Moduli space of combinatorially equivalent arrangements of hyperplanes and logarithmic Gauss-Manin connections

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Abstract

We consider a moduli space of combinatorially equivalent family of arrangements of hyperplanes (e.g., n distinct points in the complex line). A compactification of the moduli space is obtained by adding a boundary divisor. On the moduli space we study a Gauss-Manin connection and show that it has logarithmic poles along the boudary divisor. When the moduli space is one-codimensional, an explicit formula for the connection matrix is given.

Key words: Arrangements of hyperplanes; Gauss-Manin connections; local system cohomology; combinatorial equivalence; β**nbc** basis

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To the memory of Professor Nobuo Sasakura

1 Introduction

Fix a pair (ℓ, n) with $\ell \geq 1$ and $n \geq 0$. Let $\mathcal{A}_n(\mathbb{C}^\ell)$ be the set of affine arrangements of n distinct linearly ordered hyperplanes in $\mathbb{C}^\ell$. In other words, each element $\mathcal{A}$ of $\mathcal{A}_n(\mathbb{C}^\ell)$ is a collection $\{H_1, \ldots, H_n\}$ where $H_1, \ldots, H_n$ are distinct affine hyperplanes in $\mathbb{C}^\ell$. Two arrangements $\mathcal{A}^{(i)} := \{H_1^{(i)}, \ldots, H_n^{(i)}\} \in \mathcal{A}_n(\mathbb{C}^\ell) \quad (i = 1, 2)$ are said to be **combinatorially equivalent**, denoted by $\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$, if

$$\dim H_{i_1}^{(1)} \cap \cdots \cap H_{i_p}^{(1)} = \dim H_{i_1}^{(2)} \cap \cdots \cap H_{i_p}^{(2)}$$

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for each $(i_1, \dots, i_p), 1 \le i_1 < \dots i_p \le n$. Here we agree that the dimension of the empty set is equal to -1.

Fix $\mathcal{A} := \{H_1, \dots, H_n\} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Let

$$\mathsf{B}_{\mathcal{A}} = \mathsf{B} := \{\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell) \mid \mathcal{B} \sim \mathcal{A}\}.$$

In Section 2, we naturally identify B with a locally closed subset of $((\mathbb{CP}^\ell)^*)^n$, where $(\mathbb{CP}^\ell)^*$ is the ℓ-dimensional dual complex projective space. Let $\overline{\mathsf{B}}$ be the closure of B in $((\mathbb{CP}^\ell)^*)^n$. Then, as we will see in Proposition 4, the boundary $\mathsf{D} := \overline{\mathsf{B}} \setminus \mathsf{B}$ is defined by a single equation on $\overline{\mathsf{B}}$. The hypersurface D, in general, has several irreducible components. When $\mathcal{A}$ is of general position (i.e., $\dim H_{i_1} \cap \dots \cap H_{i_p} = \ell - p$ if $1 \le i_1 < \dots < i_p \le n$ and $1 \le p \le \ell + 1$), B is dense in $((\mathbb{CP}^\ell)^*)^n$: $\overline{\mathsf{B}} = ((\mathbb{CP}^\ell)^*)^n$. In this case D has $\binom{n+1}{\ell+1}$ irreducible components. We can also describe the geometry of D when the codimension of B in $((\mathbb{CP}^\ell)^*)^n$ is equal to one. In this case, D has $\binom{n+1}{\ell+1} - \ell(n - \ell - 1)$ irreducible components (if $\ell \ge 2$) or $n(n-1)/2$ irreducible components (if $\ell = 1$). The study of D is naturally related to the theory of determinantal ideals.

We assume that $\mathcal{A}$ is **essential**, i.e., there exist ℓ hyperplanes in $\mathcal{A}$ whose intersection is a point. In particular, $n = |\mathcal{A}| \ge \ell$. We have a topological fibration

$$\pi : \mathsf{M} \longrightarrow \mathsf{B}$$

such that $\pi^{-1}(\mathbf{t}) = \mathsf{M}_{\mathbf{t}} = M(\mathcal{A}_{\mathbf{t}}) := \mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}_{\mathbf{t}}} H$ for each $\mathbf{t} \in \mathsf{B}$. Here $\mathcal{A}_{\mathbf{t}} \in \mathcal{A}_n(\mathbb{C}^\ell)$ is the arrangement corresponding to $\mathbf{t} \in \mathsf{B}$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. We call λ a **weight**. Then λ defines a rank-one local system $\mathcal{L}_\lambda$ on $\mathsf{M}_{\mathbf{t}}$ so that $\mathcal{L}_\lambda$ has monodromy $\exp(-2\pi\sqrt{-1}\lambda_i)$ around $H_i \in \mathcal{A}_{\mathbf{t}}$ $\quad (1 \le i \le n)$. Let $\mathcal{L}_\lambda^\vee$ be the dual local system which has monodromy $\exp(2\pi\sqrt{-1}\lambda_i)$ around H_i. Then, for a sufficiently generic λ, there exists a local system $\mathcal{H}_\ell \to \mathsf{B}$ of rank β whose fiber at $\mathbf{t} \in \mathsf{B}$ is equal to the local system homology $H_\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_\lambda^\vee)$ because π is locally trivial. Here β is the absolute value of the Euler characteristic of $\mathsf{M}_{\mathbf{t}}$, which is independent of choice of $\mathbf{t} \in \mathsf{B}$. We can also define, in the dual manner, a local system $\mathcal{H}^\ell \to \mathsf{B}$ of rank β whose fiber at $\mathbf{t} \in \mathsf{B}$ is equal to the local system cohomology $H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_\lambda)$ so that $\mathcal{H}^\ell$ is a globally trivial local system with the β**nbc** global frame $\Xi_1, \dots, \Xi_\beta \in \Gamma(\mathsf{B}, \mathcal{H}^\ell)$ [FT, 3.9]. Let $\alpha_i = 0$ be a defining equation of $H_i \in \mathcal{A}_{\mathbf{t}}$ and let $\Phi_\lambda = \alpha_1^{\lambda_1} \dots \alpha_n^{\lambda_n}$. Then Φ_λ is a multi-valued holomorphic function on $\mathsf{M}_{\mathbf{t}}$ and gives a section of $\mathcal{L}_\lambda^\vee$. The (hypergeometric) pairing

$$\langle , \rangle : \mathcal{H}_\ell \times \mathcal{H}^\ell \longrightarrow \mathcal{O}_{\mathsf{B}}$$

is defined by the (hypergeometric) integral (in the sense of Aomoto-Gelfand)

$$\langle \sigma, \omega \rangle = \int_\sigma \Phi_\lambda \omega,$$

where $\mathcal{O}_{\mathsf{B}}$ is the sheaf of germs of holomorphic functions on (the smooth part of) B. Let σ be a flat local section on an open set $U \subseteq \mathsf{B}$ of the local system $\mathcal{H}_\ell$. Then the section $\sigma : U \to \mathcal{H}_\ell$ is represented by the vector

$$\tilde{\sigma}(\mathbf{t}) := \begin{pmatrix} \langle \sigma(\mathbf{t}), \Xi_1(\mathbf{t}) \rangle \\ .. \\ .. \\ \langle \sigma(\mathbf{t}), \Xi_\beta(\mathbf{t}) \rangle \end{pmatrix}$$

for each $\mathbf{t} \in U$. Thus the vector $\tilde{\sigma}$, which is a coordinate vector for the flat section σ, satisfies a system

$$d'\tilde{\sigma} = \Omega \wedge \tilde{\sigma},$$

of differential equations of the first order, where d' is the exterior differential on B and Ω is a $\beta \times \beta$-matrix whose entry is a differential 1-form on (the smooth part of) B. This matrix Ω is a Gauss-Manin connection matrix satisiying $d'\Omega - \Omega \wedge \Omega = 0$. In Section 3, we will show that each entry of the connection matrix Ω has at most logarithmic poles along $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Since the geometry of D is sufficiently understood when the codimension of B in $((\mathbb{CP}^\ell)^*)^n$ is one, we have an explicit formula for Ω in this case in Section 4. It turns out that each entry of Ω is a linear combination of logarithmic forms with poles along each irreducible component of D with coefficients in $\sum_{i=1}^n \mathbb{Z}\lambda_i$. It might be natural to ask if it is in the case for any B of higer codimensions.

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2 Combinatorially equivalent family of arrangements

We compactify $\mathbb{C}^\ell$ by adding the infinite hyperplane $\overline{H}_\infty$ to get complex projective space $\mathbb{CP}^\ell$.

