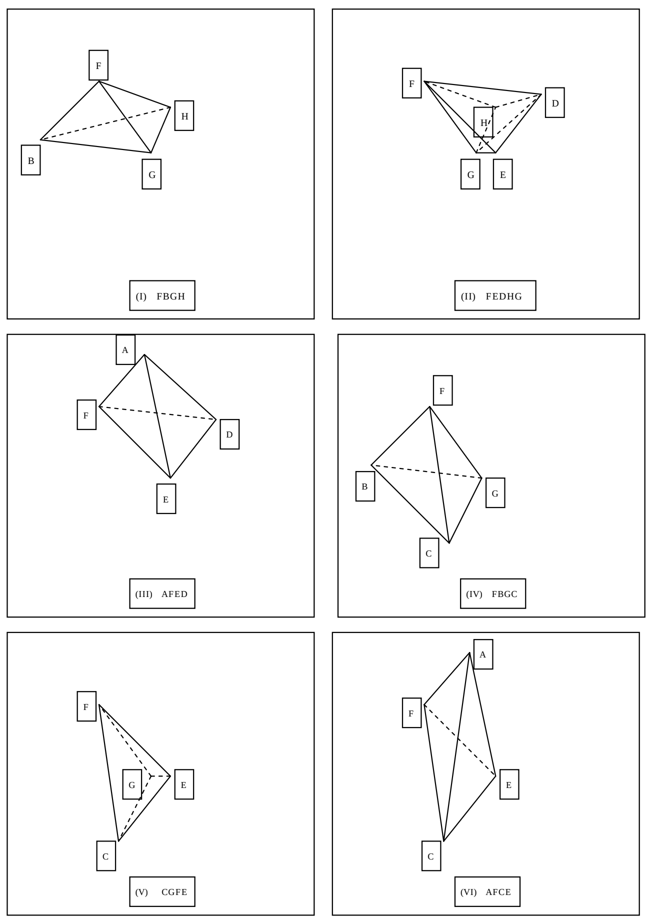

Figure 2