Ideal Free Theorem and Saturated Free Filtrations of Affine Weyl Arrangements

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at

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Credit

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Michael Cuntz (Leibniz Universität Hannover)

Torsten Hoge (Leibniz Universität Hannover)

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submitted

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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free S-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

Example.
(the braid arrangement (Weyl arrangement of type A_3))

$$\mathscr{A} := \{\ker(x_i - x_j) \mid 1 \leq i < j \leq 4\}$$

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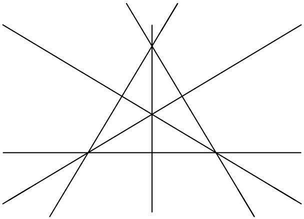

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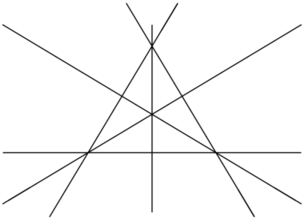

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The exponents are

$$(0, 1, 2, 3)$$

because ...

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The S-module $D(\mathcal{A})$ is a free module with a basis

$$\begin{aligned}
\theta_0 &= (\partial/\partial x_1) + (\partial/\partial x_2) + (\partial/\partial x_3) + (\partial/\partial x_4) \\
\theta_1 &= x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3) + x_4(\partial/\partial x_4) \\
\theta_2 &= x_1^2(\partial/\partial x_1) + x_2^2(\partial/\partial x_2) + x_3^2(\partial/\partial x_3) + x_4^2(\partial/\partial x_4) \\
\theta_3 &= x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2) + x_3^3(\partial/\partial x_3) + x_4^3(\partial/\partial x_4).
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\end{aligned}$$

Thus the exponents are:

$$(\deg\theta_0, \deg\theta_1, \deg\theta_2, \deg\theta_3) = (0, 1, 2, 3).$$

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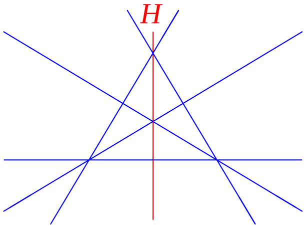

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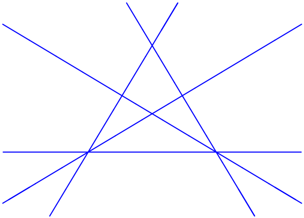

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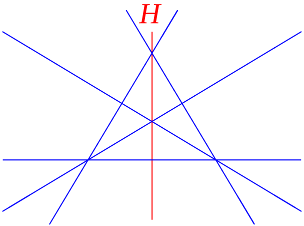

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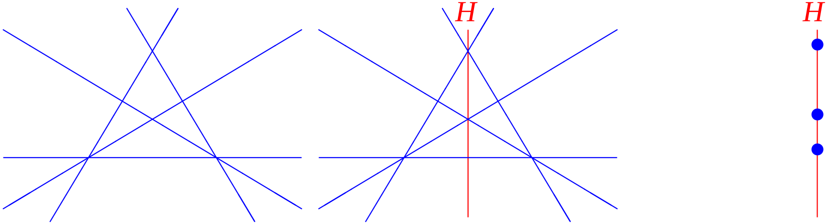

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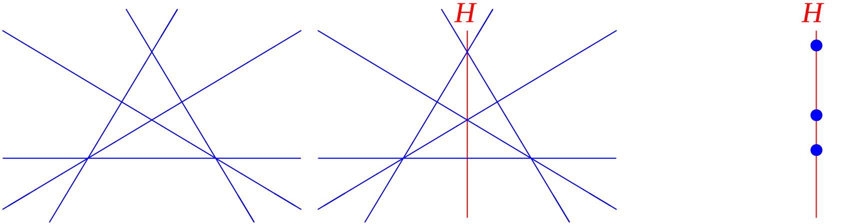

$\mathcal{A}' = \mathcal{A} \setminus \{H\}$ $\qquad$ $\mathcal{A}$ $\qquad$ $\mathcal{A}''$

braid arrangement A_3

In this case we have:

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}), \quad \exp(\mathcal{A}'') = (0, 1, 2).$$

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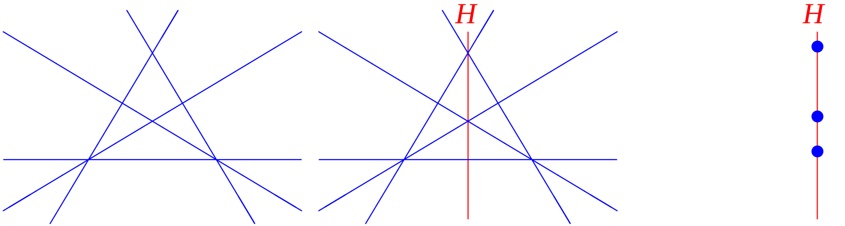

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This example is generalized into the Addition Theorem (AT) ....

