Chambers of Arrangements

and

Arrow's Impossibility Theorem

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CONTENTS

1. Arrow's Impossibility Theorem (economics version)

Assume that a society of m people have ℓ policy options and that every individual has his/her own order of preferences on the ℓ policy options.

A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference.

We require the following two requirements for a reasonable social welfare function:

Two requirements:

(A) the society prefers the option i to the option j if every individual prefers the option i to the option j (Pareto property),

(B) whether the society prefers the option i to the option j only depends which individuals prefer the option i to the option j (pairwise independence).

Arrow's impossibility theorem

For $\ell \geq 3$, the only social welfare function satisfying the two requirements (A) and (B) is a dictatorship, that is, the societal preference has to be equal to the preference of one particular individual.

(A) the society prefers the option i to the option j if every individual prefers the option i to the option j (Pareto property),

(B) whether the society prefers the option i to the option j only depends which individuals prefer the option i to the option j (pairwise independence).

Why is this theorem true?
What is the reason behind the theorem?

Condorcet's paradox by Marquis Condorcet (1743-94)

A, B, C : 3 people, $1, 2, 3$: 3 options
lists of preferences :
$A : 1 > 2 > 3$,
$B : 2 > 3 > 1$,
$C : 3 > 1 > 2$

In this situation it is very hard to decide the societal preference in a "democratic way" like the majority rule.

Roughly speaking, this is the reason why Arrow's Impossibility Theorem holds.

2. Arrow's Impossibility Theorem (arrangement version)

$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$: a real central arrangement in $\mathbf{R}^\ell$
$\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$: the set of chambers
H_j : defined by $\alpha_j = 0$
$H_j^+ := \{x \in \mathbf{R}^\ell \mid \alpha_j(x) > 0\}$: a half-space
$H_j^- := \{x \in \mathbf{R}^\ell \mid \alpha_j(x) < 0\}$: the other half-space

$B := \{+, -\}$
$\epsilon_j^\sigma : \mathbf{Ch} \longrightarrow B$ are defined by $\epsilon_j^\sigma(C) = \sigma\tau$ if $C \subseteq H_j^\tau$
$(\sigma, \tau \in B, j = 1, \ldots, n)$

m : a positive integer
$\mathbf{Ch}^m$, B^m : the m-time direct products
$\epsilon_j^\sigma : \mathbf{Ch}^m \rightarrow B^m$ is induced from $\epsilon_j^\sigma : \mathbf{Ch} \rightarrow B$ by
$\epsilon_j^\sigma(C_1, C_2, \ldots, C_m) = (\epsilon_j^\sigma(C_1), \epsilon_j^\sigma(C_2), \ldots, \epsilon_j^\sigma(C_m))$

Definition 1. A map $\Phi : \mathbf{Ch}^m \longrightarrow \mathbf{Ch}$ is called an **admissible map** if there exists a family of maps φ_j^σ $(1 \leq j \leq n,\ \sigma \in B = \{+,-\})$ which satisfies the following two conditions:

(1) $\varphi_j^\sigma(+,+,\ldots,+) = +$, and

(2) the diagram

$$\begin{array}{ccc} \mathbf{Ch}^m & \xrightarrow{\Phi} & \mathbf{Ch} \\ {\scriptstyle \epsilon_j^\sigma}\downarrow & & \downarrow{\scriptstyle \epsilon_j^\sigma} \\ B^m & \xrightarrow{\varphi_j^\sigma} & B \end{array}$$

commutes for each $j, 1 \leq j \leq n$, and $\sigma \in B = \{+,-\}$.

Let $AM(\mathcal{A}, m)$ denote the set of all admissible maps determined by $\mathcal{A}$ and m.

When Φ is an admissible map, a family of maps φ_j^σ $(1 \leq j \leq n, \sigma \in B = \{+, -\})$ satisfying the conditions in Definition 1 is uniquely determined by Φ, $\mathcal{A}$ and m.

Definition 2. For $1 \leq h \leq m$, let
Φ : the projection to the h-th component,
φ_j^σ : the projection to the h-th component.
Then Φ is an admissible map with a family of maps φ_j^σ $(1 \leq j \leq n, \sigma \in B = \{+, -\})$.

