Multiple addition theorem on arrangements of hyperplanes and a proof of the Shapiro-Steinberg-Kostant-Macdonald dual-partition formula

Hiroaki Terao

(Hokkaido University, Sapporo, Japan)

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Credit

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with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

Torsten Hoge (Leibniz Universität Hannover)

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Arrangements

- An arrangement [of hyperplanes] $\mathcal{A}$ is a finite collection of $(\ell - 1)$-dimensional vector subspaces in an ℓ-dimensional vector space V over a field $\mathbb{K}$:

$$\mathcal{A} = \{H_1, \ldots, H_n\}$$

defined by $H_i = \ker(\alpha_i)$ with $\alpha_i \in V^*(1 \leq i \leq n)$.

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- Define a graded S-module

$$D(\mathcal{A}) := \{\theta \mid \theta \text{ is an } \mathbb{R}\text{-linear derivation with } \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free S-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

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Example.
(the braid arrangement (Weyl arrangement of type A_3))

$$\mathcal{A} := \{\ker(x_i - x_j) \mid 1 \le i < j \le 4\}$$

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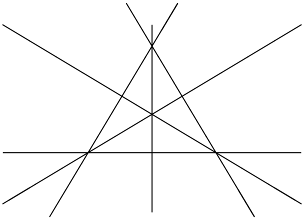

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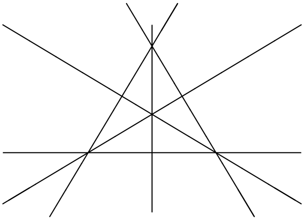

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The exponents are

$$(0, 1, 2, 3)$$

because ...

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The S-module $D(\mathcal{A})$ is a free module with a basis

$$\begin{aligned}
\theta_0 &= (\partial/\partial x_1) + (\partial/\partial x_2) + (\partial/\partial x_3) + (\partial/\partial x_4) \\
\theta_1 &= x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3) + x_4(\partial/\partial x_4) \\
\theta_2 &= x_1^2(\partial/\partial x_1) + x_2^2(\partial/\partial x_2) + x_3^2(\partial/\partial x_3) + x_4^2(\partial/\partial x_4) \\
\theta_3 &= x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2) + x_3^3(\partial/\partial x_3) + x_4^3(\partial/\partial x_4).
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Thus the exponents are:

$$(\deg\theta_0, \deg\theta_1, \deg\theta_2, \deg\theta_3) = (0, 1, 2, 3).$$

Weyl Arrangements and their Exponents

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Dynkin diagrams (root systems) and exponents

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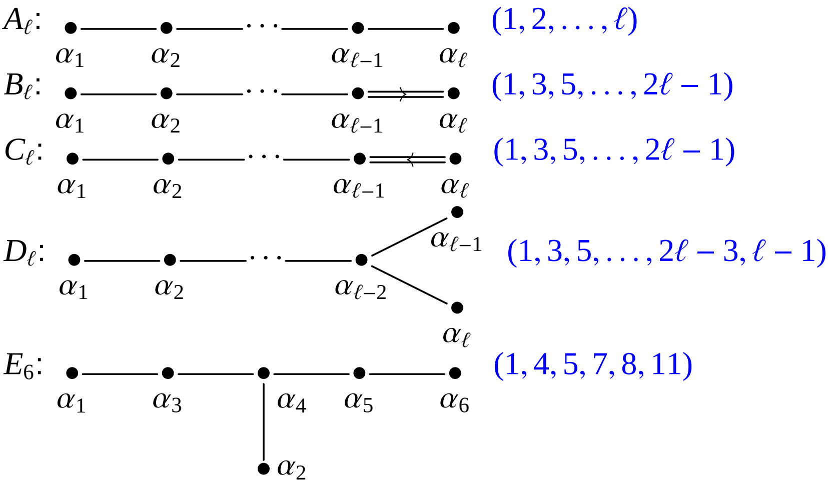
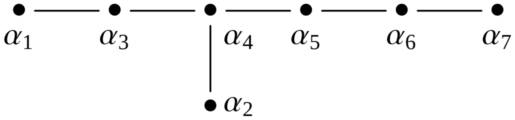
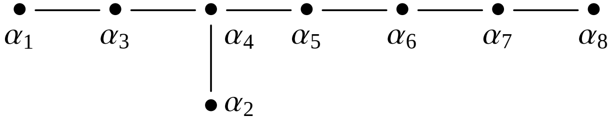