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Theorem

(H. T.(1980)) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$*, suppose that* $\mathcal{A}'$ *is free with* $\exp(\mathcal{A}') = (d_1, d_2, \ldots, d_{\ell-1}, \underline{d_\ell})$ *and* $\mathcal{A}''$ *is free with* $\exp(\mathcal{A}'') = (d_1, d_2, \ldots, d_{\ell-1})$*. Then* $\mathcal{A}$ *is also free with* $\exp(\mathcal{A}) = (d_1, d_2, \ldots, \underline{d_\ell + 1})$*.*

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Recall that, for the braid arranegemnt (the Weyl arrangement of type A_3),

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}'') = (0, 1, 2), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}).$$

Inductively free arrangements

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Definition

(H. T.(1980)) The class IF *of inductively free arrangements is characterized as the minimum class of arrangements satisfying the following two condtions:*
(1) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$*, if* $\mathcal{A}' \in IF$ *and* $\mathcal{A}'' \in IF$ *with* $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$*, then* $\mathcal{A} \in IF$*, and*
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Theorem

(Barakat-Cuntz, Adv. Math.(2012)) Every Coxeter arrangement is inductively free.

Saturated free filtrations (SFF)

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Definition

For a central arrangement $\mathcal{A}$ of countably-infinite (or finite) hyperplanes, we say that a filtration

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \cdots \subseteq \mathcal{A} \ \text{ with } \ \mathcal{A} = \bigcup_{i=1}^{|\mathcal{A}|} \mathcal{A}_i$$

is said to be saturated if $|\mathcal{A}_i| = i$ for each $i \leq |\mathcal{A}|$. We also say that the filtration is free if each $\mathcal{A}_i$ is a free arrangement. We abbreviate a saturated free filtration as an SFF.

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Every inductively free arrangement has an SFF.

Contents

What are Dual Partitions?

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$\updownarrow$ Dual Partitions

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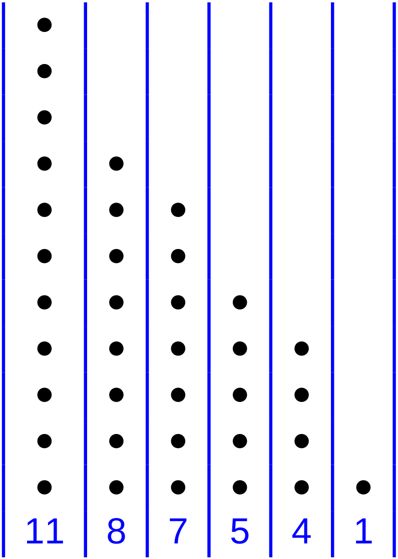

What are Dual Partitions?

1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●

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5	●	●	●	●	●	
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5	●	●	●	●	●	
6	●	●	●	●	●	●
	11	8	7	5	4	1

What are these numbers?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are these numbers?

$(1,4,5,7,8,11)$ is the exponents of the root system of the type E_6

$\updownarrow$ Dual Partitions

$(1,1,1,2,3,3,4,5,5,5,6)$ is the height distribution of the positive roots of the type E_6

the dual-partition formula by Shapiro-Steinberg, Kostant, Macdonald

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Theorem

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Remark

(1) This theorem was regarded as a method to "reading off" the exponents from the root structure.

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Remark

(1) This theorem was regarded as a method to "reading off" the exponents from the root structure.
(2) The other methods to find the exponents include: (a) from the degrees of basic invariants, (b) from the eigenvalues of a Coxeter transformation, etc.

Exponents

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Dynkin diagrams (root systems) and exponents

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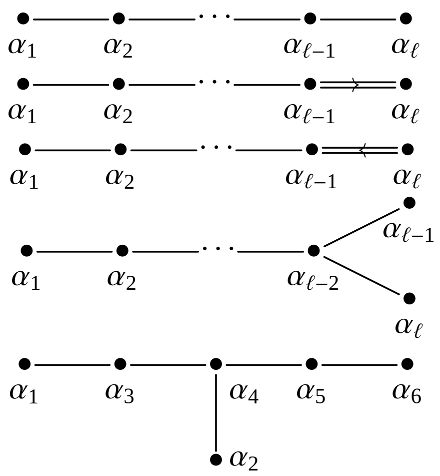


A_ℓ: $(1, 2, \dots, \ell)$

B_ℓ: $(1, 3, 5, \dots, 2\ell - 1)$

C_ℓ: $(1, 3, 5, \dots, 2\ell - 1)$

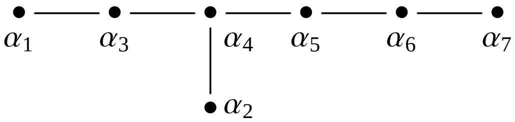
D_ℓ: $(1, 3, 5, \dots, 2\ell - 3, \ell - 1)$

E_6: $(1, 4, 5, 7, 8, 11)$

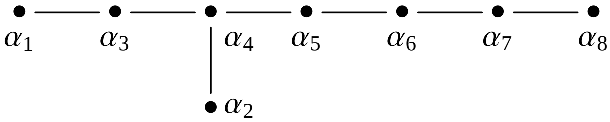
Exponents

Dynkin diagrams (root systems) and exponents

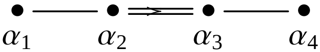
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

