Chambers of Arrangements and Arrow’s Impossibility Theorem

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at

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1. Basic concepts about hyperplane arrangements

A (central) hyperplane arrangement $\mathcal{A}$ is:

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

in an $\ell$-dimensional vector space $V$ over a field $\mathbb{K}$ defined by $H_i = \ker(\alpha_i)$ with $\alpha_i \in V^*(1 \leq i \leq n)$. 

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When \( K = \mathbb{R} \) (the real number field), the connected components of

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- Intersection lattice

Let

\[ L(\mathcal{A}) = \{ \text{all intersections of hyperplanes belonging to } \mathcal{A} \} \]

\[ = \{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \} \]

and introduce a partial order by \( X \geq Y \Leftrightarrow X \subseteq Y \) to make \( L(\mathcal{A}) \) a partially ordered set.

[Agree that \( L(\mathcal{A}) \) has the minimum \( V \).

Then \( L(\mathcal{A}) \) is called the intersection lattice.
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- Möbius function

Define

\[ \mu : L(\mathcal{A}) \rightarrow \mathbb{Z} \]

by

\[ \mu(V) := 1, \quad \mu(X) := - \sum_{Y < X} \mu(Y). \]

- Poincaré polynomial

Define the Poincaré polynomial

\[ \pi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\text{codim} X}. \]
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- **Factorization Theorem**

  Theorem. (H. T. 1981). Suppose that \( \mathcal{A} \) is a free arrangement in \( \mathbb{C}^\ell \) with exponents \( d_1, d_2, \ldots, d_\ell \). Then

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  \pi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (1 + d_i t).
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- **Zaslavsky’s Chamber-Counting Formula**

  Theorem. (Thomas Zaslavsky 1975).

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  |\text{Chambers}| = \pi(\mathcal{A}, 1).
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Catalan arrangement of type $B_2$ is free with exponents $(1, 5, 7)$

The number of chambers is

$$\pi(\mathcal{A}, 1) = (1 + 1 \times 1)(1 + 5 \times 1)(1 + 7 \times 1) = 96$$

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A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference.

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- The Two Requirements

  (A) the society prefers the option \( i \) to the option \( j \) if every individual prefers the option \( i \) to the option \( j \) (Pareto property),

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Why is Arrow’s theorem true?

What is the reason behind Arrow’s theorem?

Condorcet’s paradox by Marquis Condorcet (1743-94)

A, B, C : 3 people, 1, 2, 3 : 3 options

lists of preferences:

A : 1 > 2 > 3,
B : 2 > 3 > 1,
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In this situation it is very hard to decide the societal preference in a “democratic way” like the majority rule.

Roughly speaking, this is the reason why Arrow’s Impossibility Theorem holds.
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\[ \text{Ch} = \text{Ch}(\mathcal{A}) : \text{the set of chambers} \]

\[ H_j : \text{defined by } \alpha_j = 0 \]

\[ H_j^+ := \{x \in \mathbb{R}^\ell \mid \alpha_j(x) > 0\} : \text{a half-space} \]

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\[ B := \{+, -, 0\} \]

\[ \epsilon^\sigma_j : \text{Ch} \rightarrow B \text{ are defined by } \epsilon^\sigma_j(C) = \sigma \tau \text{ if } C \subseteq H_j^\tau \]

\[ (\sigma, \tau \in B, j = 1, \ldots, n) \]

\[ m : \text{a positive integer} \]

\[ \text{Ch}^m, B^m : \text{the } m\text{-time direct products} \]

\[ \epsilon^\sigma_j : \text{Ch}^m \rightarrow B^m \text{ is induced from } \epsilon^\sigma_j : \text{Ch} \rightarrow B \text{ by} \]

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\[ B := \{+, -, \} \]
\[ \epsilon_j^\sigma : \text{Ch} \rightarrow B \text{ are defined by } \epsilon_j^\sigma(C) = \sigma \tau \text{ if } C \subseteq H_j^\tau \]
\[ (\sigma, \tau \in B, j = 1, \ldots, n) \]
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\[ \text{Ch}^m, B^m : \text{the } m\text{-time direct products} \]
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\[ \epsilon_j^\sigma(C_1, C_2, \ldots, C_m) = (\epsilon_j^\sigma(C_1), \epsilon_j^\sigma(C_2), \ldots, \epsilon_j^\sigma(C_m)) \]
3. Arrow’s Impossibility Theorem (arrangement version)