Definition 1 *A multiset is a set which allows repetitions. A multiset $\mathcal{M}$ is a projective* **multiarrangement** *if $\mathcal{M}$ is a finite multiset of projective hyper-*

planes of $\mathbb{CP}^\ell$. *Let*

$$\mathcal{M}_n(\mathbb{CP}^\ell) = \{ \textit{ projective multiarrangements of } n+1 \textit{ linearly ordered hyperplanes of } \mathbb{CP}^\ell \textit{ where } \overline{H}_\infty \textit{ is the last hyperplane}\}.$$

Each point of $(\mathbb{CP}^\ell)^*$ corresponds to a hyperplane of $\mathbb{CP}^\ell$. Thus we indentify $\mathcal{M}_n(\mathbb{CP}^\ell)$ with $((\mathbb{CP}^\ell)^*)^n$:

$$\mathcal{M}_n(\mathbb{CP}^\ell) = ((\mathbb{CP}^\ell)^*)^n.$$

Then $\mathcal{M}_n(\mathbb{CP}^\ell)$ is a compact complex manifold biholomorphic to $(\mathbb{CP}^\ell)^n$.

Let $\mathcal{M} \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Write $\mathcal{M} = \{\overline{H}_1, \overline{H}_2, \dots, \overline{H}_{n+1} = \overline{H}_\infty\}$. We say that $\mathcal{M}$ is **essential** if $\bigcap_{H\in\mathcal{M}} H = \emptyset$. Denote the set $\{1, 2, \dots, n+1\}$ by $[n+1]$. Define

$$\left(\binom{[n+1]}{\ell+1}\right) = \{ \text{ subsets of } [n+1] \text{ of cardinality } \ell+1 \ \}.$$

Let $\wp$ denote the power set. Let

$$\mathcal{J} : \mathcal{M}_n(\mathbb{CP}^\ell) \longrightarrow \wp\left(\binom{[n+1]}{\ell+1}\right)$$

be the map defined by

$$\begin{aligned}\mathcal{J}(\mathcal{M}) &= \left\{\{i_1, \dots, i_{\ell+1}\} \in \left(\binom{[n+1]}{\ell+1}\right) \mid \overline{H}_{i_1} \cap \dots \cap \overline{H}_{i_{\ell+1}} \neq \emptyset\right\} \\ &= \left\{\{i_1, \dots, i_{\ell+1}\} \in \left(\binom{[n+1]}{\ell+1}\right) \mid \{\overline{H}_{i_1}, \dots, \overline{H}_{i_{\ell+1}}\} \text{ is not essential}\right\}.\end{aligned}$$

Recall that $\mathcal{A}_n(\mathbb{C}^\ell)$ is the set of affine arrangements of n linearly ordered distinct hyperplanes in $\mathbb{C}^\ell$. When we want to emphasize that repetitions are *not* allowed, we call an arrangement *simple*. Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. The projective closure $\mathcal{A}_\infty$ of $\mathcal{A}$ is defined by

$$\mathcal{A}_\infty = \{\overline{H} \mid H \in \mathcal{A}\} \cup \{\overline{H}_\infty\},$$

where $\overline{H}$ is the closure of H in $\mathbb{CP}^\ell$. The hyperplanes of $\mathcal{A}_\infty$ are naturally linearly ordered by regarding the infinite hyperplane $\overline{H}_\infty$ as the last, or the $(n+1)$st hyperplane. Thus $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$ and there is an injective map

$$\mathcal{A}_n(\mathbb{C}^\ell) \longrightarrow \mathcal{M}_n(\mathbb{CP}^\ell)$$

which sends $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ to its projective closure $\mathcal{A}_\infty \in \mathcal{M}_n(\mathbb{CP}^\ell)$. Through this injection, we identify $\mathcal{A}_n(\mathbb{C}^\ell)$ with its image in $\mathcal{M}_n(\mathbb{CP}^\ell)$. Then the subset $\mathcal{A}_n(\mathbb{C}^\ell)$ is open dense in $\mathcal{M}_n(\mathbb{CP}^\ell) \simeq ((\mathbb{CP}^\ell)^*)^n$ with respect to the Zariski topology because it is characterizd by the open condition that no two hyperplanes are equal.

Proposition 2 *Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Then the following three conditions are equivalent:*

(i) $\mathcal{A}$ is essential, i.e., there exist ℓ hyperplanes in $\mathcal{A}$ whose intersection is a point,

(ii) $\mathcal{A}_\infty$ is essential, i.e., the intersection of all hyperplanes of $\mathcal{A}_\infty$ is empty,

(iii) $\mathcal{J}(\mathcal{A}_\infty) \neq \left(\binom{[n+1]}{\ell+1}\right)$.

PROOF. It is clear that conditions (ii) and (iii) are equivalent because $\bigcap_{\overline{H}\in\mathcal{A}_\infty} \overline{H} = \emptyset$ implies that there exist $\ell+1$ hyperplanes $\overline{H}_{i_1}, \ldots, \overline{H}_{i_{\ell+1}} \in \mathcal{A}_\infty$ whose intersection is empty.

We also have: (iii) $\iff$ there exist $\ell+1$ hyperplanes $\overline{H}_{i_1}, \ldots, \overline{H}_{i_{\ell+1}} \in \mathcal{A}_\infty$ such that $\overline{H}_{i_1} \cap \cdots \cap \overline{H}_{i_{\ell+1}} = \emptyset \iff$ there exist ℓ hyperplanes $H_{i_1}, \ldots, H_{i_\ell} \in \mathcal{A}$ such that $\overline{H}_{i_1} \cap \cdots \cap \overline{H}_{i_\ell} \cap \overline{H}_\infty = \emptyset \iff$ there exist ℓ hyperplanes $H_{i_1}, \ldots, H_{i_\ell} \in \mathcal{A}$ such that $H_{i_1} \cap \cdots \cap H_{i_\ell}$ is a point $\iff$ (i). □

Proposition 3 *Let $\mathcal{A}^{(1)}, \mathcal{A}^{(2)} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential simple arrangements with an order-preserving bijection $\iota : \mathcal{A}^{(1)} \to \mathcal{A}^{(2)}$. Then the following two conditions are equivalent:*

(i) $\mathcal{A}^{(1)} \sim \mathcal{A}^{(2)}$, i.e., $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are combinatorially equivalent,

(ii) $\mathcal{J}(\mathcal{A}^{(1)}_\infty) = \mathcal{J}(\mathcal{A}^{(2)}_\infty)$.

PROOF. Let $\mathcal{A}^{(k)} = \{H^{(k)}_1, \ldots, H^{(k)}_n\} \quad (k = 1, 2)$.

(i) $\Rightarrow$ (ii) : Suppose $1 \leq i_1 < \cdots < i_{\ell+1} \leq n+1$.

Case 1) If $i_{\ell+1} = n+1$, then, for $k = 1, 2$, we have:

$$\begin{aligned}\{i_1, \ldots, i_{\ell+1}\} \notin \mathcal{J}(\mathcal{A}^{(k)}_\infty) &\iff \overline{H}^{(k)}_{i_1} \cap \cdots \cap \overline{H}^{(k)}_{i_\ell} \cap \overline{H}_\infty = \emptyset \\ &\iff H^{(k)}_{i_1} \cap \cdots \cap H^{(k)}_{i_\ell} \text{ is a point} \iff \dim(H^{(k)}_{i_1} \cap \cdots \cap H^{(k)}_{i_\ell}) = 0.\end{aligned}$$

Therefore $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}^{(1)}_\infty)$ if and only if $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}^{(2)}_\infty)$.

Case 2) If $i_{\ell+1} < n+1$, then let $\mathcal{B}^{(k)} = \{H_{i_1}^{(k)}, \ldots, H_{i_{\ell+1}}^{(k)}\}$ for $k = 1, 2$. Then we have:

$$
\begin{aligned}
\{i_1, \ldots, i_{\ell+1}\} \notin \mathcal{J}(\mathcal{A}_\infty^{(k)}) &\iff \overline{H}_{i_1}^{(k)} \cap \cdots \cap \overline{H}_{i_{\ell+1}}^{(k)} = \emptyset \\
&\iff H_{i_1}^{(k)} \cap \cdots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \overline{H}_{i_1}^{(k)} \cap \cdots \cap \overline{H}_{i_{\ell+1}}^{(k)} \cap \overline{H}_\infty = \emptyset \\
&\iff H_{i_1}^{(k)} \cap \cdots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \mathcal{B}_\infty^{(k)} \text{ is essential} \\
&\iff H_{i_1}^{(k)} \cap \cdots \cap H_{i_{\ell+1}}^{(k)} = \emptyset \text{ and } \mathcal{B}^{(k)} \text{ is essential} \\
&\iff \dim(H_{i_1}^{(k)} \cap \cdots \cap H_{i_{\ell+1}}^{(k)}) = -1 \text{ and there exist } \ell \text{ hyperplanes in } \mathcal{B}^{(k)} \\
&\qquad \text{whose intersection is zero-dimensional.}
\end{aligned}
$$

Here the penultimate equivalence follows from Proposition 2. Therefore $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(1)})$ if and only if $\{i_1, \ldots, i_{\ell+1}\} \in \mathcal{J}(\mathcal{A}_\infty^{(2)})$.