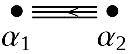


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

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- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}| \ \ (1 \leq j \leq m)$

Height of positive roots (E_6)

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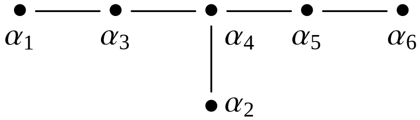
E_6: 

Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:

Height of positive roots (E_6)

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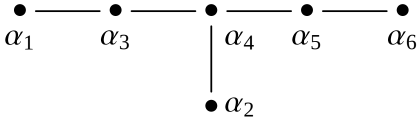

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List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

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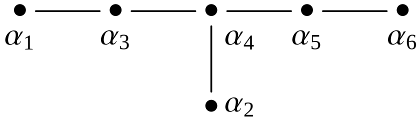

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$$\vdots$$

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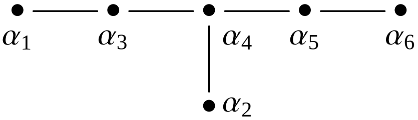


Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

.

.

.

.

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

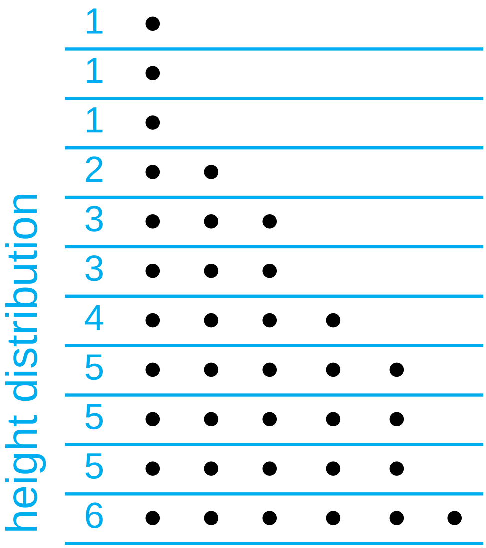
Height of positive roots (E_6)

heights

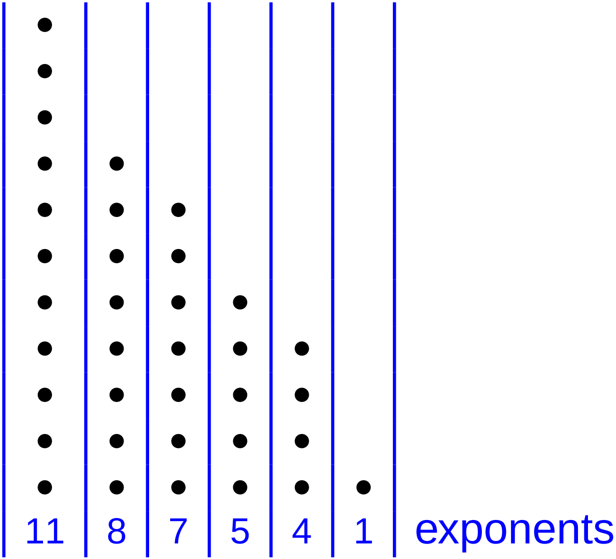
ht=11	$\tilde{\alpha}$					
ht=10	•					
ht=9	•					
ht=8	•	•				
ht=7	•	•	•			
ht=6	•	•	•			
ht=5	•	•	•	•		
ht=4	•	•	•	•	•	
ht=3	•	•	•	•	•	
ht=2	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_4$	$\alpha_3 + \alpha_4$	•	•	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)



Exponents (E_6)



The Dual-Partition Formula (E_6)

height distribution						
1	•					
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	11	8	7	5	4	1

History of the Dual-Partition Formula

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By BERTRAM KOSTANT.

. . . . . . we shall presently describe, of "reading off" the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. . . . . . . However, even though one verifies that the numbers produced by this procedure agree with the exponents . . . . . . the important question of proving that this "agreement" is more than just a coincidence remained open.

- (1959) A. Shapiro (empirical proof using the classification)
- (1959) R. Steinberg (empirical proof using the classification)
- (1959) B. Kostant (1st proof without using the classification)
- (1972) I. G. Macdonald (2nd proof: generating functions)

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↑ This is the Ideal Free Theorem (in the next Section).

Contents

Weyl arrangements

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- Φ^+ : the set of positive roots

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- $\mathcal{A} := \mathcal{A}(\Phi^+) := \{\ker(\alpha) \mid \alpha \in \Phi^+\}$: the Weyl arrangement

the root poset and ideals

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The poset is called the (positive) root poset.

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A lower-closed subset I of Φ^+ is called an ideal.
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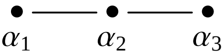
Definition

When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker\alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.

Examples of ideals/non-ideals of the root poset of A_3

A_3: 

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

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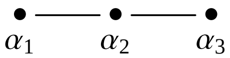

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$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

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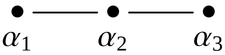

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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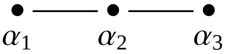
Thus $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ is an ideal, while $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not.

A_3: 

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Note that the entire set Φ^+ is always an ideal.