\( A = \{H_1, H_2, \ldots, H_n\} \) : a real central arrangement in \( \mathbb{R}^\ell \)

\( \text{Ch} = \text{Ch}(A) \) : the set of chambers

\( H_j \) : defined by \( \alpha_j = 0 \)

\( H_j^+ := \{x \in \mathbb{R}^\ell \mid \alpha_j(x) > 0\} \) : a half-space

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\( m \) : a positive integer

\( \text{Ch}^m, B^m \) : the \( m \)-time direct products

\( \epsilon_j^\sigma : \text{Ch}^m \rightarrow B^m \) is induced from \( \epsilon_j^\sigma : \text{Ch} \rightarrow B \) by

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3. Arrow’s Impossibility Theorem (arrangement version)

\[ \mathcal{A} = \{H_1, H_2, \ldots, H_n\} : \text{a real central arrangement in } \mathbb{R}^\ell \]

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$\epsilon^\sigma_j : \text{Ch} \rightarrow B$ are defined by $\epsilon^\sigma_j(C) = \sigma \tau$ if $C \subseteq H^+_j$

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3. Arrow’s Impossibility Theorem (arrangement version)

- Definition 1.

A map $\Phi : \text{Ch}^m \rightarrow \text{Ch}$ is called an **admissible map** if there exists a family of maps $\varphi_{j}^{\sigma} : B^m \rightarrow B$ ($1 \leq j \leq n$, $\sigma \in B = \{+, -\}$) which satisfies the following two conditions:

1. $\varphi_{j}^{\sigma}(+, +, \ldots, +) = +$, and
2. the diagram

\[
\begin{array}{ccc}
\text{Ch}^m & \xrightarrow{\Phi} & \text{Ch} \\
\downarrow{\varepsilon_{j}^{\sigma}} & & \downarrow{\varepsilon_{j}^{\sigma}} \\
B^m & \xrightarrow{\varphi_{j}^{\sigma}} & B
\end{array}
\]

commutes for each $j$, $1 \leq j \leq n$, and $\sigma \in B = \{+, -\}$. 
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Definition 1.

A map \( \Phi : \text{Ch}^m \longrightarrow \text{Ch} \) is called an \textit{admissible map} if there exists a family of maps \( \varphi_j^\sigma : B^m \longrightarrow B \) \( (1 \leq j \leq n, \ \sigma \in B = \{+, -\}) \) which satisfies the following two conditions:

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\[
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commutes for each \( j, 1 \leq j \leq n \), and \( \sigma \in B = \{+, -\} \).
3. Arrow’s Impossibility Theorem (arrangement version)

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A map $\Phi : \text{Ch}^m \rightarrow \text{Ch}$ is called an admissible map if there exists a family of maps $\varphi^\sigma_j : B^m \rightarrow B$ ($1 \leq j \leq n$, $\sigma \in B = \{+, -\}$) which satisfies the following two conditions:

1. $\varphi^\sigma_j(+, +, \ldots, +) = +$, and

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\[
\begin{array}{ccc}
\text{Ch}^m & \xrightarrow{\Phi} & \text{Ch} \\
\downarrow \epsilon^\sigma_j & & \downarrow \epsilon^\sigma_j \\
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\end{array}
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commutes for each $j$, $1 \leq j \leq n$, and $\sigma \in B = \{+, -\}$. 
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1. $\varphi_j^\sigma(+, +, \ldots, +) = +$, and
2. the diagram

```
\begin{align*}
\text{Ch}^m & \xrightarrow{\Phi} \text{Ch} \\
\downarrow \text{e}_j^\sigma & \quad \quad \quad \quad \quad \quad \quad \downarrow \text{e}_j^\sigma \\
B^m & \xrightarrow{\varphi_j^\sigma} B
\end{align*}
```

commutes for each $j, 1 \leq j \leq n$, and $\sigma \in B = \{+, -\}$.
3. Arrow’s Impossibility Theorem (arrangement version)

- Definition 1 (continuing).

Let $AM(\mathcal{A}, m)$ denote the set of all admissible maps determined by $\mathcal{A}$ and $m$.

When $\Phi$ is an admissible map, a family of maps $\varphi^\sigma_j (1 \leq j \leq n, \sigma \in B = \{+, -\})$ satisfying the conditions in Definition 1 is uniquely determined by $\Phi$, $\mathcal{A}$ and $m$. 
Definition 1 (continuing).

