Free Arrangements
and
Reflection Arrangements
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Free arrangements and reflection arrangements (and $K(\Pi, 1)$ arrangements) are connected for some mysterious reasons as observed since the inception of the study of free arrangements.
I Definitions
II Examples
III Fundamental results and questions
I Definitions

Def. 1. $\mathcal{A}$ is an arrangement of hyperplanes.

$\vdash$

$\mathcal{A}$ is a finite collection of hyperplanes (one-codim subspaces) of a vector space $V/K(\mathbb{R}^{n|0})$. 
Def 2. $S := \text{the symmetric algebra of } V^*$

$\cong K[x_1, \ldots, x_e]$

($x_1, \ldots, x_e$ are a basis for $V^*$)

$Ders_S := \text{the } S\text{-module of all } K\text{-linear derivations}$
\[ \text{Dens} = \{ \theta : S \to S \} \]

\( \theta \) is \( K \)-linear and
\[ \theta(fg) = f \theta(g) + g \theta(f) \quad \text{for} \]
\[ \forall f, g \in S \]

Then
\[ \text{Dens} = \bigoplus_{i=1}^{\ell} S \left( \frac{e_i}{\text{free}} \right) : \text{free} \quad \text{S-module} \]
\[ D(\mathcal{A}) := \{ \Theta \in \text{Dens} \mid \Theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \} \]

\[ (\ker(\alpha_H) = H, \alpha_H \in V^*) \]
A is a free arrangement

$D(A)$ is a free $S$-module.

When $A$ is free, we have

$\exp(A) = (\deg \Theta_1, \ldots, \deg \Theta_e)$

where $\Theta_1, \ldots, \Theta_e$ are a homogeneous basis for $D(A)$. 
For example, unless $\mathbf{A} = \Phi$, $\Theta_E := \sum_i \chi_i (\partial \chi_i) \in D(\mathbf{A})$

the Euler derivation

$\deg \Theta_E := 1$

$\deg \Theta := \deg \Theta(\chi) \text{ for } \chi \in V^* \text{ with } \Theta(\chi) \neq 0.$
II. Examples

Example 1. (Kyoji Saito '70s / IR) T '80 (/IC)

$W$: reflection group. Then

$$\text{Den}_S^{w} \otimes_{w} S = D(A(W))$$

where

$A(W) = \{\text{all reflecting hyperplanes}\}$.
Therefore $A(W)$ is a free arrangement.

Example 2 (T'80) Supersolvable (fiber-type) arrangements are free.

\[ \text{C \ (4 pts. \ fiber) } \exp (\text{(1,3,4)}) \text{ (C \ (3 pts. \ base)} \]
Fact 1. Ex. 1 & 2 are both known to be $K(11,1)$ (Bevis '14, Jambu-T '84, T '86)

Fact 2. In the Grünbaum's table of simplicial arrangements of planes, most of them are free. The first non-free is $A_{413}$. 
$K(\pi, 1)$

free

refl. arr.

SS many simplicial

$x$

skewed $B_3$

$A_4(13)$ (fixed by Deligne)

"large" overlapping

Why?
Chapter 3: Fundamental Facts

Theorem (Factorization, T'81)

\[ L(\mathcal{A}) := \left\{ \bigcap H \mid \exists B \subseteq \mathcal{A} \right\} \]

Agree \[ V = \bigcap_{H \in \mathcal{A}} H \]

\[ X \leq Y \iff X \not\subseteq Y \]

\[ L(\mathcal{A}) : \text{the intersection lattice} \]
\[ \Pi(A, t) := \sum_{X \in L(A)} \mu(\nu, X) t \]

\( X \) \in \text{the Moebius fcn}

\[ \exp(A) = (d_1, \ldots, d_e) \]

\[ \Pi(A, t) = \prod_{i=1}^e (1 + d_i t) \]
So the freeness imposes a combinatorial constraint.