Ideal-free Theorem

Theorem

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Theorem

If

- Φ *: an irreducible root system of rank* ℓ

Ideal-free Theorem

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Ideal-free Theorem

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- I *: an ideal of* Φ^+,

then

(1) $\mathcal{A}(I)$ *is free, and*

(2) the exponents of $\mathcal{A}(I)$ *and the height distribution of the positive roots in* I *are dual partitions to each other.*

Ideal-free Theorem

Theorem

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- I *: an ideal of* Φ^+*,*

then

(1) $\mathcal{A}(I)$ *is free, and*

(2) the exponents of $\mathcal{A}(I)$ *and the height distribution of the positive roots in* I *are dual partitions to each other.*

This positively settles a conjecture by Sommers-Tymoczko (2006 Trans. AMS). MR by Humphreys

Dual Partition Formula by Shapiro-Steinberg-Kostant-Macdonald

In particular, when the ideal I is equal to the entire Φ^+, the ideal-free theorem yields:

Corollary

The exponents of the entire Φ and the height distribution of the entire positive roots are dual partitions to each other.

MAT (Multiple Addition Theorem - key to our proof -)

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Theorem

(ABCHT(2015)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ and $1 \le p \le \ell$ the multiplicity of the highest exponent d. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \ldots, q$. Define $\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}$ $(j = 1, \ldots, q)$. Assume that the following three conditions are satisfied:

(1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional,

(2) $X \not\subseteq H \quad (\forall H \in \mathcal{A}')$,

(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ $(1 \le j \le q)$.

Then $q \le p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.

Ideal subarrangement has an SFF

Let I be an ideal of the root poset Φ^+.

Ideal subarrangement has an SFF

Let I be an ideal of the root poset Φ^+.
Delete an element of the maximum height in I, and we get another ideal.

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Thus we have a saturated filtration of the ideal I:

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I.$$

Therefore we obtain

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Corollary

Every ideal subarrangement has an SFF.

SFF of $\mathcal{A}(E_6)$

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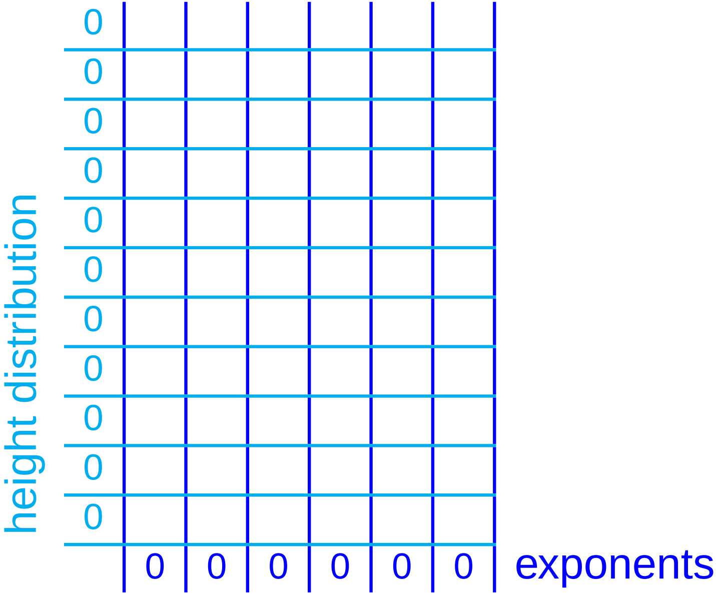

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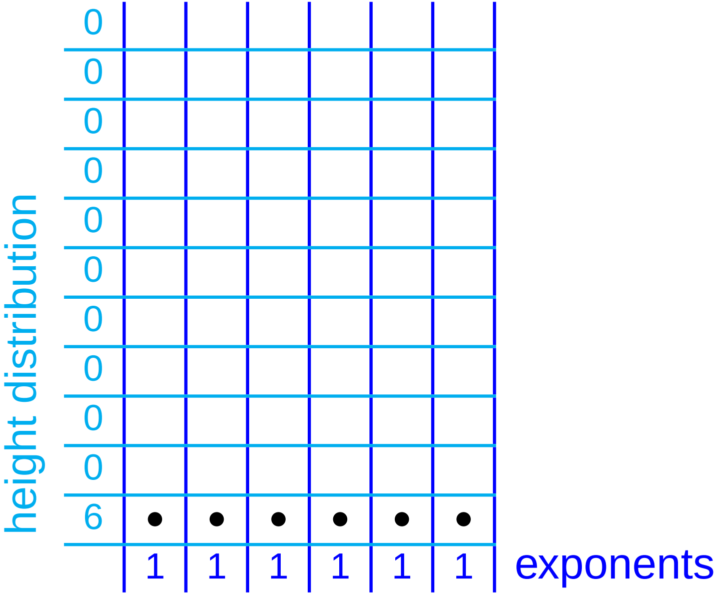