We call the admissible maps of this type **projective admissible maps.**

$\mathcal{A}$: the braid arrangement in $\mathbf{R}^\ell$ ($\ell \geq 3$)
$\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq \ell\}$ where $H_{ij} := \ker(x_i - x_j)$
$H_{ij}^+ := \{(x_1, x_2, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_i > x_j\}$
$H_{ij}^- = \{(x_1, x_2, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_i < x_j\}$.

$\mathcal{S}_\ell$: the permutation group of $\{1, 2, \ldots, \ell\}$
Then $\mathbf{Ch}(\mathcal{A}) \leftrightarrow \mathcal{S}_\ell$ (0ne-to-one correspondence) :
Each chamber of $\mathcal{A}$ can be uniquely expressed as
$\{(x_1, x_2, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \ldots < x_{\pi(\ell)}\}$
for a permutation $\pi \in \mathcal{S}_\ell$
Thus

the set of orders of preferences $\leftrightarrow \mathcal{S}_\ell \leftrightarrow \mathbf{Ch}(\mathcal{A})$

$$\mathcal{S}_\ell^m \leftrightarrow \quad \begin{array}{ccc} \mathbf{Ch}^m & \xrightarrow{\Phi} & \mathbf{Ch} \\ \downarrow{\scriptstyle \epsilon_j^\sigma} & & \downarrow{\scriptstyle \epsilon_j^\sigma} \\ B^m & \xrightarrow{\varphi_j^\sigma} & B \end{array} \quad \leftrightarrow \mathcal{S}_\ell$$

Other correspondences are:

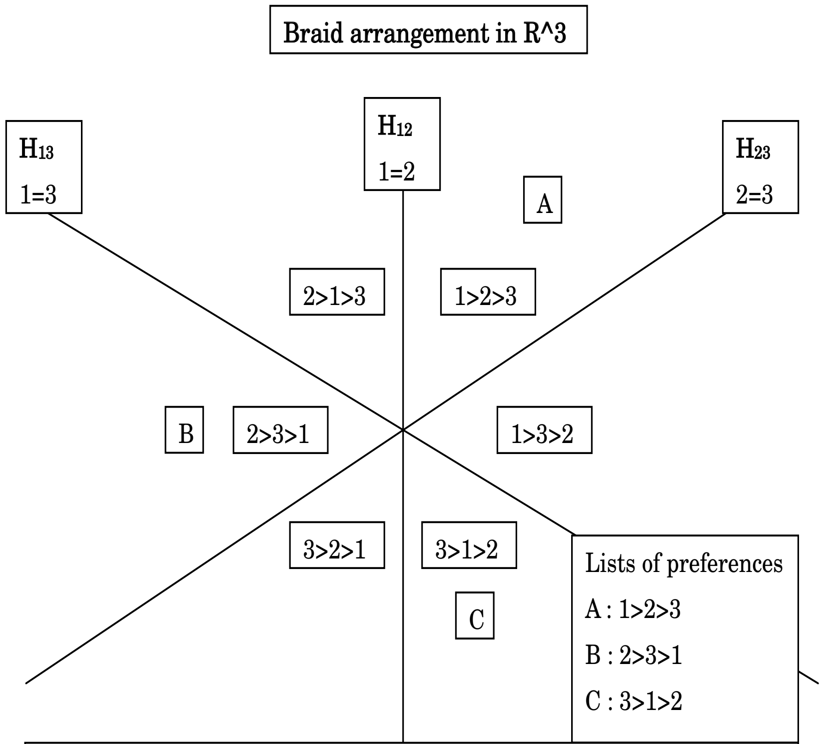
a social welfare function $\leftrightarrow \Phi$
a dictatorship $\leftrightarrow$ the projection to a component

(A) (Pareto property) $\leftrightarrow$ (1) $(\varphi_j^\sigma(+, \ldots, +) = +)$
(B) (pairwise independence) $\leftrightarrow$ (2) (commutativity) $\varphi_j^\sigma \circ \epsilon_j^\sigma = \epsilon_j^\sigma \circ \Phi$ $(\forall j)$

Arrow's impossibility theorem can be formulated as:

Arrow's Impossibility Theorem (arrangement version)
If $\mathcal{A}$ is a braid arrangement with $\ell \geq 3$, then every admissible map is projective.