Dynkin diagrams (root systems) and exponents

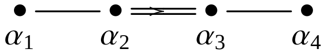
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

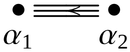


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

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The exponents are

$$d_1 = \deg\theta_1 = 1, \ \ d_2 = \deg\theta_2 = 3.$$

Exponents and Betti Numbers

Theorem

(H. T.(1981)) Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(\mathcal{A})$ splits as

$$\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

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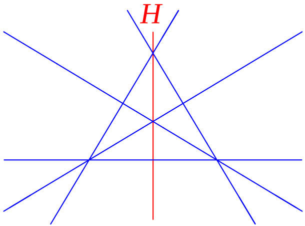

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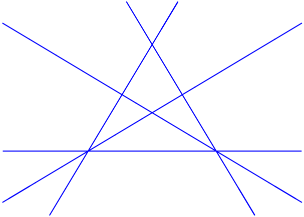

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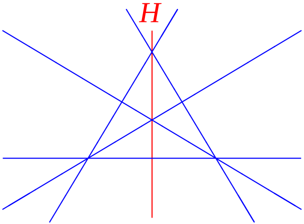

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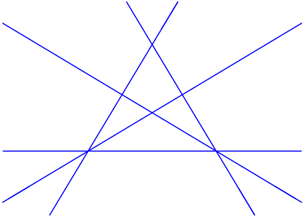

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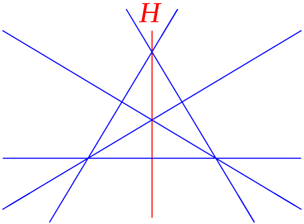

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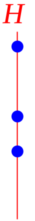

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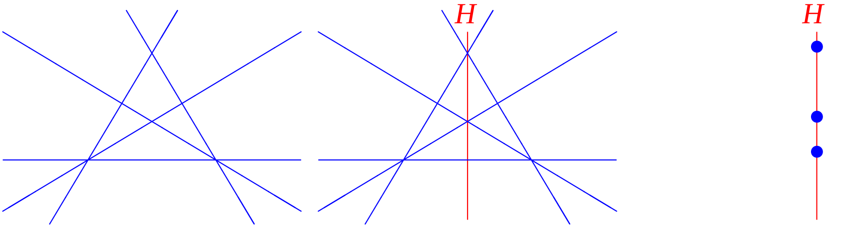

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In this case we have:

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}), \quad \exp(\mathcal{A}'') = (0, 1, 2).$$

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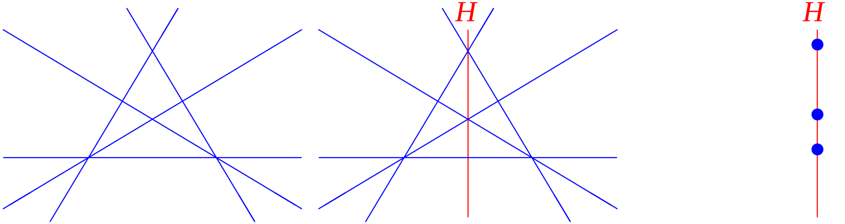

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This example is generalized into the Addition Theorem (AT) ....

Addition Theorem (AT)

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Theorem

(H. T.(1980)) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$*, suppose that* $\mathcal{A}'$ *is free with* $\exp(\mathcal{A}') = (d_1, d_2, \ldots, d_{\ell-1}, \underline{d_\ell})$ *and* $\mathcal{A}''$ *is free with* $\exp(\mathcal{A}'') = (d_1, d_2, \ldots, d_{\ell-1})$*. Then* $\mathcal{A}$ *is also free with* $\exp(\mathcal{A}) = (d_1, d_2, \ldots, \underline{d_\ell + 1})$*.*

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Remark. In the AT, d_ℓ is not necessarily the maximum exponent in $\exp(\mathcal{A}')$.

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Then (a) $q \le p$

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(ABCHT(2016?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \dots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ and $1 \le p \le \ell$ the multiplicity of the maximum exponent d.
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Define $\mathcal{A}_j'' := \{H \cap H_j \mid H \in \mathcal{A}'\}$ $(j = 1, \dots, q)$.
Assume
(1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional,
(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and
*(3) $|\mathcal{A}'| - |\mathcal{A}_j''| = d$ $(j = 1, \dots, q)$ (**Remark:** $\le$ always holds true) .*
Then (a) $q \le p$ and (b) $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ is free with exponents $(d_1, \dots, d_{\ell-q}, (d+1)^q)$.

