Chambers of Arrangements and Arrow's Impossibility Theorem

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at

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1. Basic concepts about hyperplane arrangements

- Hyperplane Arrangement

A (central) hyperplane arrangement $\mathcal{A}$ is:

$$\mathcal{A} := \{H_1, \ldots, H_n\}$$

in an ℓ-dimensional vector space V over a field $\mathbb{K}$ defined by $H_i = \ker(\alpha_i)$ with $\alpha_i \in V^*(1 \leq i \leq n)$.

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Let

$$L(\mathcal{A}) = \{\text{all intersections of hyperplanes belonging to } \mathcal{A}\}$$
$$= \{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$$

and introduce a partial order by $X \geq Y \Leftrightarrow X \subseteq Y$ to make $L(\mathcal{A})$ a partially ordered set.
[Agree that $L(\mathcal{A})$ has the minimum V.]
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- Möbius function

Define

$$\mu : L(\mathcal{A}) \to \mathbb{Z}$$

by

$$\mu(V) := 1, \quad \mu(X) := -\sum_{Y<X} \mu(Y).$$

- Poincaré polynomial

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  Theorem. (H. T. 1981). Suppose that $\mathcal{A}$ is a free arrangement in $\mathbb{C}^\ell$ with exponents $d_1, d_2, \ldots, d_\ell$. Then

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(1 + d_i t).$$

- Zaslavsky's Chamber-Counting Formula

Theorem.(Thomas Zaslavsky 1975).

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The number of chambers is
$\pi(\mathcal{A}, 1) = (1 + 1 \times 1)(1 + 5 \times 1)(1 + 7 \times 1) = 96$

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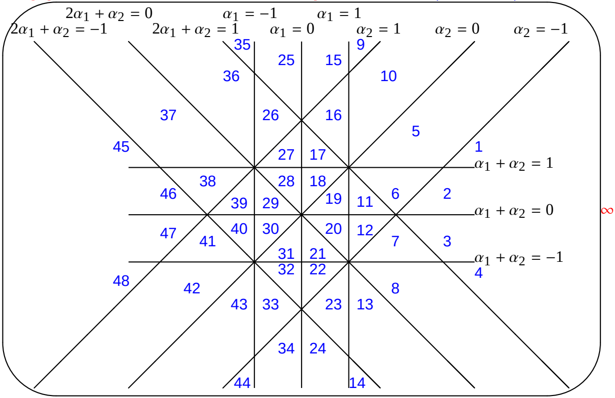

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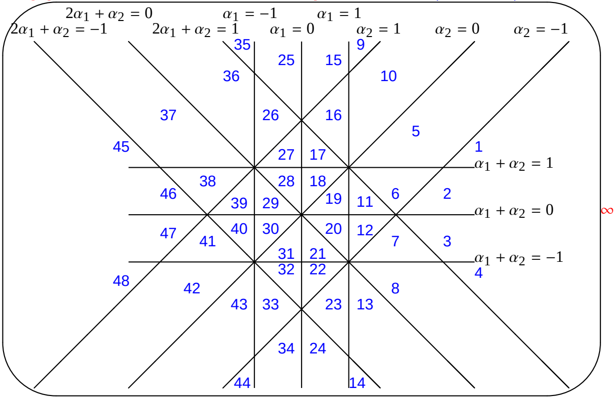

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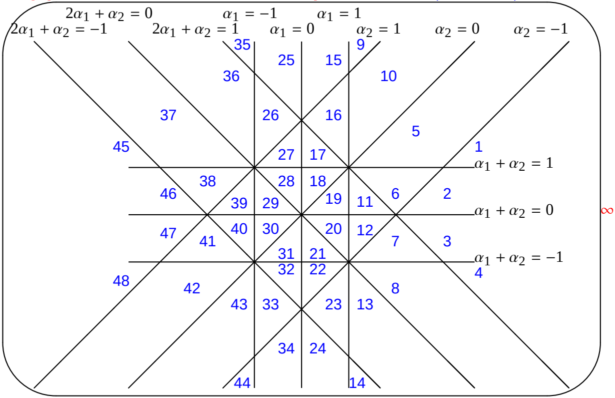

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Assume that a society of m people have ℓ policy options and that every individual has his/her own order of preferences on the ℓ policy options.

A social welfare function can be interpreted as a voting system by which the individual preferences are aggregated into a single societal preference.

