

Ideal Free Theorem and Saturated Free Filtrations of Affine Weyl Arrangements

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Credit

Credit

with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität
Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

Torsten Hoge (Leibniz Universität Hannover)

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submitted

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- 3 Ideal Free Theorem
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Free Arrangements and their Exponents

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- \mathcal{A} is said to be a **free arrangement** if $D(\mathcal{A})$ is a free S -module.
- When \mathcal{A} is free, then $\exists \theta_1, \theta_2, \dots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers d_1, d_2, \dots, d_ℓ are called the **exponents** of \mathcal{A} .

Free Arrangements and their Exponents

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Example.

(the braid arrangement (Weyl arrangement of type A_3))

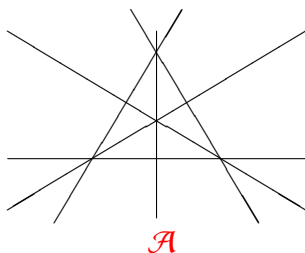
$$\mathcal{A} := \{\ker(x_i - x_j) \mid 1 \leq i < j \leq 4\}$$

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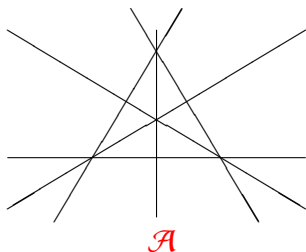


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$$\mathcal{A} := \{\ker(x_i - x_j) \mid 1 \leq i < j \leq 4\}$$



The **exponents** are

$$(0, 1, 2, 3)$$

because ...

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The S -module $D(\mathcal{A})$ is a free module with a basis

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Thus the **exponents** are:

$$(\deg \theta_0, \deg \theta_1, \deg \theta_2, \deg \theta_3) = (0, 1, 2, 3).$$

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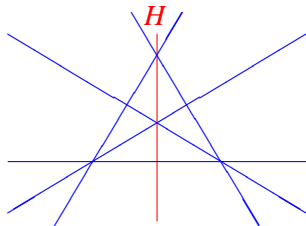
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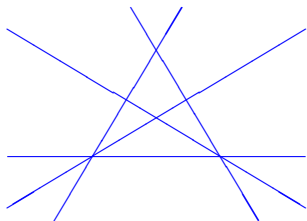


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braid arrangement A_3

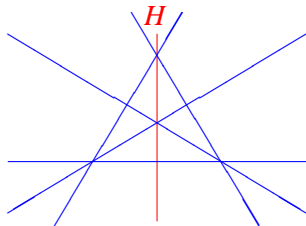
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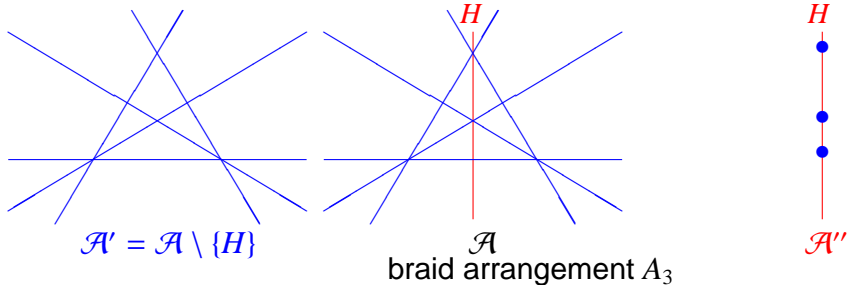


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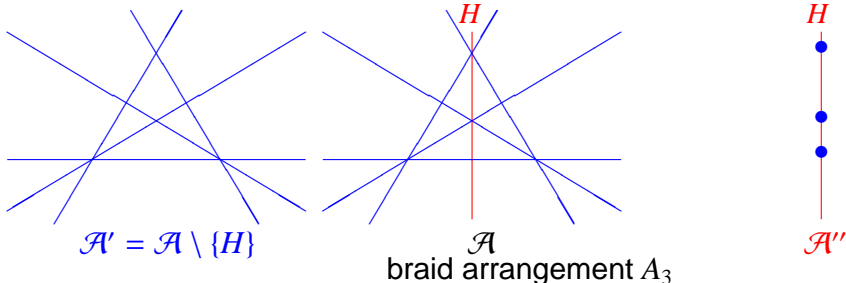
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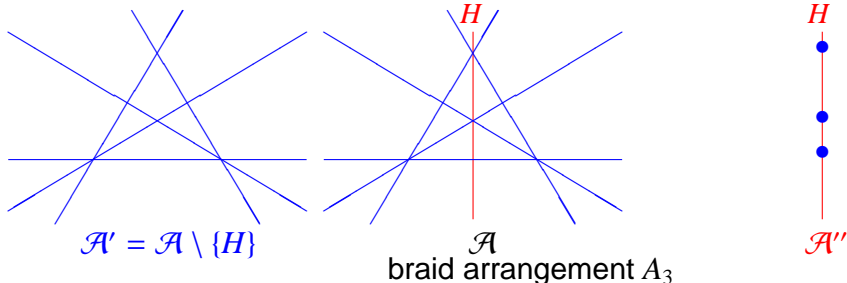
In this case we have:

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}), \quad \exp(\mathcal{A}'') = (0, 1, 2).$$

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This example is generalized into the **Addition Theorem (AT)**

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Theorem

(H. T.(1980)) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, suppose that \mathcal{A}' is free with $\exp(\mathcal{A}') = (d_1, d_2, \dots, d_{\ell-1}, \underline{d_\ell})$ and \mathcal{A}'' is free with $\exp(\mathcal{A}'') = (d_1, d_2, \dots, d_{\ell-1})$. Then \mathcal{A} is also free with $\exp(\mathcal{A}) = (d_1, d_2, \dots, \underline{d_\ell + 1})$.