Addition Theorem (AT)

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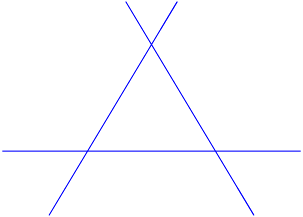

$\mathcal{A}'$

$(0, 1, \underline{1}, \underline{1})$

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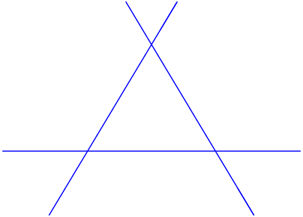

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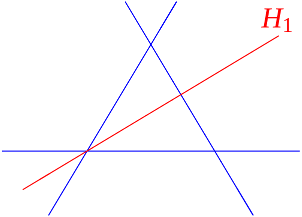

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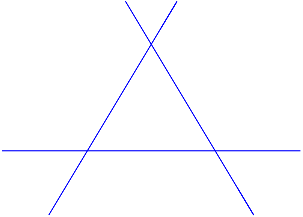

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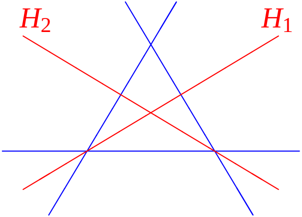

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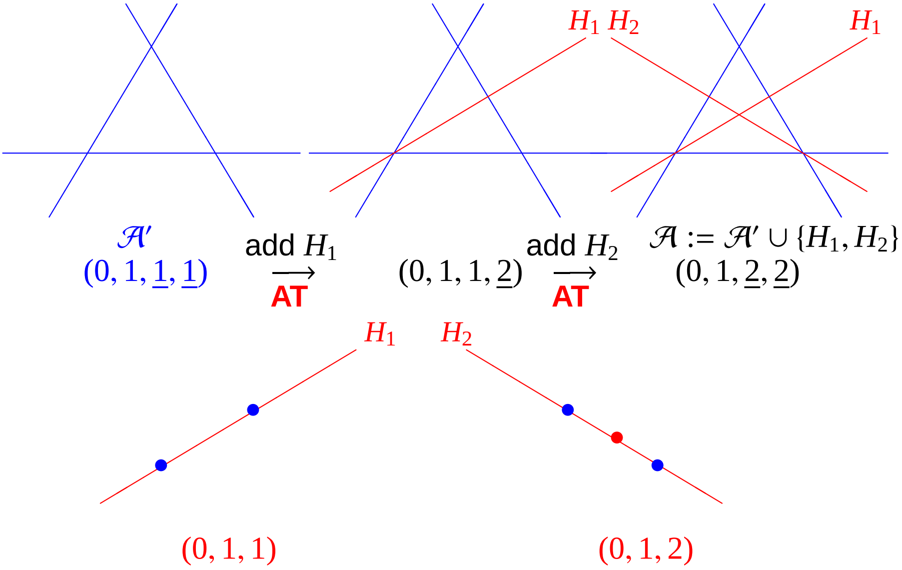

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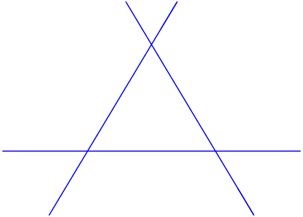

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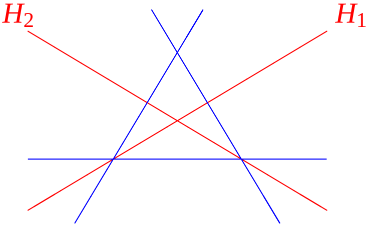

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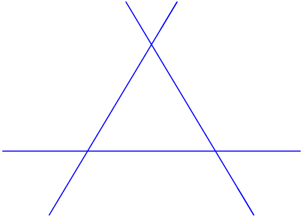

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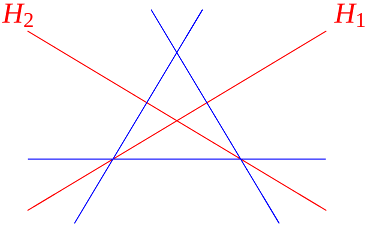
add 2 hyperplanes

$\Longrightarrow$

MAT



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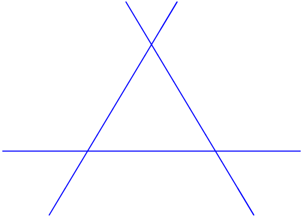