(0) cohomological when $k = \mathbb{C}$

because Poincaré poly.

\[ \pi(A, t) = \pi(V \setminus U H, t) \]

To construct free arrangements from scratch we will frequently use:

\[ \pi(A, t) = \pi(V \setminus U H, t) \]
Theorem 2. (Addition - Deletion $\text{ T'}80$) 16

$H \subseteq \mathcal{A}$, $\mathcal{A}' = \mathcal{A} \setminus \{H\}$, 
$\mathcal{A}'' = \mathcal{A}^H = \{H \cap K | K \subseteq \mathcal{A}'\}$

Let

1. $\exp \mathcal{A}' = (d_1, \ldots, d_e)$
2. $\exp \mathcal{A} = (d_1, \ldots, d_i - 1, \ldots, d_e)$
3. $\exp \mathcal{A}'' = (d_1, \ldots, \hat{d_i}, \ldots, d_e)$

Then

$$(1) + (3) \Rightarrow (2) \quad \text{Addition}$$

$$(2) + (3) \Rightarrow (1) \quad \text{Deletion}.$$
Theorem 3 (T. Abe 2016)

\[ \pi^A = \text{free} \]
\[ \pi^A \mid \pi^A \]
\[ \Rightarrow \pi^A = \text{free} \]

[No need of the freeness of \( A' \)]

\[ \text{big improvement} \]
Question 1.
Any topological (homotopical) constraint by the freeness (other than Betti numbers)?

Question 2.
Any new insight concerning reflection groups by the freeness?
Theorem 4. (Hoge-Röhrle ’13)
All restrictions of any finite complex reflection arrangement are free.

Theorem 5. (Abe-Barakat-Cuntz-Hoge-T ’16)
All ideal subarrangements of any Weyl arr. are free (and proof of the dual partition)
To prove Th. 5, we used: $(20$ Theorem 6 (Multiple Addition Theorem)

$A$: free, $\exp(A) = (1, d_1, \cdots, d_{e-1}) \leq B = \{H_{e-9}, \cdots, H_{e-1}\}$: Boolean $\forall \beta \in U \beta$, $A^+ = A \cup B$

$A^+_j = A \cup \{H_j, \cdots, H_{q+1}\}$

$|A_j^+| - 1 \geq d_j (H_j)$

$\Rightarrow \exp(A^+) = (1, d_1, \cdots, d_{e-q+1}, d_{e-q+1}, \cdots, d_{e-1+1})$
\[(1, 1, 1) \rightarrow (1, 2, 2)\]
\[x_1 = 0 \quad (\text{h+1}) \quad x_1 + x_2 = 0 \quad (\text{h+2})\]
\[x_2 = 0 \quad (\text{h+1}) \quad x_2 + x_3 = 0 \quad (\text{h+2})\]
\[x_3 = 0 \quad (\text{h+1}) \quad x_1 + x_2 + x_3 = 0 \quad (\text{h+3})\]

"the dual partition"

(Arnold-Steinberg-Kostant-... ) \[(1, 2, 3)\]

(type A_3)
Theorem 7 (Multiple Deletion Theorem)

\[ A: \text{ free, } \exp(A) = (1, d_1, \ldots, d_{e-1}) \leq \]
\[ B = \{ H_1, \ldots, H_q \}: \text{ Boolean} \]
\[ A^- := A \setminus B, \quad \forall B \in \Omega(A) \]
\[ A_j^- := A \setminus \{ H_1, \ldots, H_j \} \]
\[ \left| \bigwedge_{j=1}^{q-1} \left( A_j^- \right)^{H_j} \right| \leq d_j (4j) \]
\[ \Rightarrow \exp(A^-) = (1, d_1, \ldots, d_{e-1}, d_1+1, \ldots, d_{e-1}) \]
The assumptions of MAT is purely combinatorial. So it is still natural to ask:

Question 3. Is the fineness a combinatorial property? (T's conjecture)