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Recall that, for the braid arrangement (the Weyl arrangement of type A_3),

$$\exp(\mathcal{A}') = (0, 1, 2, \underline{2}), \quad \exp(\mathcal{A}'') = (\underline{0}, 1, 2), \quad \exp(\mathcal{A}) = (0, 1, 2, \underline{3}).$$

Inductively free arrangements

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Definition

(H. T.(1980)) The class *IF* of *inductively free* arrangements is characterized as the minimum class of arrangements satisfying the following two conditions:

- (1) For a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, if $\mathcal{A}' \in IF$ and $\mathcal{A}'' \in IF$ with $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A} \in IF$, and
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Theorem

(Barakat-Cuntz, Adv. Math.(2012)) Every *Coxeter arrangement* is *inductively free*.

Saturated free filtrations (SFF)

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For a central arrangement \mathcal{A} of countably-infinite (or finite) hyperplanes, we say that a filtration

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is said to be **saturated** if $|\mathcal{A}_i| = i$ for each $i \leq |\mathcal{A}|$. We also say that the filtration is **free** if each \mathcal{A}_i is a free arrangement. We abbreviate a **saturated free filtration** as an **SFF**.

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Every **inductively free arrangement** has an **SFF**.

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Dual-Partition Formula
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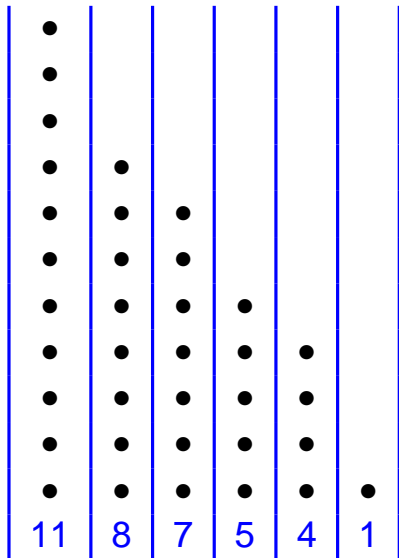
What are Dual Partitions?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

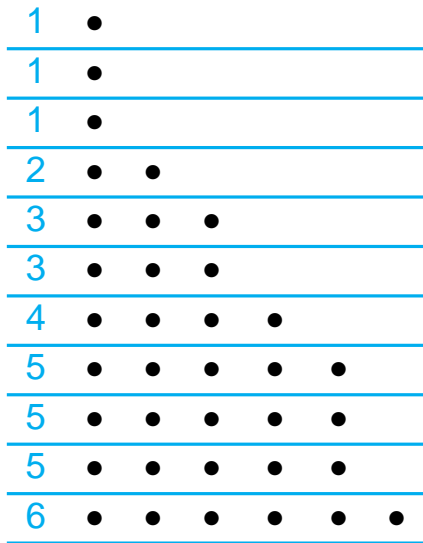
↕ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

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| | | | | | | |
|---|----|---|---|---|---|---|
| 1 | • | | | | | |
| 1 | • | | | | | |
| 1 | • | | | | | |
| 2 | • | • | | | | |
| 3 | • | • | • | | | |
| 3 | • | • | • | | | |
| 4 | • | • | • | • | | |
| 5 | • | • | • | • | • | |
| 5 | • | • | • | • | • | |
| 5 | • | • | • | • | • | |
| 6 | • | • | • | • | • | • |
| | 11 | 8 | 7 | 5 | 4 | 1 |

What are these numbers?

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What are these numbers?

(1, 4, 5, 7, 8, 11) is the **exponents** of the root system of the type E_6

↕ Dual Partitions

(1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 6) is the **height distribution** of the positive roots of the type E_6

the dual-partition formula by Shapiro-Steinberg, Kostant, Macdonald

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Theorem

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The *exponents* of an irreducible root system and the *height distribution* of positive roots are *dual partitions to each other*.

Remark

(1) *This theorem was regarded as a method to “reading off” the exponents from the root structure.*

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Remark

(1) This theorem was regarded as a method to “*reading off*” the *exponents from the root structure*.

(2) The other methods to find the exponents include: (a) from the degrees of *basic invariants*, (b) from the eigenvalues of a *Coxeter transformation*, etc.