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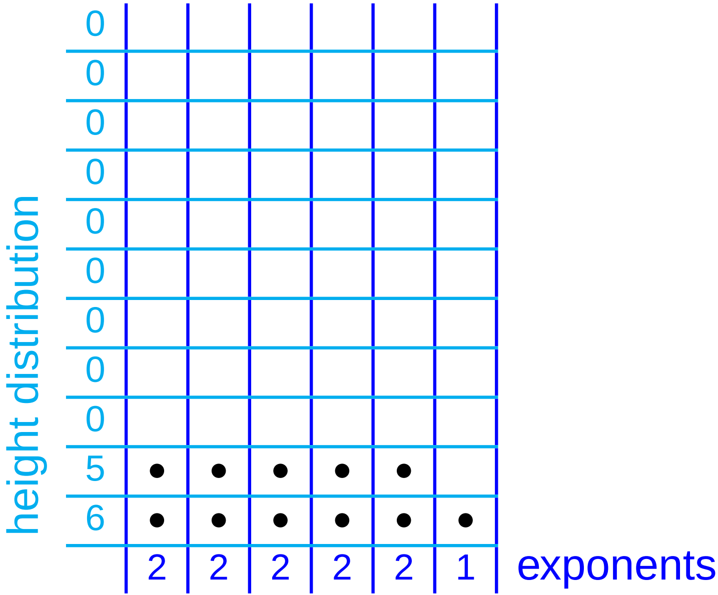

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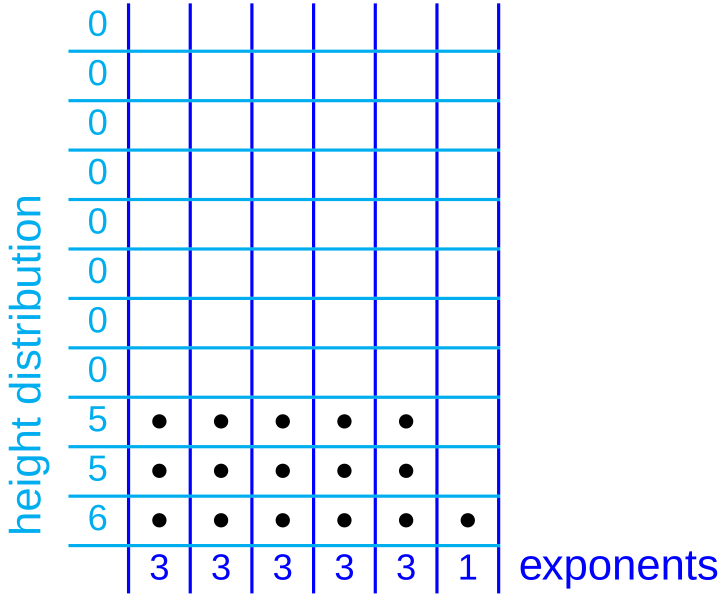

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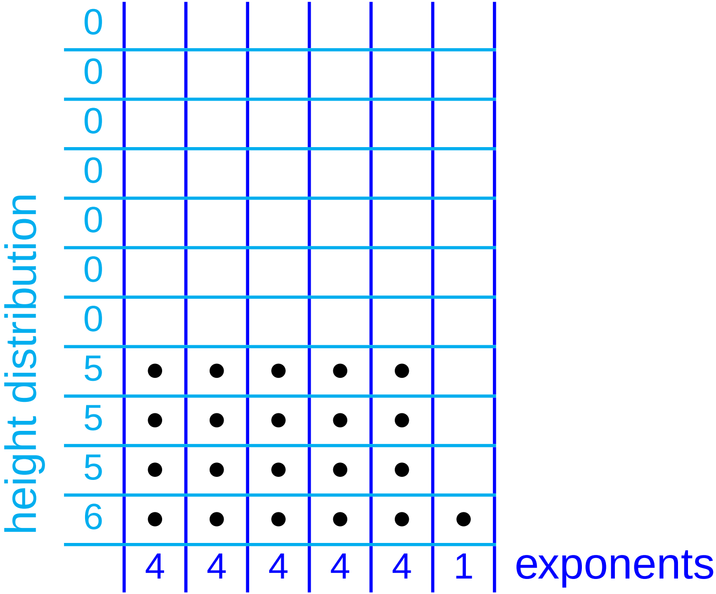