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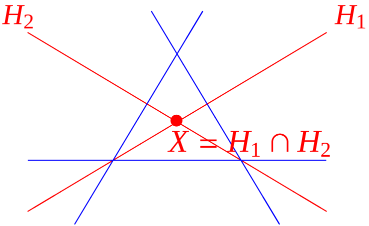

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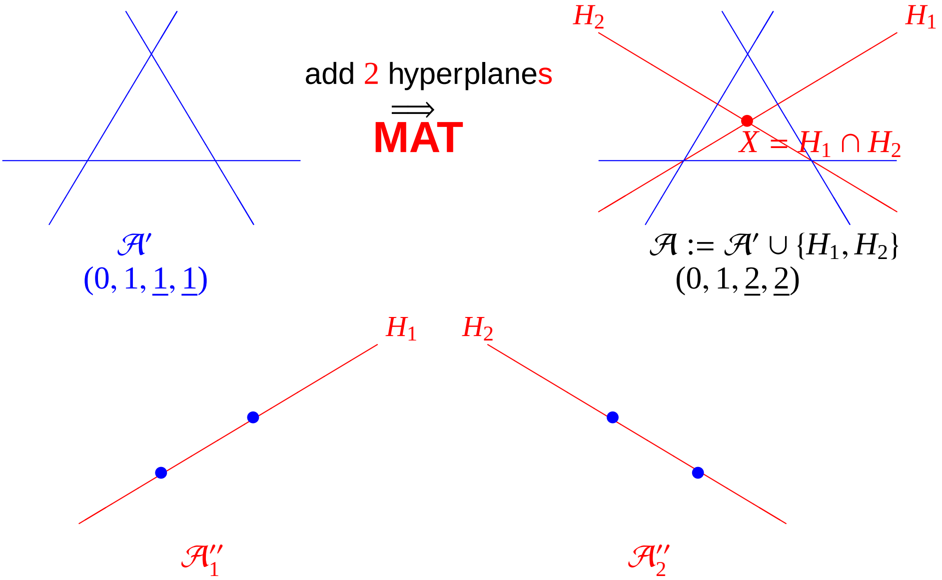

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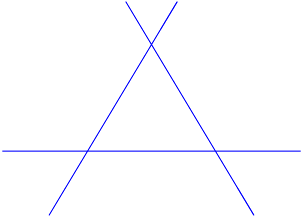

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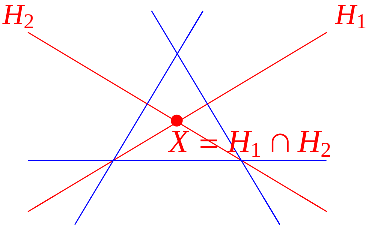
add 2 hyperplanes $\Longrightarrow$ **MAT**



$\mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\}$

$(0, 1, \underline{2}, \underline{2})$

$|\mathcal{A}'| - |\mathcal{A}''_1| = 3 - 2 = \underline{1}$

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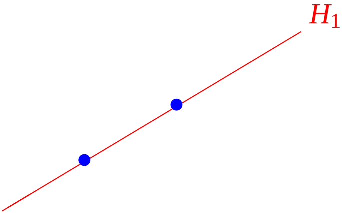
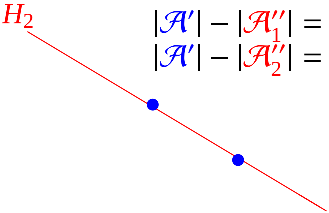


$\mathcal{A}''_1$



$\mathcal{A}''_2$

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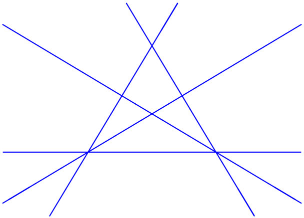

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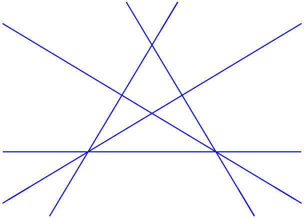

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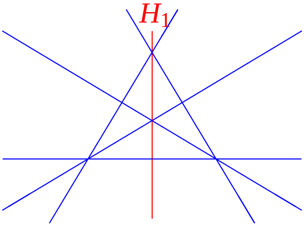