(ii) $\Rightarrow$ (i) : Let $1 \le i_1 < \cdots < i_p \le n$. Since $\mathcal{A}$ is essential, we have

$$
\begin{aligned}
&\dim H_{i_1}^{(k)} \cap \cdots \cap H_{i_p}^{(k)} = p \\
&\iff \text{there exist } 1 \le i_{p+1} < \cdots < i_\ell \le n \text{ such that } \dim H_{i_1}^{(k)} \cap \cdots \cap H_{i_\ell}^{(k)} = 0 \\
&\iff \text{there exist } 1 \le i_{p+1} < \cdots < i_\ell \le n \text{ such that } \overline{H}_{i_1}^{(k)} \cap \cdots \cap \overline{H}_{i_\ell}^{(k)} \cap \overline{H}_\infty = \emptyset \\
&\iff \text{there exist } 1 \le i_{p+1} < \cdots < i_\ell \le n \text{ such that} \\
&\qquad \{i_1, \ldots i_\ell, n+1\} \in \mathcal{J}(\mathcal{A}_\infty^{(1)}) = \mathcal{J}(\mathcal{A}_\infty^{(2)})
\end{aligned}
$$

for $k = 1, 2$. Therefore $\dim H_{i_1}^{(1)} \cap \cdots \cap H_{i_p}^{(1)} = p$ if and only if $\dim H_{i_1}^{(2)} \cap \cdots \cap H_{i_p}^{(2)} = p$. The condition (i) easily follows from this. $\square$

Let $(u_0 : \cdots : u_\ell)$ be the homogeneous coordinates for $\mathbb{CP}^\ell = \mathbb{C}^\ell \cup \overline{H}_\infty$ so that the equation $u_0 = 0$ defines $\overline{H}_\infty$. Let $\mathbf{t}$ denote the ordered $(n+1)$-tuple of homogeneous coordinates:

$$
\mathbf{t} = \left((t_1^{(0)} : \cdots : t_1^{(\ell)}), (t_2^{(0)} : \cdots : t_2^{(\ell)}), \ldots, (t_n^{(0)} : \cdots : t_n^{(\ell)}) \right).
$$

Use $\mathbf{t}$ as the homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. The point $\mathbf{t}$ of $((\mathbb{CP}^\ell)^*)^n$ corresponds to the projective multiarrangement $\mathcal{M}_\mathbf{t}$ whose hyperplanes are $\overline{H}_i$ defined by $\overline{\alpha}_i := \sum_{j=0}^{\ell} t_i^{(j)} u_j = 0 \quad (i = 1, \ldots, n)$ and $\overline{H}_{n+1} = \overline{H}_\infty$ defined by $\overline{\alpha}_{n+1} := u_0 = 0$. Define the $(\ell+1) \times (n+1)$-matrix T by

$$
\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1 \\ t_1^{(1)} & \cdots & t_n^{(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.
$$

Note that the ith column of T gives the coefficients of $\overline{\alpha}_i \quad (i = 1, \ldots, n+1)$. Let $S \in \left(\binom{[n+1]}{\ell+1}\right)$. Denote by Δ_S the $(\ell+1)$-minor using the columns of T corresponding to S. Then it is easy to see

$$\mathcal{J}(\mathcal{M}_{\mathbf{t}}) = \{S \in \left(\binom{[n+1]}{\ell+1}\right) \mid \Delta_S(\mathbf{t}) = 0\}$$

by definition.

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$. Define

$$\mathsf{B}_{\mathcal{A}} := \mathcal{J}^{-1}(\mathcal{J}(\mathcal{A}_\infty)).$$

Let $\mathcal{J} \subseteq \left(\binom{[n+1]}{\ell+1}\right)$. Define

$$\mathsf{C}_{\mathcal{J}} = \{\mathbf{t} \in ((\mathbb{CP}^\ell)^*)^n \mid \Delta_S(\mathbf{t}) = 0 \text{ for } S \in \mathcal{J}\}.$$

Then $\mathsf{C}_{\mathcal{J}}$ is Zariski closed in $((\mathbb{CP}^\ell)^*)^n$ and thus compact.

Proposition 4 *Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. Write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$. Then*

(i) $\mathsf{B} = \{\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell) \mid \mathcal{A} \sim \mathcal{B}\}$,

(ii) $\mathsf{B} = \{\mathbf{t} \in ((\mathbb{CP}^\ell)^*)^n \mid \Delta_S(\mathbf{t})$ *vanishes exactly when* $S \in \mathcal{J}(\mathcal{A}_\infty)\}$,

(iii) B *is a locally closed subset of* $((\mathbb{CP}^\ell)^*)^n$,

(iv) Let $\overline{\mathsf{B}}$ be the closure of B in $((\mathbb{CP}^\ell)^*)^n$. Define $\mathsf{D}_T = \overline{\mathsf{B}} \cap \mathsf{C}_{\{T\}}$ for $T \in \mathcal{J}(\mathcal{A}_\infty)^c := \left(\binom{[n+1]}{\ell+1}\right) \setminus \mathcal{J}(\mathcal{A}_\infty)$. Then

$$\mathsf{D} := \overline{\mathsf{B}} \setminus \mathsf{B} = \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_T$$

and each D_T is a hypersurface in $\overline{\mathsf{B}}$.

PROOF. (i) Suppose $\mathcal{A} = \{H_1, \ldots, H_n\}$ and $\mathcal{M} = \{\overline{K}_1, \ldots, \overline{K}_{n+1}\} \in \mathsf{B}$. Then $\mathcal{J}(\mathcal{M}) = \mathcal{J}(\mathcal{A}_\infty)$. We will first show that $\mathcal{M} = \mathcal{B}_\infty$ for some $\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell)$. If not, $\mathcal{M}$ has a hyperplane of multiplicity more than one. Suppose $\overline{K_i} = \overline{K_j} (i \neq j)$. Then $S \in \mathcal{J}(\mathcal{M}) = \mathcal{J}(\mathcal{A}_\infty)$ whenever S contains i and j. Since $\mathcal{A}$ is essential, this implies $\overline{H_i} = \overline{H_j} \in \mathcal{A}_\infty (i \neq j)$, which contradicts the fact that $\mathcal{A}$ is simple. Thus there exists $\mathcal{B} \in \mathcal{A}_n(\mathbb{C}^\ell)$ with $\mathcal{M} = \mathcal{B}_\infty$. Since $\mathcal{A}$ is essential, so is $\mathcal{B}$ by Proposition 2. Then apply Proposition 3.

(ii) One has

$$\mathcal{M}_{\mathbf{t}} \in \mathsf{B} \iff \mathcal{J}(\mathcal{A}_\infty) = \mathcal{J}(\mathcal{M}_{\mathbf{t}})$$
$$\iff \mathcal{J}(\mathcal{A}_\infty) = \{S \in \left(\binom{[n+1]}{\ell+1}\right) \mid \Delta_S(\mathbf{t}) = 0\}.$$

(iii) By (ii), one has

$$\mathsf{B} = \mathsf{C}_{\mathcal{J}(\mathcal{A}_\infty)} \setminus \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{C}_{\{T\}}.$$

Thus B is locally closed.

(iv) One has

$$\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B} = \overline{\mathsf{B}} \cap \left(\bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{C}_{\{T\}} \right) = \bigcup_{T \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_T.$$

Note that $\mathsf{D}_T \quad (T \in \mathcal{J}(\mathcal{A}_\infty)^c)$ is defined by a single equation in $\overline{\mathsf{B}}$. If D_T is not of codimension one in $\overline{\mathsf{B}}$, then there exists an irreducible component C_0 of $\overline{\mathsf{B}}$ which lies in D_T. Thus $C_0 \cap \mathsf{B} = \emptyset$. On the other hand, since B is dense in $\overline{\mathsf{B}}$, B meets any irreducible component of $\overline{\mathsf{B}}$. This is a contradiction, which proves (iv). □

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential in the rest of this section. In particular, $\ell \leq n$. Because of Propostion 4 (i), we can regard $\mathsf{B}_\mathcal{A}$ as a moduli space of the affine arrangements which are combinatorially equivalent to $\mathcal{A}$. When the codimension of $\mathsf{B}_\mathcal{A}$ in $((\mathbb{CP}^\ell)^*)^n$ is less than two, we can explicitly describe the geometry of $\mathsf{B}_\mathcal{A}$ and $\mathsf{D}_\mathcal{A}$.