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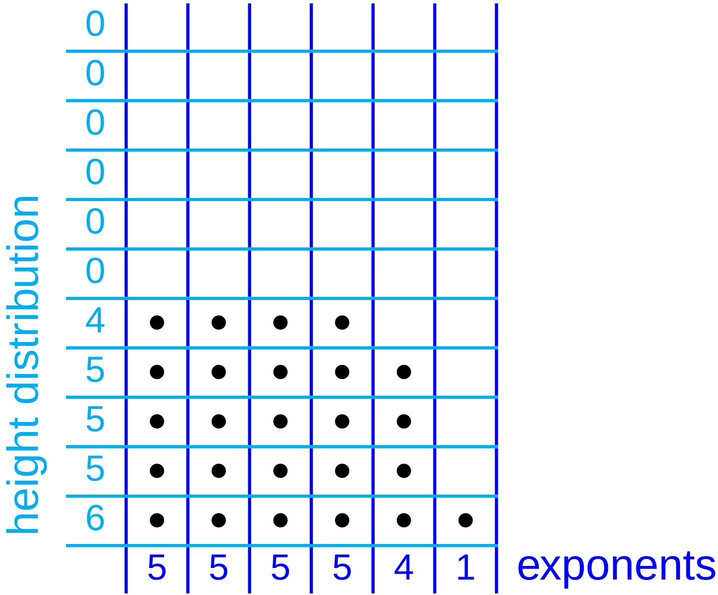
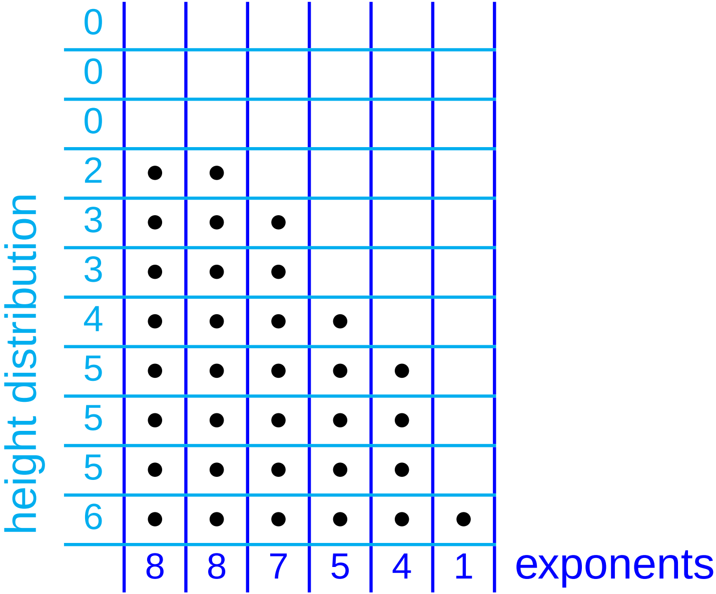


height distribution						
0						
0						
0						
0						
0						
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	6	6	6	5	4	1

SFF of $\mathcal{A}(E_6)$

height distribution						
0						
0						
0						
0						
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	7	7	7	5	4	1

SFF of $\mathcal{A}(E_6)$



SFF of $\mathcal{A}(E_6)$

height distribution							
0							
0							
1	●						
2	●	●					
3	●	●	●				
3	●	●	●				
4	●	●	●	●			
5	●	●	●	●	●		
5	●	●	●	●	●		
5	●	●	●	●	●		
6	●	●	●	●	●	●	
	9	8	7	5	4	1	exponents

SFF of $\mathcal{A}(E_6)$

height distribution						
0						
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	10	8	7	5	4	1

SFF of $\mathcal{A}(E_6)$

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
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Contents

Affine Weyl arrangement has an SFF

For a root system Φ, choose a linear order $(\alpha_1, \dots, \alpha_n)$ on the set Φ^+ of positive roots in such a way that $\{\alpha_i\}_{i=1}^k$ is an ideal of Φ^+ for any $1 \le k \le n$.

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$$\{\{\alpha_i - j = 0\} \mid j \in \mathbb{Z}, 1 \le i \le n\}.$$

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Let $\mathcal{A}_\infty(\Phi)$ denote the cone of the affine Weyl arrangement:

$$\mathcal{A}_\infty(\Phi) := \{H_z\} \cup \{H_i^j \mid j \in \mathbb{Z},\ 1 \le i \le n\}.$$

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Put

$$K_0 := H_z, \quad K_p := \begin{cases} H_r^{-q} & \text{if } 1 \le r \le n, \\ H_{2n+1-r}^{q+1} & \text{if } n+1 \le r \le 2n, \end{cases}$$

where $p \in \mathbb{Z}_{>0}$ with $p = 2nq + r \ \ (1 \le r \le 2n, \ q \in \mathbb{Z}_{\ge 0})$.

Affine Weyl arrangement has an SFF

Then the filtration $\mathcal{A}_i := \{K_0 = H_z, K_1, K_2, \ldots, K_i\} \quad (i \in \mathbb{Z}_{>0})$ is called the standard filtration of $\mathcal{A}_\infty(\Phi)$.

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Theorem

The standard filtration $\mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \ldots$ of $\mathcal{A}_\infty(\Phi)$ is an SFF. Moreover, the exponents of $\mathcal{A}_i$ is the dual partition of the pair

$$(\{z\} \cup \{\alpha - jz \mid \alpha \in \Phi^+, j \in \mathbb{Z}, \{\alpha - jz = 0\} \in \mathcal{A}_i\}, \widetilde{\mathrm{ht}}).$$

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$$(\{z\} \cup \{\alpha - jz \mid \alpha \in \Phi^+, j \in \mathbb{Z}, \{\alpha - jz = 0\} \in \mathcal{A}_i\}, \widetilde{\mathrm{ht}}).$$

Here, for $\alpha \in \Phi^+$ and $j \in \mathbb{Z}$, define the extended height function $\widetilde{\mathrm{ht}}$ by

$$\widetilde{\mathrm{ht}}(\alpha - jz) = \begin{cases} -\mathrm{ht}(\alpha) + jh + 1 & \text{if } j > 0, \\ \mathrm{ht}(\alpha) - jh & \text{if } j \leq 0. \end{cases}$$

We also define $\widetilde{\mathrm{ht}}(z) = 1$.