Condorcet's paradox can be interpreted in terms of arrangements and their chambers:



$(H_{12})^{+}$ ¥cap $(H_{23})^{+}$ ¥cap $(H_{13})^{-}$ would satisfy all of the three, but it is empty. (Condorcet's paradox)

3. A theorem on arramgements

For a central arrangement $\mathcal{A}$, define the rank of $\mathcal{A}$
$r(\mathcal{A}) = \mathrm{codim}_{\mathbf{R}^\ell} \bigcap_{1\le j\le n} H_j$

Definition 3. A central arrangement $\mathcal{A}$ is said to be **decomposable** if there exist nonempty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and $r(\mathcal{A}) = r(\mathcal{A}_1) + r(\mathcal{A}_2)$. In this case, write $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$

A central arrangement $\mathcal{A}$ is said to be **indecomposable** if it is not decomposable.

Remark 1. $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ if and only if the defining polynomials for $\mathcal{A}_1$ and $\mathcal{A}_2$ have no common variables after an appropriate linear coordinate change.

Remark 2. It is also known that $\mathcal{A}$ is decomposable if and only if its characteristic polynomial $\chi(\mathcal{A}, t)$ is divisible by $(t-1)^2$.

An arrangement of only one hyperplane is always indecomposable.

An arrangement of two hyperplanes is always decomposable.

The Boolean arrangement is always decomposable into arrangements with only one hyperplane.

Any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements:

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \ldots \uplus \mathcal{A}_r.$$

The following two theorems completely determine the set $AM(\mathcal{A}, m)$ of admissible maps.

Theorem 1. For a nonempty real central arrangement $\mathcal{A}$ with the decomposition

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \ldots \uplus \mathcal{A}_r.$$

there exists a natural bijection

$$AM(\mathcal{A}, m) \simeq AM(\mathcal{A}_1, m) \times AM(\mathcal{A}_2, m) \times \ldots \times AM(\mathcal{A}_r, m)$$

for each positive integer m.

Theorem 2. Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and m be a positive integer. Then,

(1) if $|\mathcal{A}| = 1$, $AM(\mathcal{A}, m) = \{\Phi : \mathbf{Ch}^m \to \mathbf{Ch} \mid \Phi(C, C, \ldots, C) = C$ for each chamber $C\}$,

(2) if $|\mathcal{A}| \geq 3$, every admissible map is projective.

Corollary. Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as
$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \ldots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \ldots \uplus \mathcal{B}_b$ with $|\mathcal{A}_p| = 1$ $(1 \leq p \leq a)$ and $|\mathcal{B}_q| \geq 3$ $(1 \leq q \leq b)$. Then, for each positive integer m,

$|AM(\mathcal{A}, m)| = (2^{a(2^m-2)})m^b$

4. Implications (?)

What do Theorems 1 and 2 imply?

Theorem 1. For a nonempty real central arrangement $\mathcal{A}$ with the decomposition $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \ldots \uplus \mathcal{A}_r$. there exists a natural bijection
$AM(\mathcal{A}, m) \simeq AM(\mathcal{A}_1, m) \times AM(\mathcal{A}_2, m) \times \ldots \times AM(\mathcal{A}_r, m)$
for each positive integer m.

Theorem 2. Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and m be a positive integer. Then,
(1) if $|\mathcal{A}| = 1$, $AM(\mathcal{A}, m) = \{\Phi : \mathbf{Ch}^m \to \mathbf{Ch} \mid \Phi(C, C, \ldots, C) = C$ for each chamber $C\}$,
(2) if $|\mathcal{A}| \geq 3$, every admissible map is projective.

hyperplane $\leftrightarrow$ a political issue

arrangement $\leftrightarrow$ a set of political issues

$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \ldots \uplus \mathcal{A}_r$. $\leftrightarrow$ a set of political issues is grouped into certain subsets

For each $\mathcal{A}_i$ with ($|\mathcal{A}_i| \geq 3$), there is a "mini-dictator."

For each $\mathcal{A}_i$ with ($|\mathcal{A}_i| = 1$), any voting system (e. g., the simple majority rule) works as long as the unanimous decision is respected.

This is random thoughts which might mean nothing.

However, Theorems mean something mathematically.

I stop here. Thank you!