Codimension-zero case: The moduli space $\mathsf{B}_\mathcal{A}$ is zero-codimensional in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 0$. We say that an affine arrangement $\mathcal{A}$ is of general position if $\mathcal{J}(\mathcal{A}_\infty) = \emptyset$. Thus $\mathsf{B}_\mathcal{A}$ is a moduli space of affine arrangements of general position. In this case $\mathsf{B}_\mathcal{A}$ is a dense open subset of $((\mathbb{CP}^\ell)^*)^n$ and

$$\mathsf{D}_\mathcal{A} = \bigcup_T \mathsf{C}_{\{T\}},$$

where T runs over $\left(\binom{[n+1]}{\ell+1}\right)$. Since $\mathsf{C}_{\{T\}}$ is defined by the single equation $\Delta_T = 0$ and the determinant function is an irreducible polynomial (a special case of Theorem 6), each $\mathsf{C}_{\{T\}}$ is an irreducible hypersurface. Therefore $\mathsf{D}_\mathcal{A}$ is composed of $\binom{n+1}{\ell+1}$ irreducible components. When $\ell = 1$, $\mathsf{B}_\mathcal{A} = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid t_i \neq t_j (i \neq j)\}$ is the pure braid space.

Codimension-one case: The moduli space $\mathsf{B}_{\mathcal{A}}$ is one-codimensional in $((\mathbb{CP}^\ell)^*)^n$ if and only if $|\mathcal{J}(\mathcal{A}_\infty)| = 1$.

Proposition 5 *Suppose $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$. Write $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$, $\mathsf{C} = \mathsf{C}_{\{S\}}$ and $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Then*

(i) $\overline{\mathsf{B}} = \mathsf{C}$ is irreducible,

(ii) B is smooth,

(iii) the irreducible components of D are:

type I $\mathsf{C}_{\{S,S'\}}$ *for* $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ *with* $|S \cap S'| \leq \ell - 1$,

type II $\mathsf{C}_{\langle S-p\rangle}$ *for* $p \in S$, *where* $\langle S - p\rangle := \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \supseteq S \setminus \{p\}\}$, *and*

type III $\mathsf{C}_{\langle S+q\rangle}$ *for* $q \in [n+1] \setminus S$, *where* $\langle S + q\rangle := \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \subseteq S \cup \{q\}\}$.

In all, there exist $\binom{n+1}{\ell+1} - \ell(n - \ell - 1)$ irreducible components of D. When $\ell = 1$, the type II does not appear and the number of irreducible components of D is equal to $n(n-1)/2$.

In order to prove this Proposition we need the following fundamental result on determinantal ideals:

Theorem 6 (Hochter-Eagon[HE]) *Let $X = (X_{ij})$ be a matrix of indeterminates over an integral domain R of size $m \times n$. Let $I_t(X)$ be the ideal in the polynomial ring $R[X_{ij}]$ generated by the t-minors of X. Then $I_t(X)$ is a prime ideal of height $(m - t + 1)(n - t + 1)$.* □

Recall the $(\ell + 1) \times (n + 1)$-matrix T:

$$\mathsf{T} = \begin{pmatrix} t_1^{(0)} & \cdots & t_n^{(0)} & 1 \\ t_1^{(1)} & \cdots & t_n^{(1)} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ t_1^{(\ell)} & \cdots & t_n^{(\ell)} & 0 \end{pmatrix}.$$

Let $\mathbb{C}[\mathsf{T}]$ be the polynomial ring over $\mathbb{C}$ with indeterminates $\{t_j^{(i)}\}_{0\leq i\leq \ell, 1\leq j\leq n}$. For $S \subseteq [n+1]$, define T_S to be the submatrix of T consisting of the columns corresponding to S. When $|S| = \ell + 1$, $\Delta_S = \det(\mathsf{T}_S)$.

Lemma 7 *Let $S, S' \in \left(\binom{[n+1]}{\ell+1}\right)$ and $I := (\Delta_S, \Delta_{S'})\mathbb{C}[\mathsf{T}]$. Then*

(i)

$$I = I_\ell(\mathsf{T}_{S\cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S\cup S'})$$

when $|S \cap S'| = \ell$*. Here* I_t *is defined in the same manner as in Theorem 6.*

(ii) I *is a prime ideal of height two when* $|S \cap S'| \leq \ell - 1$*.*

PROOF. (i) Let $A = \mathsf{T}_{S\cap S'}, B = \mathsf{T}_{S\cup S'}$. Write $B = (b_{ij})_{0\leq i\leq \ell, 0\leq j\leq \ell+1}$. Define

$$\Delta_j = (-1)^j \det(B_j) \quad (j = 0, \ldots, \ell + 1),$$

where B_j is obtained from B by deleting the jth column of B. We may assume that $\Delta_S = \Delta_0$ and $\Delta_{S'} = \Delta_{\ell+1}$. Let $P_1 := I_\ell(A)$ and $P_2 := I_{\ell+1}(B) = (\Delta_0, \ldots, \Delta_{\ell+1})$. We will show $I = P_1 \cap P_2$. If $\ell = 1$ and $S \cap S' = \{n+1\}$, then $P_1 = \mathbb{C}[\mathsf{T}]$. In this case $I = P_2$ and (i) holds true. In the other cases, both P_1 and P_2 are prime ideals of height two by Theorem 6. By elementary linear algebra, one has

$$\sum_{j=0}^{\ell+1} b_{ij}\Delta_j = 0 \quad (i = 0, \ldots, \ell).$$

Thus

$$\sum_{j=1}^{\ell} b_{ij}\Delta_j \in I \quad (i = 0, \ldots, \ell).$$

By applying Cramer's rule, one obtains $P_1P_2 \subseteq I$. Since I is generated by two irreducible polynomials, every associated prime of I is of height two or less. If P is an associated prime of I, then $P_1P_2 \subseteq I \subseteq P$. Thus either $P_1 \subseteq P$ or $P_2 \subseteq P$. So we have $P \in \{P_1, P_2\}$. Write a primary decomposition of I as

$$I = Q_1 \cap Q_2$$

with $\sqrt{Q_i} = P_i \quad (i = 1, 2)$. Note that there is no inclusion relation between P_1 and P_2. Since $P_1P_2 \subseteq Q_i$, we have $P_i = Q_i \quad (i = 1, 2)$.

(ii) (K. Kurano) Case 1): Suppose $n + 1 \notin S \cap S'$. Choose $S'' \in \left(\!\binom{[n+1]}{\ell+1}\!\right)$ such that $S \cap S' \subset S'' \subset S \cup S'$ and $|S \cap S''| = \ell$. Let $\Delta = \Delta_S$, $\Delta' = \Delta_{S'}$, and $\Delta'' = \Delta_{S''}$. By abuse of notation, let a matrix also denote the set of its entries. So the ring $R := \mathbb{C}[\mathsf{T}_{S''}, (\Delta'')^{-1}]$ stands for the subring of $\mathbb{C}(\mathsf{T}_{S''})$ generated by $(\Delta'')^{-1}$ and the entries of $\mathsf{T}_{S''}$ over $\mathbb{C}$. Let $\mathsf{Z} := (\mathsf{T}_{S''})^{-1}$. Then each entry of Z lies in R. Let $S''' := (S \cup S') \setminus S''$. Since the entries of $\mathsf{T}_{S'''}$ are algebraically independent over $\mathbb{C}(\mathsf{T}_{S''})$, so are the entries of $\mathsf{Z}\mathsf{T}_{S'''}$. Note that there exists an entry of $\mathsf{Z}\mathsf{T}_{S'''}$ which is equal either to $\det(\mathsf{Z}\mathsf{T}_S)$ or to $-\det(\mathsf{Z}\mathsf{T}_S)$ and that there exists a minor of $\mathsf{ZT}_{S'''}$ which is equal either to $\det(\mathsf{ZT}_{S'})$ or to $-\det(\mathsf{ZT}_{S'})$. Thus the ideal

$$(\Delta, \Delta')R\left[\mathsf{T}_{S'''}\right] = (\det(\mathsf{T}_S), \det(\mathsf{T}_{S'}))R\left[\mathsf{T}_{S'''}\right] = (\det(\mathsf{ZT}_S), \det(\mathsf{ZT}_{S'}))R\left[\mathsf{ZT}_{S'''}\right]$$

is a prime ideal of

$$R\left[\mathsf{ZT}_{S'''}\right] = R\left[\mathsf{T}_{S'''}\right] = \mathbb{C}\left[\mathsf{T}_{S\cup S'}, (\Delta'')^{-1}\right]$$

by Theorem 6. On the other hand, the associated primes of $(\Delta, \Delta')R\left[\mathsf{T}_{S\cup S'}\right]$ are $I_\ell(\mathsf{T}_{S\cap S''})$ and $I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Since $(S\cap S'')\setminus S' \neq \emptyset$ and $|(S\cup S'')\setminus S'| \geq 2$, one has $\Delta' \notin I_\ell(\mathsf{T}_{S\cap S''})$ and $\Delta' \notin I_{\ell+1}(\mathsf{T}_{S\cup S''})$. Therefore $(\Delta, \Delta'') : (\Delta') = (\Delta, \Delta'')$. This implies $(\Delta, \Delta') : (\Delta'') = (\Delta, \Delta')$. Thus Δ'' is a non-zero divisor of $\mathbb{C}\left[T_{S\cup S'}\right]/(\Delta, \Delta')$. Because the factor ring $\mathbb{C}\left[T_{S\cup S'}, (\Delta'')^{-1}\right]/(\Delta, \Delta')$ is a domain, so is the factor ring $\mathbb{C}\left[T_{S\cup S'}\right]/(\Delta, \Delta')$. This shows (ii).