Shi arrangements and Catalan arrangements

There exist two kinds of subarrangements appearing in the standard filtration

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$$\mathrm{Shi}^q(\Phi) := \mathcal{A}_{2qn} \quad (q \in \mathbb{Z}_{>0}), \quad \mathrm{Cat}^q(\Phi) := \mathcal{A}_{(2q+1)n} \quad (q \in \mathbb{Z}_{\geq 0}).$$

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There exist two kinds of subarrangements appearing in the standard filtration

$$\mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_i \subseteq \cdots \subseteq \mathcal{A}_\infty(\Phi).$$

$$\mathrm{Shi}^q(\Phi) := \mathcal{A}_{2qn} \quad (q \in \mathbb{Z}_{>0}), \quad \mathrm{Cat}^q(\Phi) := \mathcal{A}_{(2q+1)n} \quad (q \in \mathbb{Z}_{\geq 0}).$$

Then it is known (M. Yoshinaga (2004)) that

$$\exp(\mathrm{Shi}^q(\Phi)) = (1, qh, qh, \ldots, qh)$$

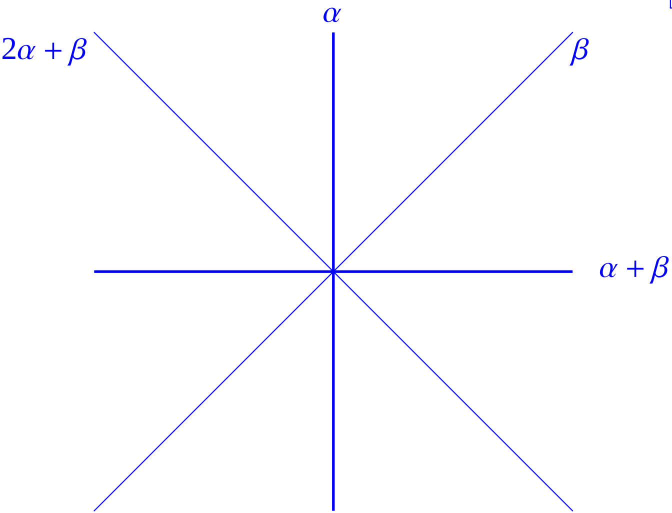
$$\exp(\mathrm{Cat}^q(\Phi)) = (1, qh + m_1, qh + m_2, \ldots, qh + m_\ell),$$

where h is the Coxeter number and

$$\exp(\mathcal{A}(\Phi)) = (m_1, m_2, \ldots, m_\ell).$$

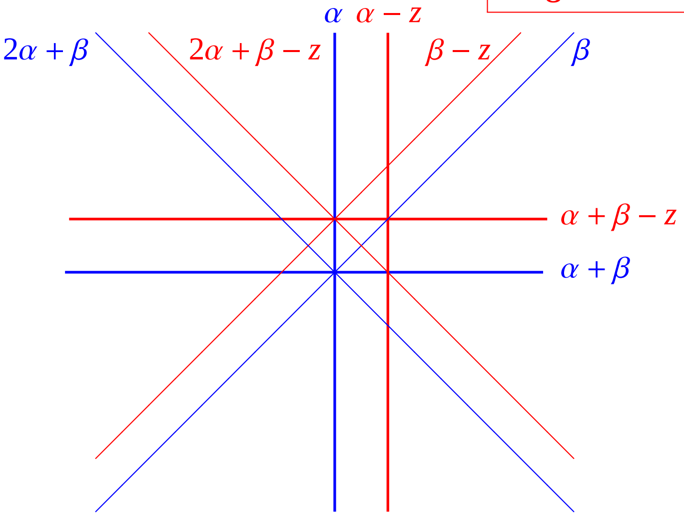
SFF of the Affine Weyl arrangement

$\mathcal{A}(B_2)$



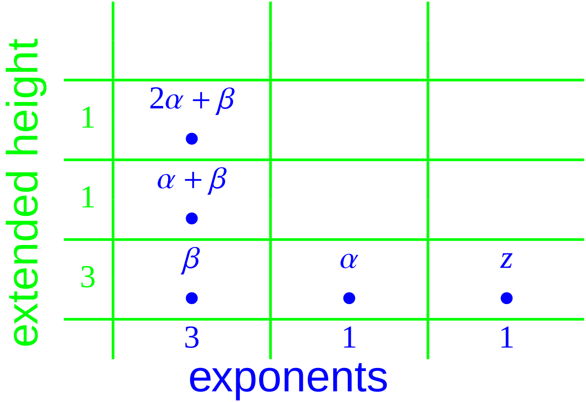
SFF of the Affine Weyl arrangement

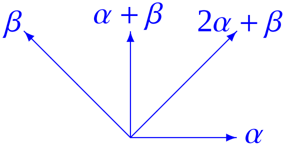
$\mathcal{A}_8 = \mathrm{Shi}^1(B_2)$



Dual-Partition Formula

$$\mathcal{A}_4 = \mathcal{A}(B_2) \cup \{H_z\}$$



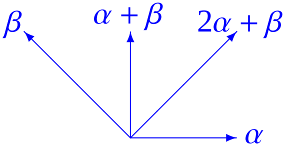


Dual-Partition Formula

$\mathcal{A}_5$

extended height			
1	$2\alpha+\beta$ ●		
2	$\alpha+\beta$ ●	$2\alpha+\beta-z$ ●	
3	β ●	α ●	z ●
	3	2	1