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When $\Phi$ is an admissible map, a family of maps $\varphi^\sigma_j$ ($1 \leq j \leq n, \sigma \in B = \{+, -\}$) satisfying the conditions in Definition 1 is uniquely determined by $\Phi$, $\mathcal{A}$ and $m$. 
3. Arrow’s Impossibility Theorem (arrangement version)

Definition 2.
For $1 \leq h \leq m$, let $\Phi :$ the projection to the $h$-th component, $\varphi^\sigma_j :$ the projection to the $h$-th component.

Then $\Phi$ is an admissible map with a family of maps
$\varphi^\sigma_j$ ($1 \leq j \leq n, \sigma \in B = \{+, -\}$).

We call the admissible maps of this type projective admissible maps.
Definition 2.

For $1 \leq h \leq m$, let $\Phi :$ the projection to the $h$-th component, $\varphi^\sigma_j :$ the projection to the $h$-th component. Then $\Phi$ is an admissible map with a family of maps $\varphi_j^\sigma$ ($1 \leq j \leq n, \sigma \in B = \{+,-\}$).

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Definition 2.

For $1 \leq h \leq m$, let $\Phi :$ the projection to the $h$-th component, $\varphi_j^{\sigma} :$ the projection to the $h$-th component. Then $\Phi$ is an admissible map with a family of maps $\varphi_j^{\sigma} (1 \leq j \leq n, \sigma \in B = \{+, -\})$.

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3. Arrow’s Impossibility Theorem (arrangement version)

Definition 2.

For $1 \leq h \leq m$, let $\Phi$ : the projection to the $h$-th component, $\varphi^\sigma_j$ : the projection to the $h$-th component. Then $\Phi$ is an admissible map with a family of maps $\varphi^\sigma_j$ ($1 \leq j \leq n, \sigma \in B = \{+, -, \}$).
We call the admissible maps of this type **projective admissible maps**.
3. Arrow’s Impossibility Theorem (arrangement version)

- The Braid Arrangement Case

\( \mathcal{A} : \) the braid arrangement in \( \mathbb{R}^\ell \) \((\ell \geq 3)\)

\( \mathcal{A} = \{ H_{ij} \mid 1 \leq i < j \leq \ell \} \) where \( H_{ij} := \ker(x_i - x_j) \)

\( H_{ij}^+ := \{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i > x_j \} \)

\( H_{ij}^- = \{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i < x_j \} \).

\( S_\ell : \) the permutation group of \( \{1, 2, \ldots, \ell\} \)

Then \( \text{Ch}(\mathcal{A}) \leftrightarrow S_\ell \) (One-to-one correspondence):

Each chamber of \( \mathcal{A} \) can be uniquely expressed as

\( \{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(\ell)} \} \) for a permutation \( \pi \in S_\ell \)

Thus the set of orders of preferences \( \leftrightarrow S_\ell \leftrightarrow \text{Ch}(\mathcal{A}) \)
3. Arrow’s Impossibility Theorem (arrangement version)

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\( \mathcal{A} = \{ H_{ij} \mid 1 \leq i < j \leq \ell \} \) where \( H_{ij} := \ker(x_i - x_j) \)

\( H_{ij}^+ := \{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i > x_j \} \)

\( H_{ij}^- = \{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_i < x_j \}. \)

\( S_\ell : \) the permutation group of \( \{1, 2, \ldots, \ell\} \)

Then \( \text{Ch}(\mathcal{A}) \leftrightarrow S_\ell \) (One-to-one correspondence):

Each chamber of \( \mathcal{A} \) can be uniquely expressed as

\( \{(x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(\ell)} \} \) for a permutation \( \pi \in S_\ell \)

Thus the set of orders of preferences \( \leftrightarrow S_\ell \leftrightarrow \text{Ch}(\mathcal{A}) \)
3. Arrow’s Impossibility Theorem (arrangement version)

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\[ S^m_\ell \leftrightarrow \text{Ch}^m \xrightarrow{\Phi} \text{Ch} \leftrightarrow S_\ell \]

Other correspondences are:

- a social welfare function \( \leftrightarrow \Phi \)
- a dictatorship \( \leftrightarrow \) the projection to a component
- (A) (Pareto property) \( \leftrightarrow (1) (\varphi^\sigma_j (+, \ldots, +) = +) \)
- (B) (pairwise independence) \( \leftrightarrow (2) \) (commutativity)

\[ \varphi^\sigma_j \circ \epsilon^\sigma_j = \epsilon^\sigma_j \circ \Phi \ (\forall j) \]
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Arrow’s impossibility theorem can be formulated as:

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Braid arrangement in $\mathbb{R}^3$

$H_{13}$
$1=3$

$H_{12}$
$1=2$

$H_{23}$
$2=3$

$2>1>3$

$1>2>3$

$2>3>1$

$1>3>2$

$3>2>1$

$3>1>2$

Lists of preferences

A: $1>2>3$
B: $2>3>1$
C: $3>1>2$

$(H_{12})^+ \cap (H_{23})^+ \cap (H_{13})^-$ would satisfy all of the three, but it is empty. (Condorcet’s paradox)
Decomposability/Indecomposability of an Arrangement

For a central arrangement $\mathcal{A}$, define the rank of $\mathcal{A}$

$$ r(\mathcal{A}) = \text{codim}_{\mathbb{R}^\ell} \bigcap_{1 \leq j \leq n} H_j $$

**Definition 3.** A central arrangement $\mathcal{A}$ is said to be **decomposable** if there exist nonempty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and $r(\mathcal{A}) = r(\mathcal{A}_1) + r(\mathcal{A}_2)$. In this case, write $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2$

A central arrangement $\mathcal{A}$ is said to be **indecomposable** if it is not decomposable.
4. Two theorems on arrangements

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Remark 1. $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ if and only if the defining polynomials for $\mathcal{A}_1 \neq \emptyset$ and $\mathcal{A}_2 \neq \emptyset$ have no common variables after an appropriate linear coordinate change.

Remark 2. It is also known that $\mathcal{A}$ is decomposable if and only if its Poincaré polynomial $\pi(\mathcal{A}, t)$ is divisible by $(t + 1)^2$.

An arrangement of only one hyperplane is always indecomposable.

An arrangement of two hyperplanes is always decomposable.

The Boolean arrangement is always decomposable into arrangements with only one hyperplane.
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Any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements:

$$\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_r.$$
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**Theorem 2.** Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and $m$ be a positive integer. Then,

1. if $|\mathcal{A}| = 1$, $AM(\mathcal{A}, m) = \{\Phi : \text{Ch}^m \to \text{Ch} | \Phi(C, C, \ldots, C) = C \}$ for each chamber $C$,
2. if $|\mathcal{A}| \geq 3$, every admissible map is projective.
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Corollary. Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as 
$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_a \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_b$ with 
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Then, for each positive integer $m$, 

$$|AM(\mathcal{A}, m)| = (2^{a(2^m-2)})^b$$
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5. Implications

What do Theorems 1 and 2 imply?

**Theorem 1.** For a nonempty real central arrangement $\mathcal{A}$ with the decomposition $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_r$, there exists a natural bijection

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for each positive integer $m$.

**Theorem 2.** Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and $m$ be a positive integer. Then,

1. if $|\mathcal{A}| = 1$, $AM(\mathcal{A}, m) = \{ \Phi : \text{Ch}^m \rightarrow \text{Ch} | \Phi(C, C, \ldots, C) = C \text{ for each chamber } C \}$,
2. if $|\mathcal{A}| \geq 3$, every admissible map is projective.
5. Implications

hyperplane $\leftrightarrow$ a political issue

arrangement $\leftrightarrow$ a set of political issues

$\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \cdots \sqcup \mathcal{A}_r$.  $\leftrightarrow$ a set of political issues is grouped into certain subsets

For each $\mathcal{A}_i$ with ($|\mathcal{A}_i| \geq 3$), there is a "mini-dictator."

For each $\mathcal{A}_i$ with ($|\mathcal{A}_i| = 1$), any voting system (e.g., the simple majority rule) works as long as unanimous decisions are respected.
5. Implications

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5. Implications

- hyperplane ↔ a political issue
- arrangement ↔ a set of political issues
- $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r$. ↔ a set of political issues is grouped into certain subsets

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\[ \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r. \leftrightarrow \text{a set of political issues is grouped into certain subsets} \]

For each \( \mathcal{A}_i \) with \( |\mathcal{A}_i| \geq 3 \), there is a “mini-dictator.”

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I stop here.
Thank you!
5. Implications

I stop here.

Thank you!
5. Implications

I stop here.
Thank you!