Case 2): Suppose $n+1 \in S\cap S'$. Then this case reduces into Case 1).

Case 3): Suppose $n+1 \in S\setminus S'$. Choose $S'' \in \left(\binom{[n+1]}{\ell+1}\right)$ such that $S\cap S' \subset S'' \subset S\cup S'$, $|S\cap S''| = \ell$, and $n+1 \in S''$. The rest of the proof is exactly the same as Case 1). □

Proof of Proposition 5 Since $\mathcal{A}$ is essential and not of general position, one has $\ell + 1 \leq n$.

(i) By Theorem 6, Δ_S is an irreducible polynomial. Thus C is irreducible and $\overline{\mathsf{B}} = \mathsf{C}$.

(ii) Let $n+1 \notin S$. Let J be the ideal generated by the partial derivatives of Δ_S. Because of the Laplace expansion formula for $\det(\mathsf{T}_S)$, J is generated by the ℓ-minors of T_S. Thus any singular point $\mathbf{t}$ of B lies in $\mathsf{C}_{\{S'\}}$ for any $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S\cap S'| = \ell$. Thus $\mathbf{t} \notin \mathsf{B}$. We can similarly prove the assertion when $n+1 \in S$.

(iii) Let $S' \in \left(\binom{[n+1]}{\ell+1}\right) \setminus \{S\}$. Note $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$. If $|S\cap S'| \leq \ell - 1$, then $(\Delta_S, \Delta_{S'})$ is a prime ideal by Lemma 7 (i). Thus $\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}}$ is irreducible. If $|S\cap S'| = \ell$, then $(\Delta_S, \Delta_{S'}) = I_\ell(\mathsf{T}_{S\cap S'}) \cap I_{\ell+1}(\mathsf{T}_{S\cup S'})$ by Lemma 7 (ii). If $\ell \geq 2$, this is a primary decomposition of $(\Delta_S, \Delta_{S'})$. Let $\{p\} = S\setminus S'$ and $\{q\} = S'\setminus S$. Then

$$\mathsf{D}_{S'} = \mathsf{C}_{\{S,S'\}} = \mathsf{C}_{\langle S-p\rangle} \cup \mathsf{C}_{\langle S+q\rangle}$$

is the decomposition of $\mathsf{D}_{S'}$ into irreducible components. The cardinality of the set $\{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid |S\cap S'| \leq \ell - 1\}$ is equal to $\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1)$.

Thus the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{\ell+1} - 1 - (n-\ell)(\ell+1) + (\ell+1) + (n-\ell) = \binom{n+1}{\ell+1} - \ell(n-\ell-1).$$

If $\ell = 1 = |S \cap S'|$, then the ideal $I_\ell(\mathsf{T}_{S\cap S'})$ does not define a subvariety of $((\mathbb{CP}^\ell)^*)^n$. Thus $\mathsf{D}_{S'} = \mathsf{C}_{\langle S+q \rangle}$ where $\{q\} = S' \setminus S$. Therefore the total number of irreducible components of $\mathsf{D} = \bigcup_{S' \in \mathcal{J}(\mathcal{A}_\infty)^c} \mathsf{D}_{S'}$ is equal to

$$\binom{n+1}{2} - 1 - 2(n-1) + (n-1) = n(n-1)/2. \square$$

3 Logarithmic Gauss-Manin connections

Let $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ be essential. We fix $\mathcal{A}$ in the rest of this section and write $\mathsf{B} = \mathsf{B}_\mathcal{A}$. Then, as we saw in the previous section, B may be considered a moduli space of the family of essential simple affine ℓ-arrangements which are combinatorially equivalent to $\mathcal{A}$. Let

$$\mathbf{t} = \left((t_1^{(0)} : \cdots : t_1^{(\ell)}), (t_2^{(0)} : \cdots : t_2^{(\ell)}), \ldots, (t_n^{(0)} : \cdots : t_n^{(\ell)}) \right).$$

be the homogeneous coordinates for $((\mathbb{CP}^\ell)^*)^n$. Let $\mathbf{u} = (u_1, \ldots, u_\ell)$ be the standard coordinates for $\mathbb{C}^\ell$. Define

$$\mathsf{M} = \{(\mathbf{u}, \mathbf{t}) \in \mathbb{C}^\ell \times ((\mathbb{CP}^\ell)^*)^n \mid \mathbf{t} \in \mathsf{B},\ t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j \neq 0 \ (i = 1, \ldots, n)\}.$$

Let

$$\pi : \mathsf{M} \longrightarrow \mathsf{B}$$

be the projection defined by $\pi(\mathbf{u}, \mathbf{t}) = \mathbf{t}$. Then the fiber $\mathsf{M}_\mathbf{t} := \pi^{-1}(\mathbf{t})$ is the complement of the affine arrangement $\mathcal{A}_\mathbf{t}$ whose hyperplanes are defined by the equations $\alpha_i := t_i^{(0)} + \sum_{j=1}^{\ell} t_i^{(j)} u_j = 0$ $(i = 1, \ldots, n)$. Thus $\pi : \mathsf{M} \longrightarrow \mathsf{B}$ is the complete family of essential simple affine arrangements in $\mathbb{C}^\ell$ which are combinatorially equivalent to $\mathcal{A}$. A result of Randell [Ra] implies that π is a fiber bundle over (the smooth part of) B.

We assume that d is the exterior differential operator with respect to the coordinates $\mathbf{u} = (u_1, \ldots, u_\ell)$ of $\mathbb{C}^\ell$ of the fiber and that $\omega_i := d \log \alpha_i = d\alpha_i / \alpha_i$ for $1 \le i \le n$ and

$$\omega_\lambda := \sum_{i=1}^{n} \lambda_i \omega_i, \qquad \nabla_\lambda : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda \eta := d\eta + \omega_\lambda \wedge \eta.$$

In this section we will compute covariant derivatives of differential forms in the fiber along the direction of the base.

Definition 8 *Let d' be the exterior differential operator with respect to the homogeneous coordinates $\mathbf{t}$ of $((\mathbb{CP}^\ell)^*)^n$. For $1 \leq i \leq n$ define $\omega_i' := d' \log(\alpha_i/t_i^{(0)}) = (d'\alpha_i/\alpha_i) - (d't_i^{(0)}/t_i^{(0)})$ and*

$$\omega_\lambda' := \sum_{i=1}^n \lambda_i \omega_i', \qquad \nabla_\lambda' : \Omega_M^p \to \Omega_M^{p+1}, \qquad \nabla_\lambda' \eta := d'\eta + \omega_\lambda' \wedge \eta.$$

Our next aim is to compute the operator ∇_λ' explicitly. For $S = (j_1, \ldots, j_m)$, $j_1 < \cdots < j_m$, write $S_k = (j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_m)(1 \leq k \leq m)$ and $(S, j) = (j_1, \ldots, j_m, j)$ for $j \in [n+1] \setminus S$.

Definition 9 *Let $T = (i_1, \ldots, i_\ell)$, with $i_k \in [n]$ $(1 \leq k \leq \ell)$. Write*

$$\omega_T := \omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}, \qquad \zeta_T := \sum_{k=1}^{\ell} (-1)^{k+1} \omega_{i_k}' \wedge \omega_{T_k}.$$

The following computation was suggested by a method employed in [AK].

Proposition 10

$$\nabla_\lambda' \omega_T = -\nabla_\lambda \zeta_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge d' \log\left(\frac{\Delta_{(T,j)}}{\Delta_{((T,j)_k, n+1)}} \right).$$

This Proposition is an immediate consequence of the following two lemmas.