exponents

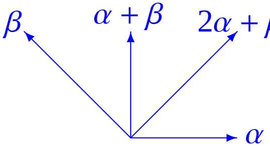


SFF of the Affine Weyl arrangement

Dual-Partition Formula

$\mathcal{A}_6$

extended height			
2	$2\alpha+\beta$ •	$\alpha+\beta-z$ •	
2	$\alpha+\beta$ •	$2\alpha+\beta-z$ •	
3	β •	α •	z •
exponents	3	3	1



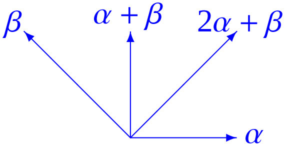
Dual-Partition Formula

$\mathcal{A}_7$

extended height

1		$\beta - z$ ●	
2	$2\alpha + \beta$ ●	$\alpha + \beta - z$ ●	
2	$\alpha + \beta$ ●	$2\alpha + \beta - z$ ●	
3	β ●	α ●	z ●
	3	4	1

exponents



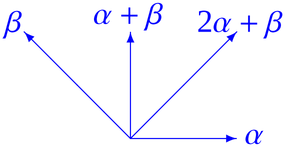
Dual-Partition Formula

$$\mathcal{A}_8 = \mathrm{Shi}^1(B_2)$$

extended height

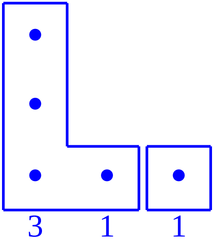
2	$\alpha - z$	$\beta - z$	
2	$2\alpha + \beta$	$\alpha + \beta - z$	
2	$\alpha + \beta$	$2\alpha + \beta - z$	
3	β	α	z
	4	4	1

exponents



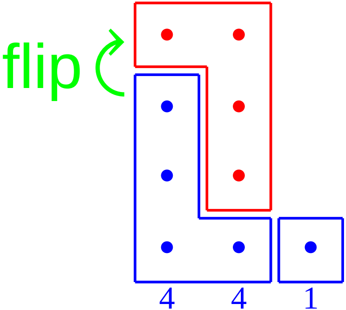
SFF of the Affine Weyl arrangement

$$\boxed{\mathcal{A}_4 = \mathcal{A}(B_2)}$$



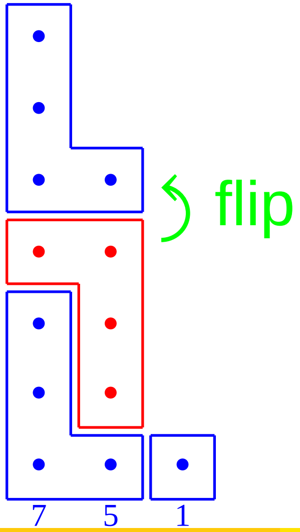
SFF of the Affine Weyl arrangement

$$\mathcal{A}_8 = \mathrm{Shi}^1(B_2)$$



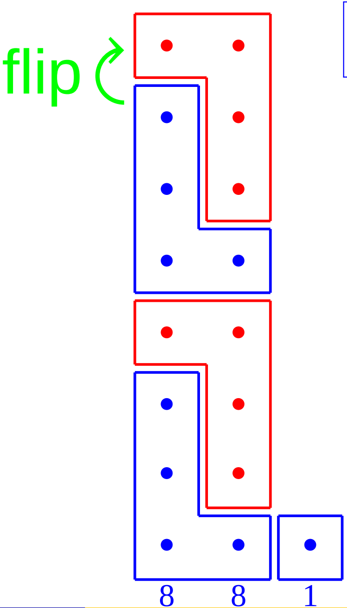
SFF of the Affine Weyl arrangement

$\mathcal{A}_{12} = \mathrm{Cat}^1(B_2)$



SFF of the Affine Weyl arrangement

$$\mathcal{A}_{16} = \mathrm{Shi}^2(B_2)$$



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Summary

1. The Addition Theorem gives the concept of inductively free arrangements.
2. The Addition Theorem also gives the concept of saturated free filtrations (SFF) .
3. Each ideal subarrangement of the Weyl arrangements has an SFF. Each filter satisfies the celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald.
4. Each affine Weyl arrangement has an SFF. Each filter satisfies the dual-partition formula with respect to the extended height function.

I stop here.

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Thanks for your attention!