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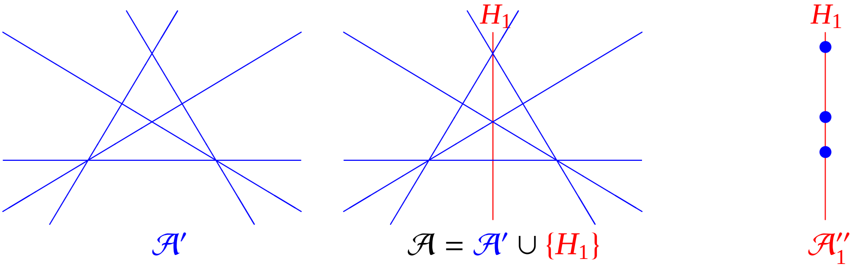

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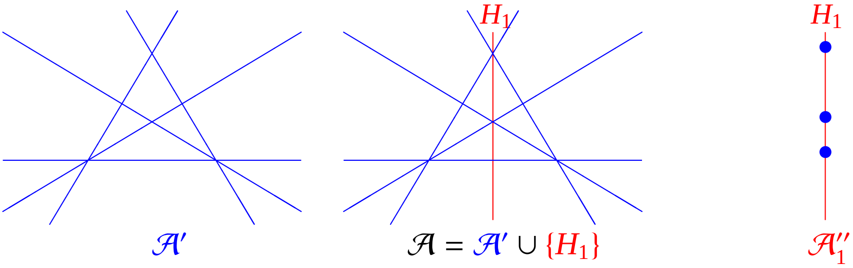

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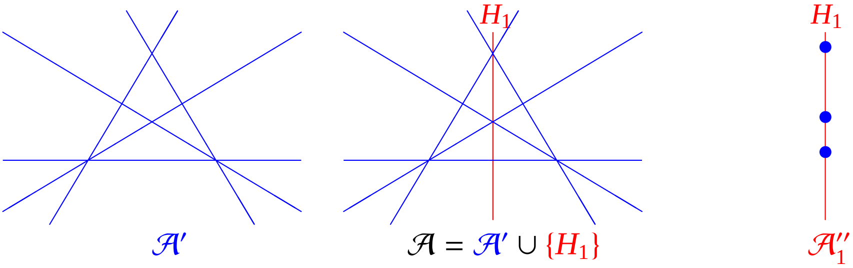

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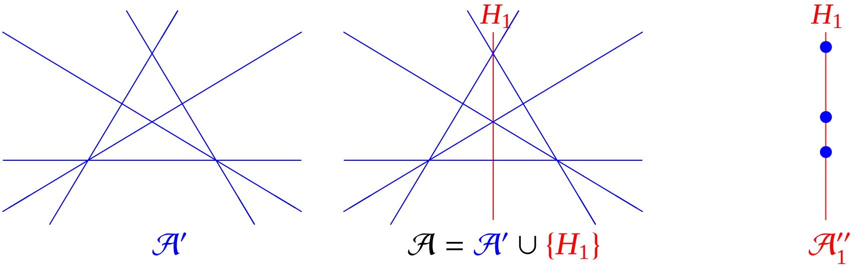

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Thus $\mathcal{A} = \mathcal{A}' \cup \{H_1\}$ with exponents $(1, 2, \underline{3})$

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So it is natural to ask the following

Question. Is there any significant application of MAT?

Contents

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2
3 Shapiro-Steinberg-Kostant-Macdonald Dual-partition Formula
4

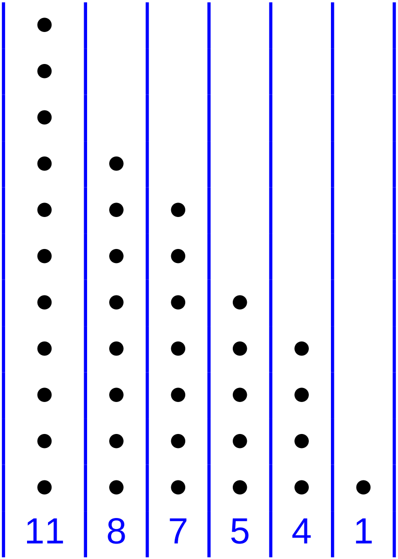
What are Dual Partitions?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are Dual Partitions?



What are Dual Partitions?

1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●

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1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
	11	8	7	5	4	1

What are these numbers?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are these numbers?

$(1, 4, 5, 7, 8, 11)$ is the exponents of the root system of the type E_6

$\updownarrow$ Dual Partitions

$(1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 6)$ is the height distribution of the positive roots of the type E_6

the dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald

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Theorem

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(1) This theorem can be (was) regarded as a method to "reading off" the exponents from the root structure.