Lemma 11

$$\nabla_\lambda' \omega_T + \nabla_\lambda \zeta_T = \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge \omega_{i_k}'.$$

PROOF. Since d and d' operate in different variables, $dd' + d'd = 0$. This gives $d'\omega_T + d\zeta_T = 0$ used in the calculation below.

$$
\begin{aligned}
&\nabla'_\lambda \omega_T + \nabla_\lambda \zeta_T \\
=&d'\omega_T + \omega'_\lambda \wedge \omega_T + d\zeta_T + \omega_\lambda \wedge \zeta_T \\
=&\sum_{k=1}^{\ell} (-1)^{k+1} (d'\omega_{i_k}) \wedge \omega_{T_k} + \omega'_\lambda \wedge \omega_T + \sum_{k=1}^{\ell} (-1)^{k+1} (d\omega'_{i_k}) \wedge \omega_{T_k} \\
&- \sum_{k=1}^{\ell} \lambda_{i_k} \omega'_{i_k} \wedge \omega_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell} (-1)^{\ell+k+1} \omega'_{i_k} \wedge \omega_{(T_k, j)} \\
=&\left(\sum_{j \in [n] \setminus T} \lambda_j \omega'_j \right) \wedge \omega_T + \sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell} (-1)^{\ell+k+1} \omega'_{i_k} \wedge \omega_{(T_k, j)} \\
=&\sum_{j \in [n] \setminus T} \lambda_j \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{(T,j)_k} \wedge \omega'_{i_k}. \square
\end{aligned}
$$

Lemma 12 *For $S \in \left(\binom{[n+1]}{\ell+1}\right)$, we have*

$$
\sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge \omega'_{j_k} = \sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge d' \log \left(\frac{\Delta_S}{\Delta_{(S_k, n+1)}} \right).
$$

PROOF. Note that

$$
\Delta_S = \sum_{k=1}^{\ell+1} (-1)^{k+1} t_{j_k}^{(0)} \Delta_{(S_k, n+1)} = \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k, n+1)}
$$

by the Laplace expansion. Let

$$
\alpha_S := \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_{\ell+1}}, \qquad d\mathbf{u} := du_1 \wedge \cdots \wedge du_\ell.
$$

We compute

$$
\begin{aligned}
&\sum_{k=1}^{\ell+1} (-1)^{k+1} \omega_{S_k} \wedge (d' \log \Delta_S) \\
=&\frac{1}{\alpha_S} \sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{i_k} \Delta_{(S_k, n+1)} (d\mathbf{u}) \wedge (d' \log \Delta_S) \\
=&\frac{1}{\alpha_S} \Delta_S (d\mathbf{u}) \wedge (d' \log \Delta_S) = \frac{1}{\alpha_S} (d\mathbf{u}) \wedge (d' \Delta_S) \\
=&\frac{1}{\alpha_S} (d\mathbf{u}) \wedge d' \left(\sum_{k=1}^{\ell+1} (-1)^{k+1} \alpha_{j_k} \Delta_{(S_k, n+1)} \right)
\end{aligned}
$$

$$=\frac{1}{\alpha_S}(d\mathbf{u}) \wedge \left[\sum_{k=1}^{\ell+1}(-1)^{k+1}\left\{(d'\alpha_{j_k})\Delta_{(S_k,n+1)} + \alpha_{j_k}\left(d'\Delta_{(S_k,n+1)}\right)\right\}\right]$$
$$=\sum_{k=1}^{\ell+1}(-1)^{k+1}\left\{\omega_{S_k} \wedge \omega'_{j_k} + \omega_{S_k} \wedge \left(d' \log \Delta_{(S_k,n+1)}\right)\right\}.$$

This shows the lemma.□

For $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mathbf{t} \in \mathsf{B}$, recall the rank-one local system $\mathcal{L}_\lambda$ on $\mathsf{M}_\mathbf{t} = \pi^{-1}(\mathbf{t})$ so that $\mathcal{L}_\lambda$ has monodromy $\exp(-2\pi\sqrt{-1}\lambda_i)$ around $H_i = \{\alpha_i = 0\}(i = 1, \ldots, n)$.

Theorem 13 *([ESV] [STV]) For a "generic" $\lambda \in \mathbb{C}^n$,*

(1) $H^p(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}) = 0 (p \neq \ell)$ and $\dim H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$ is equal to $\beta = |\chi(M_\mathbf{t})|$, where χ stands for the Euler Poincaré characteristic,

(2) there exisits a natural (twisted) de Rham isomorphism

$$A^\ell/\omega_\lambda \wedge A^{\ell-1} \xrightarrow{\sim} H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}),$$

where $A^\cdot = \oplus_{q=0}^{\ell} A^q$ is the Orlik-Solomon algebra [OSo] [OT, 3.45] of $\mathcal{A}_\mathbf{t}$. □

For explicit conditions for the "genericity," see [ESV] and [STV]. In the rest of the paper, we assume that λ is generic in the sense of [STV, 4.3 (Mon)**].

Since $\nabla_\lambda \circ \nabla'_\lambda + \nabla'_\lambda \circ \nabla_\lambda = 0$ and

$$H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}) \simeq A^\ell/\omega_\lambda \wedge A^{\ell-1} = A^\ell/\nabla_\lambda A^{\ell-1},$$

the operator ∇'_λ induces a $\mathbb{C}$-minear map

$$\nabla'_\lambda : H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}) \to H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}) \otimes \Omega^1(\log \mathsf{D})$$

by Proposition 10. Here $\Omega^1(\log \mathsf{D})$ is the space of meromorphic 1-forms on (the smooth part of) $\overline{\mathsf{B}}$ with logarithmic poles along $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$. Let

$$\mathsf{D} = \bigcup_{s=1}^{t} \mathsf{D}_s$$

be the irreducible decomposition. For each irreducible component D_s and $S' \in \mathcal{J}(\mathcal{A}_\infty)^c$, define

$$\mathrm{mult}(S', \mathsf{D}_s) := \text{the order of zeros of } \ \Delta_{S'}|_{\overline{B}} \text{ along } \mathsf{D}_s$$

and

$$\begin{aligned}\Gamma(\mathsf{D}_s) :=& \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \mathrm{mult}(S', \mathsf{D}_s) \geq 1\} \\ =& \{S' \in \mathcal{J}(\mathcal{A}_\infty)^c \mid \Delta_{S'}|_{\overline{B}} \text{ vanishes on } \mathsf{D}_s\}.\end{aligned}$$

We denote the logarithmic 1-form on $\overline{B}$ with simple logarithmic pole along D_s by $d' \log \mathsf{D}_s$ by abuse of notation. It can be locally expressed as $d \log f$ where $f = 0$ is a local defining equation for D_s. For $\omega \in A^\ell$, let $[\omega] \in H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$ be its (twisted) de Rham cohomology class. Then, by Proposition 10, we immediately have

Theorem 14 *We have*

$$\nabla'_\lambda = \sum_{s=1}^{t} \nabla'_{\lambda,s} \otimes d' \log \mathsf{D}_s,$$

where $\nabla'_{\lambda,s} \in \mathrm{End}(H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}))$ *and, for* $T \in \left(\binom{[n+1]}{\ell+1}\right)$,

$$\begin{aligned}\nabla'_{\lambda,s}[\omega_T] =& \sum_{(T,j)\in\Gamma(\mathsf{D}_s)} \mathrm{mult}((T,j),\mathsf{D}_s)\lambda_j \sum_{k=1}^{\ell+1}(-1)^{k+1}[\omega_{(T,j)_k}] \\ & - \sum_{((T,j)_k,n+1)\in\Gamma(\mathsf{D}_s)} \mathrm{mult}(((T,j)_k, n+1),\mathsf{D}_s)(-1)^{k+1}\lambda_j[\omega_{(T,j)_k}].\square\end{aligned}$$