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Arrow's Impossibility Theorem (Kenneth Arrow 1950).
For $\ell \geq 3$, the only social welfare function satisfying the two requirements (A) and (B) is a dictatorship, that is, the societal preference has to be equal to the preference of one particular individual.

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- Why is Arrow's theorem true?

What is the reason behind Arrow's theorem?

Condorcet's paradox by Marquis Condorcet (1743-94)

A, B, C : 3 people, $1, 2, 3$: 3 options

lists of preferences :

$A : 1 > 2 > 3,$

$B : 2 > 3 > 1,$

$C : 3 > 1 > 2$

In this situation it is very hard to decide the societal preference in a "democratic way" like the majority rule.

Roughly speaking, this is the reason why Arrow's Impossibility Theorem holds.

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3. Arrow's Impossibility Theorem (arrangement version)

$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$: a real central arrangement in $\mathbf{R}^\ell$
$\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$: the set of chambers
H_j : defined by $\alpha_j = 0$
$H_j^+ := \{x \in \mathbf{R}^\ell \mid \alpha_j(x) > 0\}$: a half-space
$H_j^- := \{x \in \mathbf{R}^\ell \mid \alpha_j(x) < 0\}$: the other half-space
$B := \{+, -\}$
$\epsilon_j^\sigma : \mathbf{Ch} \longrightarrow B$ are defined by $\epsilon_j^\sigma(C) = \sigma\tau$ if $C \subseteq H_j^\tau$
$(\sigma, \tau \in B, j = 1, \ldots, n)$
m : a positive integer
$\mathbf{Ch}^m$, B^m : the m-time direct products
$\epsilon_j^\sigma : \mathbf{Ch}^m \to B^m$ is induced from $\epsilon_j^\sigma : \mathbf{Ch} \to B$ by
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3. Arrow's Impossibility Theorem (arrangement version)

$\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$: a real central arrangement in $\mathbf{R}^\ell$
$\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$: the set of chambers
H_j : defined by $\alpha_j = 0$
$H_j^+ := \{x \in \mathbf{R}^\ell \mid \alpha_j(x) > 0\}$: a half-space
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- Definition 1.

A map $\Phi : \mathbf{Ch}^m \longrightarrow \mathbf{Ch}$ is called an admissible map if there exists a family of maps $\varphi_j^\sigma : B^m \longrightarrow B$ $(1 \leq j \leq n, \ \ \sigma \in B = \{+, -\})$ which satisfies the following two conditions:
(1) $\varphi_j^\sigma(+, +, \ldots, +) = +$, and
(2) the diagram

$$\begin{array}{ccc} \mathbf{Ch}^m & \xrightarrow{\Phi} & \mathbf{Ch} \\ \downarrow{\scriptstyle \epsilon_j^\sigma} & & \downarrow{\scriptstyle \epsilon_j^\sigma} \\ B^m & \xrightarrow{\varphi_j^\sigma} & B \end{array}$$

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3. Arrow's Impossibility Theorem (arrangement version)

- Definition 1 (continuing).

Let $AM(\mathcal{A}, m)$ denote the set of all admissible maps determined by $\mathcal{A}$ and m.
When Φ is an admissible map, a family of maps φ_j^σ $(1 \leq j \leq n, \sigma \in B = \{+, -\})$ satisfying the conditions in Definition 1 is uniquely determined by Φ, $\mathcal{A}$ and m.

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3. Arrow's Impossibility Theorem (arrangement version)

- Definition 2.

For $1 \leq h \leq m$, let Φ : the projection to the h-th component,
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Then Φ is an admissible map with a family of maps
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We call the admissible maps of this type **projective admissible maps.**

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3. Arrow's Impossibility Theorem (arrangement version)