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Remark

(1) This theorem can be (was) regarded as a method to "reading off" the exponents from the root structure.
(2) The other methods to find the exponents include: (a) from the degrees of basic invariants, (b) from the eigenvalues of a Coxeter transformation, etc.

Height of positive roots

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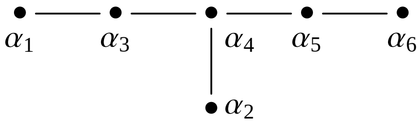
- $\mathrm{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (height) for a positive root $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$ $(c_i \in \mathbb{Z}_{\geq 0})$
- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}|$ $(1 \leq j \leq m)$

Height of positive roots (E_6)

E_6:



Exponents: $(1, 4, 5, 7, 8, 11)$

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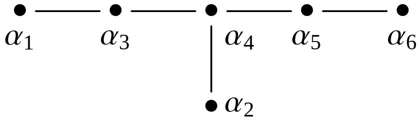

Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

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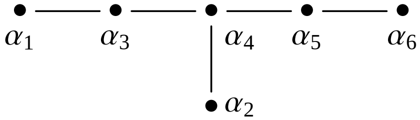

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$\vdots$

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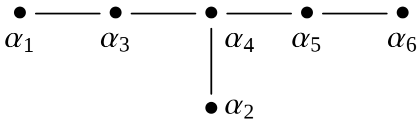

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height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

.

.

.

.

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

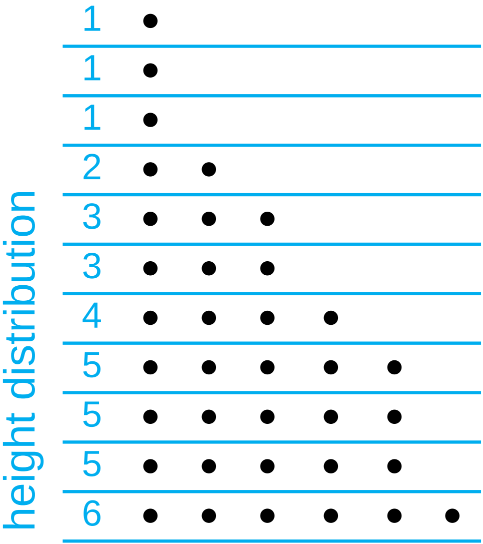
Height of positive roots (E_6)

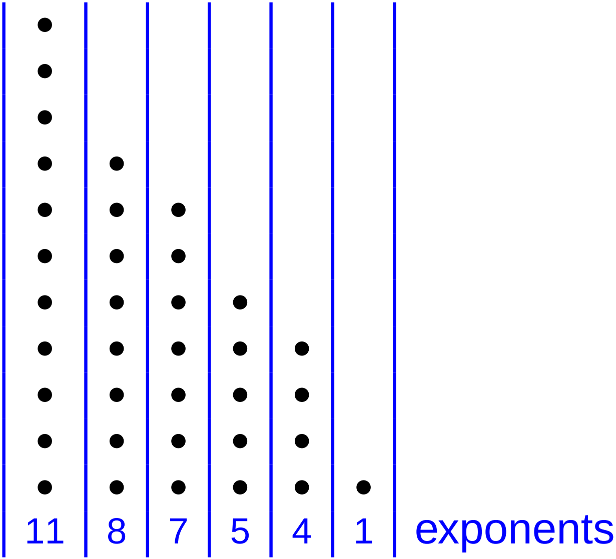
heights

ht=11	$\tilde{\alpha}$					
ht=10	•					
ht=9	•					
ht=8	•	•				
ht=7	•	•	•			
ht=6	•	•	•			
ht=5	•	•	•	•		
ht=4	•	•	•	•	•	
ht=3	•	•	•	•	•	
ht=2	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_4$	$\alpha_3 + \alpha_4$	•	•	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)




11
8
7
5
4
1
exponents

The Dual-Partition Formula (E_6)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

History of the Dual-Partition Formula

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By Bertram Kostant.

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- (1959) A. Shapiro (empirical proof using the classification)
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- (2016?) ABCHT (for ideal subarr.: using free arrangements)

Contents

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4 Ideal Subarrangement Theorem

Subarrangements of a Weyl arrangement

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- For any subset I of Φ^+, let

$$\mathcal{A}(I) := \{\ker(\alpha) \mid \alpha \in I\}$$

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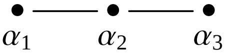
Definition

When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker\alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.