Although Theorem 14 determines ∇'_λ and $\nabla'_{\lambda,s}$ completely, it is desirable to express each $\nabla'_{\lambda,s}$ explicitly in terms of a basis for $H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$. We propose to use the β**nbc** basis for this purpose. The β**nbc** basis is a combinatorially constructed basis for $H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$ in [FT, 3.9]. When $\mathcal{A}$ is of general position, the set

$$\{[\eta_T] \in H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t}) \mid T = (i_1, \ldots, i_\ell), 2 \leq i_1 < \cdots < i_\ell \leq n\}$$

is the β**nbc** basis, where

$$\eta_T := \lambda_{i_1} \cdots \lambda_{i_\ell} \omega_T.$$

In this case the expression of each $\nabla'_{\lambda,s}$ in terms of the β**nbc** basis is obtained in [AK, Ch. 3 §8]. When B is one-codimensional in $((\mathbb{CP}^\ell)^*)^n$, the explicit formula is given in the next section. In general, it is not difficult to see from [FT, 3.9] that $[\omega_T] \in H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$ is uniquely expresed as a linear combination of the β**nbc** basis $[\Xi_1], \ldots, [\Xi_\beta] \in H^\ell(\mathsf{M}_\mathbf{t}, \mathcal{L}_\mathbf{t})$ with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]} := \mathbb{Q}[\lambda_1, \ldots, \lambda_n, \{\lambda_X^{-1}\}]$, where $\lambda_X^{-1} = 1/(\sum_{X\subseteq H_j} \lambda_j)$ runs over the set $\{X \mid X \text{ is a dense edge }\}$. Recall that $\mathcal{H}_\ell$ is the rank β local system coming from the topological fibration $\pi : \mathsf{M} \to \mathsf{B}$. Then we have

Theorem 15 *The* $\beta \times \beta$*-matrix* Ω*, which satisfies the system of differential* *quations*

$$d' \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ .. \\ .. \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} = \Omega \wedge \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ .. \\ .. \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix}$$

*for any (local) section σ of $\mathcal{H}_\ell$, the β**nbc** basis $[\Xi_1], \ldots, [\Xi_\beta] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ and $\Phi_\lambda = \alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n}$, has logarithmic poles along D with coefficients lying in $\widetilde{\mathbb{Q}[\lambda]}$.*

PROOF. The integral $\int_\sigma \Phi_\lambda \Xi$ depends only upon the cohomology class $[\Xi] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$. By Proposition 10, there exists a unique $\beta \times \beta$-matrix Ω such that

$$\begin{pmatrix} \nabla'_{\lambda,s}[\Xi_1] \\ .. \\ .. \\ \nabla'_{\lambda,s}[\Xi_\beta] \end{pmatrix} = \Omega \wedge \begin{pmatrix} [\Xi_1] \\ .. \\ .. \\ [\Xi_\beta] \end{pmatrix}.$$

Then Ω satisfies the desired properties. $\square$

Thus the connection $d' - \Omega\wedge$ on $\mathcal{O}_{\mathsf{B}}^\beta$ is a logarithmic Gauss-Manin connection and its flat sections are given by

$$\{ \begin{pmatrix} \int_\sigma \Phi_\lambda \Xi_1 \\ .. \\ .. \\ \int_\sigma \Phi_\lambda \Xi_\beta \end{pmatrix} \mid \sigma \text{ is a local section of } \mathcal{H}^\ell \}.$$

4 The codimension one case

Suppose that the codimension of $\mathsf{B}_{\mathcal{A}}$ in $((\mathbb{CP}^\ell)^*)^n$ is one in this section. Then $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$ for some $S \in \left(\binom{[n+1]}{\ell+1}\right)$. There are two cases : $n+1 \notin S$ (Case A) or $n+1 \in S$ (Case B). By permuting the hyperplanes if necessary, one can assume that $S = (1, 2, \ldots, \ell+1)$ (Case A) or $S = (n-\ell+1, n-\ell+2, \ldots, n+1)$

(Case B). It is easy to see that the β**nbc** basis for $H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}})$ is given by $\{[\eta_T] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T \in \beta\mathbf{nbc}\}$, where

$$\beta\mathbf{nbc} = \{(j_1, \dots, j_\ell) \mid 2 \le j_1 < \cdots < j_\ell \ne \ell + 1\} \quad \text{(Case A)}$$

or

$$\beta\mathbf{nbc} = \{(j_1, \dots, j_\ell) \mid 2 \le j_1 < \cdots < j_\ell, j_1 \ne n - \ell + 1\} \quad \text{(Case B).}$$

We will express $\nabla'_{\lambda,s}[\eta_T]$, $T \in \beta\mathbf{nbc}$, as a linear combination of $\{[\eta_{T'}] \in H^\ell(\mathsf{M}_{\mathbf{t}}, \mathcal{L}_{\mathbf{t}}) \mid T' \in \beta\mathbf{nbc}\}$ with coefficients in $\widetilde{\mathbb{Q}[\lambda]}$. (It will turn out that all the coefficients lie in $\sum_{i=1}^n \mathbb{Z}\lambda_i$.) In the following formulas for $\nabla'_{\lambda,s}[\eta_T]$ we use the notation

$$\epsilon(T, T') = (-1)^{p+q}$$

if $T, T' \subseteq [n]$, $|T| = |T'| = \ell$, $|T \cap T'| = \ell - 1$, $U = T \cup T'$, $T = U_p$ and $T' = U_q$. Define $\epsilon(T, T') = 1$ if $T = T'$. For example, $\epsilon(23, 35) = 1$ because $U = 235, T = 23 = U_3, T' = 35 = U_1$.

Case A.: Let $S = (1, 2, \dots, \ell + 1)$.

Type A.I.: Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S \cap S'| \le \ell - 1$ (Proposition 5 (iii)). In this case, $\Gamma(\mathsf{D}_s) = \{S, S'\}$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ because the ideal $(\Delta_S, \Delta_{S'})$ is prime by Lemma 7 (ii).

Case A.I.1.: Suppose $S' \cap \{1, n + 1\} = \emptyset$.

- If $S' \supset T \in \beta\mathbf{nbc}$, then

$$\nabla'_{\lambda,s}[\eta_T] = \sum_{k=1}^{\ell+1} \epsilon(T, S'_k)\lambda_{S' \setminus S'_k}[\eta_{S'_k}],$$

  where $S'_k = (i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_{\ell+1})$ and $\lambda_{S' \setminus S'_k} = \lambda_{i_k}$ if $S' = (i_1, \dots, i_{\ell+1})$.
- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.2.: Suppose $S' \cap \{1, n + 1\} = \{n + 1\}$.

- If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_{\ell+1} = S' \setminus \{n + 1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j \in [n] \setminus T} \lambda_j\right)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\epsilon(T, S'_{\ell+1})\lambda_{T \setminus S'}[\eta_{S'_{\ell+1}}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.3.: Suppose $S' \cap \{1, n+1\} = \{1\}$.

- If $S' \supset T \in \beta\mathbf{nbc}$, then $T = S'_1 = S' \setminus \{1\}$ and

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j \in S'} \lambda_j\right) [\eta_T] - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1}} \epsilon(T, T') \lambda_{T \setminus T'} [\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.I.4.: Suppose $S' \cap \{1, n+1\} = \{1, n+1\}$.

- If $T \in \beta\mathbf{nbc}$ with $|T \cap S'| = \ell - 1$, then $S' \setminus \{1, n+1\} \subset T$ and

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T \setminus S'} \sum_{\substack{T' \in \beta\mathbf{nbc} \\ T' \supset T \cap S'}} [\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.II.: Suppose $\ell \geq 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p \rangle}$ where $p \in S = (1, 2, \ldots, \ell+1)$, $S - p := S \setminus \{p\}$, and $\langle S - p \rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \supset S - p\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S - p \rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S - p \rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 7 (i).

Case A.II.1.: Suppose $p \neq 1$.

- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S - p)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j \in S-p} \lambda_j\right) [\eta_T] - \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ |T' \cap (S-p)| = \ell - 2}} \epsilon(T, T') \lambda_{T \setminus T'} [\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.II.2.: Suppose $p = 1$.

- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S - 1)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = \lambda_{(S-1) \setminus T} [\eta_T] + \sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell - 1 \\ T' \subset S \cup T}} \epsilon(T, T') \lambda_{T \setminus T'} [\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Type A.III.: Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q\rangle}$ where $q \in [n+1]\setminus S = (\ell+2, \ell+3, \dots, n+1)$, and $S+q := S \cup \{q\}$, and $\langle S+q\rangle = \{S' \in \left(\binom{[n+1]}{\ell+1}\right) \mid S' \subset S+q\}$. In this case, $\Gamma(\mathsf{D}_s) = \langle S+q\rangle$ and $\mathrm{mult}(S', \mathsf{D}_s) = 1$ for $S' \in \langle S+q\rangle$, $S' \neq S$, because the ideal $(\Delta_S, \Delta_{S'})$ is a radical ideal by Lemma 7 (i).

Case A.III.1.: Suppose $q \neq n+1$.

- If $T \in \beta\mathbf{nbc}$ with $T \subset S+q$, then

$$\nabla'_{\lambda,s}[\eta_T] = \left(\sum_{j\in S+q} \lambda_j\right)[\eta_T] - \sum_{\substack{T'\in\beta\mathbf{nbc} \\ |T\cap T'|=\ell-1 \\ |T'\cap(S+q)|=\ell-1}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case A.III.2.: Suppose $q = n+1$.