- The Braid Arrangement Case

$\mathcal{A}$: the braid arrangement in $\mathbf{R}^\ell$ ($\ell \geq 3$)
$\mathcal{A} = \{H_{ij} \mid 1 \leq i < j \leq \ell\}$ where $H_{ij} := \ker(x_i - x_j)$
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$\mathcal{S}_\ell$: the permutation group of $\{1, 2, \ldots, \ell\}$
Then $\mathbf{Ch}(\mathcal{A}) \leftrightarrow \mathcal{S}_\ell$ (One-to-one correspondence) :
Each chamber of $\mathcal{A}$ can be uniquely expressed as
$\{(x_1, x_2, \ldots, x_\ell) \in \mathbf{R}^\ell \mid x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(\ell)}\}$ for a permutation $\pi \in \mathcal{S}_\ell$
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Other correspondences are:
a social welfare function $\leftrightarrow \Phi$
a dictatorship $\leftrightarrow$ the projection to a component
(A) (Pareto property) $\leftrightarrow$ (1) ($\varphi_j^\sigma(+,\ldots,+) = +$)
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$\varphi_j^\sigma \circ \epsilon_j^\sigma = \epsilon_j^\sigma \circ \Phi$ ($\forall j$)

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\mathcal{S}_\ell^m \leftrightarrow \quad
\begin{array}{ccc}
\mathbf{Ch}^m & \xrightarrow{\ \Phi\ } & \mathbf{Ch} \\
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\quad \leftrightarrow \mathcal{S}_\ell
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Other correspondences are:

a social welfare function $\leftrightarrow \Phi$

a dictatorship $\leftrightarrow$ the projection to a component

(A) (Pareto property) $\leftrightarrow$ (1) ($\varphi_j^\sigma(+,\ldots,+) = +$)

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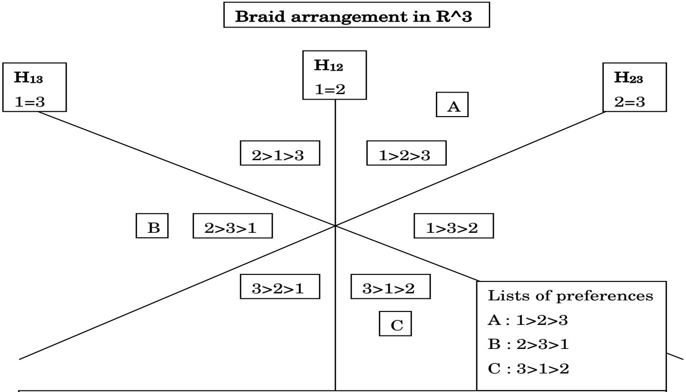

$(H_{12})^{+}$ ¥cap $(H_{23})^{+}$ ¥cap $(H_{13})^{-}$ would satisfy all of the three, but it is empty. (Condorcet's paradox)

4. Two theorems on arramgements

- Decomposability/Indecomposability of an Arrangement

For a central arrangement $\mathcal{A}$, define the rank of $\mathcal{A}$
$r(\mathcal{A}) = \mathrm{codim}_{\mathbf{R}^\ell} \bigcap_{1 \le j \le n} H_j$

Definition 3. A central arrangement $\mathcal{A}$ is said to be **decomposable** if there exist nonempty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ (disjoint) and $r(\mathcal{A}) = r(\mathcal{A}_1) + r(\mathcal{A}_2)$. In this case, write $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$

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Remark 1. $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2$ if and only if the defining polynomials for $\mathcal{A}_1 \neq \emptyset$ and $\mathcal{A}_2 \neq \emptyset$ have no common variables after an appropriate linear coordinate change.

Remark 2. It is also known that $\mathcal{A}$ is decomposable if and only if its Poincaré polynomial $\pi(\mathcal{A}, t)$ is divisible by $(t+1)^2$.

An arrangement of only one hyperplane is always indecomposable.

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Any nonempty real central arrangement $\mathcal{A}$ can be uniquely (up to order) decomposed into nonempty indecomposable arrangements:

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Theorem 2. Let $\mathcal{A}$ be a nonempty indecomposable real central arrangement and m be a positive integer. Then,
(1) if $|\mathcal{A}| = 1$, $AM(\mathcal{A}, m) = \{\Phi : \mathbf{Ch}^m \to \mathbf{Ch} \mid \Phi(C, C, \ldots, C) = C$ for each chamber $C\}$,
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Corollary. Decompose a nonempty real central arrangement $\mathcal{A}$ into nonempty indecomposable arrangements as $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots \uplus \mathcal{A}_a \uplus \mathcal{B}_1 \uplus \mathcal{B}_2 \uplus \cdots \uplus \mathcal{B}_b$ with $|\mathcal{A}_p| = 1$ $(1 \le p \le a)$ and $|\mathcal{B}_q| \ge 3$ $(1 \le q \le b)$.
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This work appeared in Advances in Math. 214 (2007) 366–378

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