Examples of ideals/non-ideals of the root poset of A_3

A_3: 

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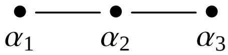

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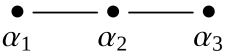
A_3: 

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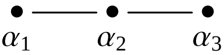

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Note that the entire set Φ^+ is always an ideal.

Main Theorem

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This positively settles a conjecture by Sommers-Tymoczko (2006).

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Let $H_1, \ldots, H_q$ *be (new) hyperplanes.*
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Assume
(1) $X := H_1 \cap \cdots \cap H_q$ *is q-codimensional,*
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(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ $(j = 1, \ldots, q)$ *(**Remark:** $\le$ always holds true) .*
Then (a) $q \le p$ *and (b)* $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ *is free with exponents* $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$*.*

Inductive use of MAT (E_6) : $I = \Phi_0^+$

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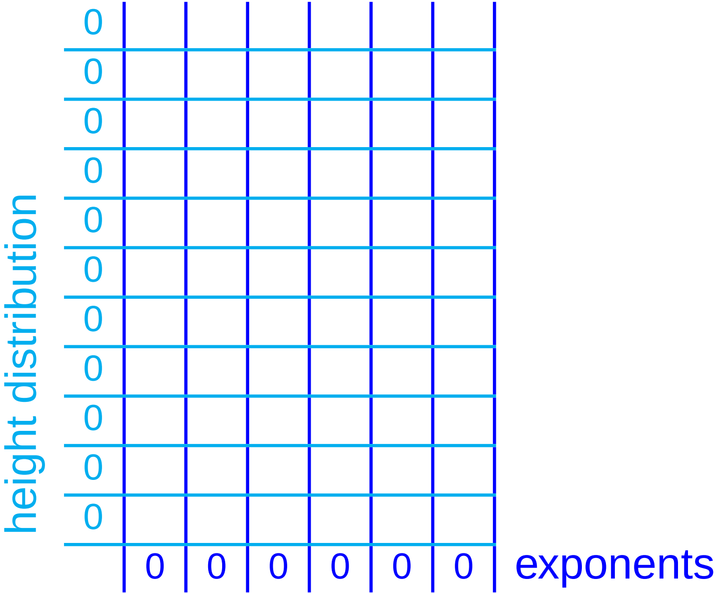

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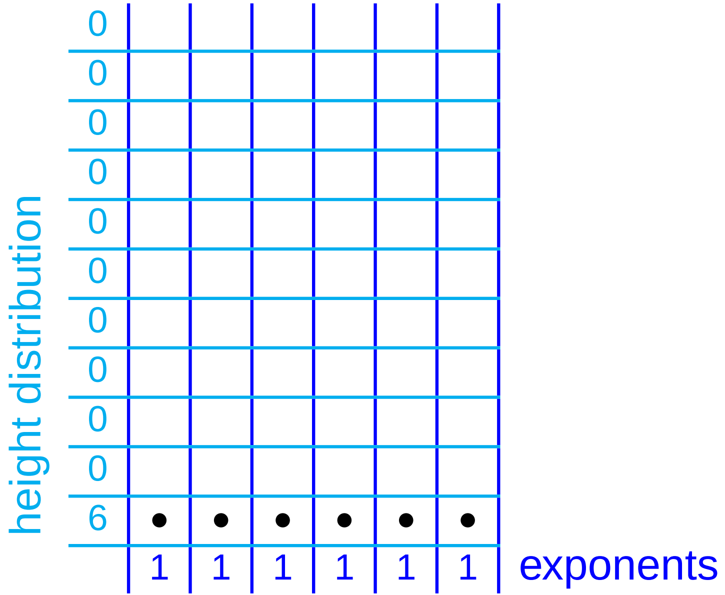
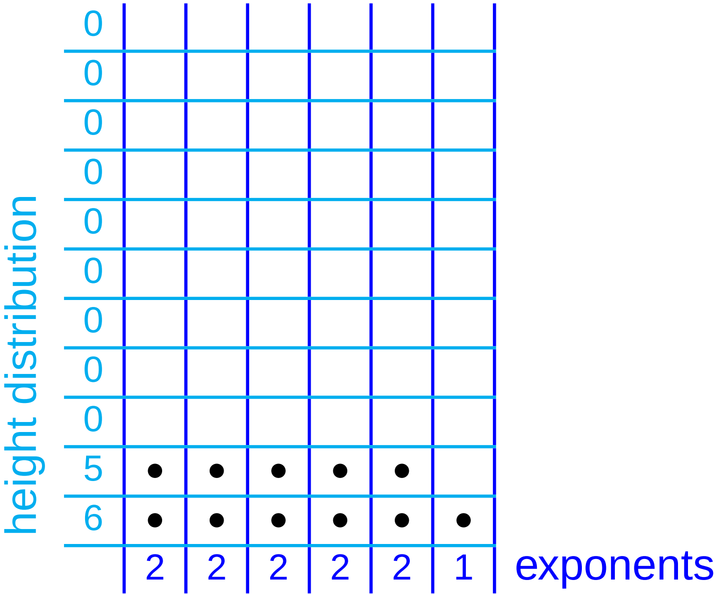