- If $T \in \beta\mathbf{nbc}$ with $|T\cap S| = \ell-1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\lambda_{T\setminus S} \sum_{\substack{T'\in\beta\mathbf{nbc} \\ T'\cap S = T\cap S}} [\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.: Let $S = (n-\ell+1, \dots, n+1)$.

Type B.I.: Let $\mathsf{D}_s = \mathsf{C}_{\{S,S'\}}$ for $S' \in \left(\binom{[n+1]}{\ell+1}\right)$ with $|S\cap S'| \leq \ell-1$. For this type, we have the exact same formulas as Case A.

Type B.II.: Suppose $\ell \geq 2$. Let $\mathsf{D}_s = \mathsf{C}_{\langle S-p\rangle}$.

Case B.II.1.: Suppose $p \neq n+1$.

- If $T \in \beta\mathbf{nbc}$ with $|T\cap(S-p)| = \ell-1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j\notin S-p} \lambda_j\right)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T\cap(S-p)| = \ell-2$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\sum_{\substack{T'\in\beta\mathbf{nbc} \\ |T\cap T'|=\ell-1 \\ |T'\cap(S-p)|=\ell-1}} \epsilon(T,T')\lambda_{T\setminus T'}[\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.II.2.: Same formulas as Case A. II. 2.

Type B.III.: Let $\mathsf{D}_s = \mathsf{C}_{\langle S+q\rangle}$ where $q \in [n+1] \setminus S = (\ell+2, \ell+3, \dots, n+1)$.

Case B.III.1.: Suppose $q \neq 1$.

- If $T \in \beta\mathbf{nbc}$ with $T \subset S+q$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\left(\sum_{j \notin S+q} \lambda_j\right)[\eta_T].$$

- If $T \in \beta\mathbf{nbc}$ with $|T \cap (S+q)| = \ell - 1$, then

$$\nabla'_{\lambda,s}[\eta_T] = -\sum_{\substack{T' \in \beta\mathbf{nbc} \\ |T \cap T'| = \ell-1 \\ T' \subset (S+q)}} \epsilon(T,T')\lambda_{T \setminus T'}[\eta_{T'}].$$

- Otherwise $\nabla'_{\lambda,s}[\eta_T] = 0$.

Case B.III.2.: Same formulas as Case A. III. 2.

Summarizing Cases A and B above, we have

Theorem 16 *Suppose that* $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$ *is one-codimensional in* $((\mathbb{CP}^\ell)^*)^n$. *Let* $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$, $\mathsf{D} = \overline{\mathsf{B}} \setminus \mathsf{B}$ *and* $\mathsf{D} = \cup_{s=1}^t \mathsf{D}_s$ *be the irreducible decomposition. Then*

(1) the logarithmic Gauss-Manin connection matrix Ω *in Theorem 15 can be expressed as* $\Omega = \sum_{s=1}^t \Omega_s \otimes d' \log \mathsf{D}_s$ *such that each* Ω_s *has its entries in* $\sum_{i=1}^n \mathbb{Z}\lambda_i$.

(2) The eigenvalues of Ω_s *are:*

(i) $\sum_{j \in S} \lambda_j$ *with multiplicity one and the rest are zero (if* D_s *is of type I in Proposition 5),*

(ii) $\sum_{j \in S-p} \lambda_j$ *with multiplicity* $n - \ell - 1$ *and the rest are zero (if* D_s *is of type II), or*

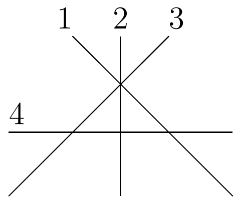
(iii) $\sum_{j \in S+q} \lambda_j$ *with multiplicity* ℓ *and the rest are zero (if* D_s *is of type III),*

where we define $\lambda_{n+1} := -\lambda_1 - -\lambda_2 - \cdots - \lambda_n$.

(The explicit formulas for Ω_s *are given above when* $S = (1, 2, \dots, \ell+1)$ *(Case A) or* $S = (n-\ell+1, n-\ell+2, \dots, n+1)$ *(Case B).)*

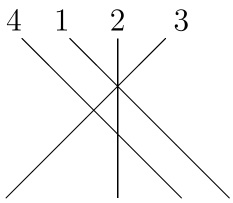
PROOF. Although the β**nbc** basis depends on the linear order on $\mathcal{A}$. it is known [FT, 3.11] that two β**nbc** bases are connected by an integral unimodular matrix (without λ). Thus one can assume that $S = (1, 2, \ldots, \ell + 1)$ (when $n + 1 \notin S$) or $S = (n - \ell + 1, n - \ell + 2, \ldots, n + 1)$ (when $n + 1 \in S$). Use the above-mentioned explicit formulas for Cases A and B. $\square$

Example 17 *Let $\ell = 2$, $n = 4$, $S = (1, 2, 3)$ and $\mathcal{J}(\mathcal{A}_\infty) = \{S\}$.*

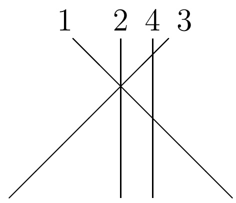


$\mathcal{A}$

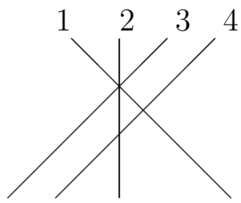
Write 123 for $(1, 2, 3)$ etc. The boundary divisor $\mathsf{D} = \overline{\mathsf{B}}_\mathcal{A} \backslash \mathsf{B}_\mathcal{A}$ has the following eight irreducible components:



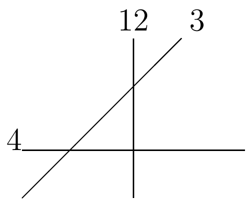
$\mathsf{D}_1 = \mathsf{C}_{\{123,145\}}$



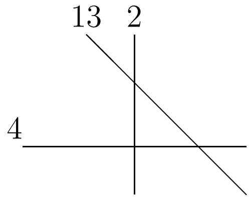
$\mathsf{D}_2 = \mathsf{C}_{\{123,245\}}$



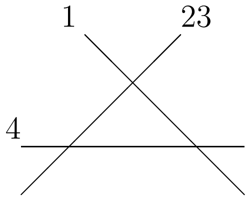
$\mathsf{D}_3 = \mathsf{C}_{\{123,345\}}$



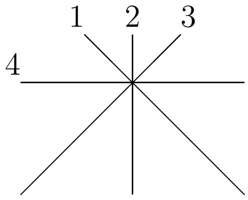
$\mathsf{D}_4 = \mathsf{C}_{\langle 123-3 \rangle}$



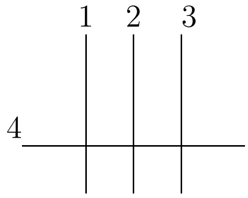
$\mathsf{D}_5 = \mathsf{C}_{\langle 123-2 \rangle}$



$\mathsf{D}_6 = \mathsf{C}_{\langle 123-1 \rangle}$



$\mathsf{D}_7 = \mathsf{C}_{\langle 123+4 \rangle}$



$\mathsf{D}_8 = \mathsf{C}_{\langle 123+5 \rangle}$

The matrices $\Omega_s (s = 1, \ldots, 8)$ *in terms of the* $\beta\mathbf{nbc}$ *basis* $\{[\eta_{24}], [\eta_{34}]\}$, *are*

$$\Omega_1 = \begin{pmatrix} -\lambda_2 & -\lambda_2 \\ -\lambda_3 & -\lambda_3 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} -\lambda_1 - \lambda_3 & 0 \\ \lambda_3 & 0 \end{pmatrix}, \quad \Omega_3 = \begin{pmatrix} 0 & \lambda_2 \\ 0 & -\lambda_1 - \lambda_2 \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad \Omega_5 = \begin{pmatrix} 0 & 0 \\ \lambda_3 & \lambda_1 + \lambda_3 \end{pmatrix}, \quad \Omega_6 = \begin{pmatrix} \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_2 \end{pmatrix},$$

$$\Omega_7 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & 0 \\ 0 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \end{pmatrix}, \quad \Omega_8 = \begin{pmatrix} -\lambda_4 & 0 \\ 0 & -\lambda_4 \end{pmatrix}.$$

For an arbitrary arrangement $\mathcal{A} \in \mathcal{A}_n(\mathbb{C}^\ell)$ and $\mathsf{B} = \mathsf{B}_{\mathcal{A}}$, it seems to be difficult to find explicit matrix presentations for ∇'_λ. Based upon our result for the codimension one case, it might be natural to ask the following questions:

Question 1. Does each entry of the matrix Ω_s lie in $\sum_{i=1}^n \mathbb{Z}\lambda_i$?

Question 2. Is B smooth?

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