Inductive use of MAT (E_6) : $I = \Phi_2^+$



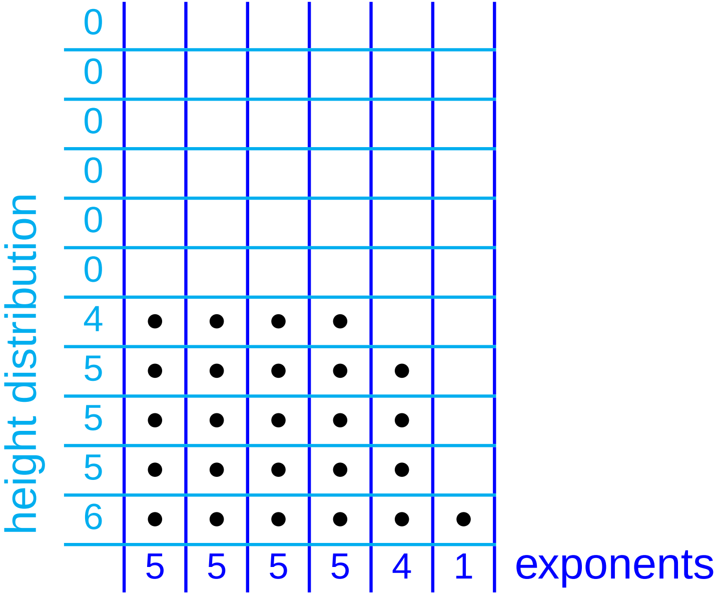
Inductive use of MAT (E_6) : $I = \Phi_3^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
0						
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	3	3	3	3	3	1

Inductive use of MAT (E_6) : $I = \Phi_4^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	4	4	4	4	4	1

Inductive use of MAT (E_6) : $I = \Phi_5^+$



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height distribution						
0						
0						
0						
0						
0						
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	6	6	6	5	4	1

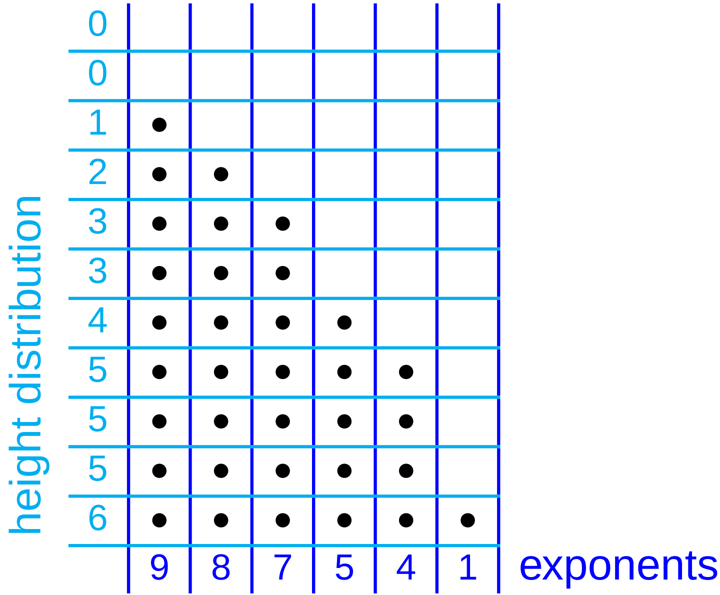
Inductive use of MAT (E_6) : $I = \Phi_7^+$

height distribution						
0						
0						
0						
0						
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	7	7	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_8^+$

height distribution						
0						
0						
0						
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	8	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_9^+$



Inductive use of MAT (E_6) : $I = \Phi^+_{10}$

height distribution						
0						
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	10	8	7	5	4	1

The Dual-Partition Formula (E_6) (again)

height distribution						
1	•					
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	11	8	7	5	4	1

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4. Moreover, we have the dual-partion formula for any ideal subarrangements.

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Thanks for your attention!