Exponents

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Dynkin diagrams (root systems) and exponents

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$$A_\ell: \begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \text{---} & \bullet \\ \alpha_1 & & \alpha_2 & & & & \alpha_{\ell-1} & & \alpha_\ell \end{array} \quad (1, 2, \dots, \ell)$$

$$B_\ell: \begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \rightleftarrows & \bullet \\ \alpha_1 & & \alpha_2 & & & & \alpha_{\ell-1} & & \alpha_\ell \end{array} \quad (1, 3, 5, \dots, 2\ell - 1)$$

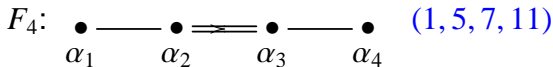
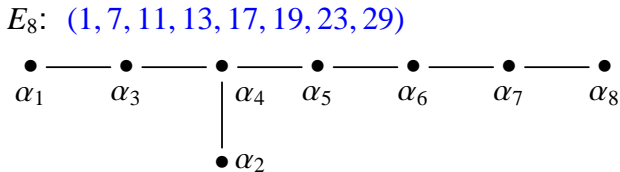
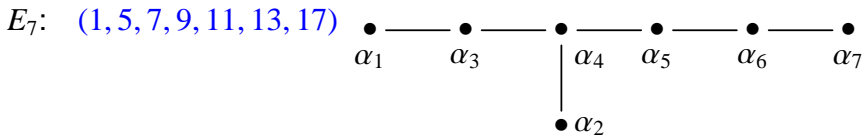
$$C_\ell: \begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \leftleftarrows & \bullet \\ \alpha_1 & & \alpha_2 & & & & \alpha_{\ell-1} & & \alpha_\ell \end{array} \quad (1, 3, 5, \dots, 2\ell - 1)$$

$$D_\ell: \begin{array}{ccccccc} & & & & & & \bullet & & \\ & & & & & & \alpha_{\ell-1} & & \\ \bullet & \text{---} & \bullet & \text{---} & \cdots & \text{---} & \bullet & \begin{array}{l} \diagup \\ \diagdown \end{array} & \\ \alpha_1 & & \alpha_2 & & & & \alpha_{\ell-2} & & \\ & & & & & & & & \bullet \\ & & & & & & & & \alpha_\ell \end{array} \quad (1, 3, 5, \dots, 2\ell - 3, \ell - 1)$$

$$E_6: \begin{array}{ccccccc} \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet & \text{---} & \bullet \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \\ & & & & | & & & & \\ & & & & \bullet & & & & \\ & & & & \alpha_2 & & & & \end{array} \quad (1, 4, 5, 7, 8, 11)$$

Exponents

Dynkin diagrams (root systems) and exponents



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Height of positive roots

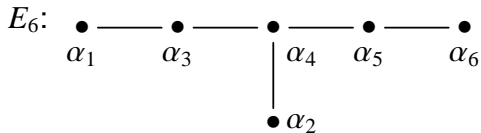
- Φ : an irreducible **root system** of rank ℓ
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- Φ^+ : the set of **positive roots**
- $\text{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (**height**) for a positive root
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 $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ ($c_i \in \mathbb{Z}_{\geq 0}$)
- The **height distribution** in Φ^+ is a sequence of positive integers (i_1, i_2, \dots, i_m) , where
 $i_j := |\{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = j\}|$ ($1 \leq j \leq m$)

Height of positive roots (E_6)

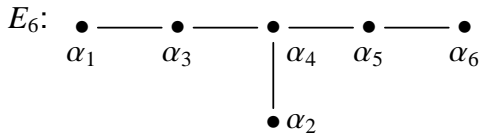
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List of positive roots:

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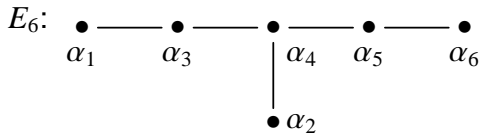
List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \dots$

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List of positive roots:

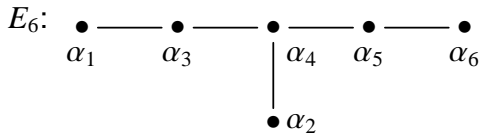
height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \dots$

.
. .
. . .
. . . .

Height of positive roots (E_6)



Exponents: (1, 4, 5, 7, 8, 11)

List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

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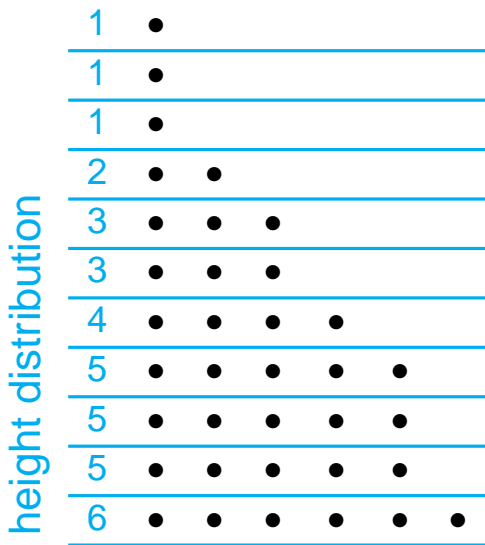
height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

Height of positive roots (E_6)

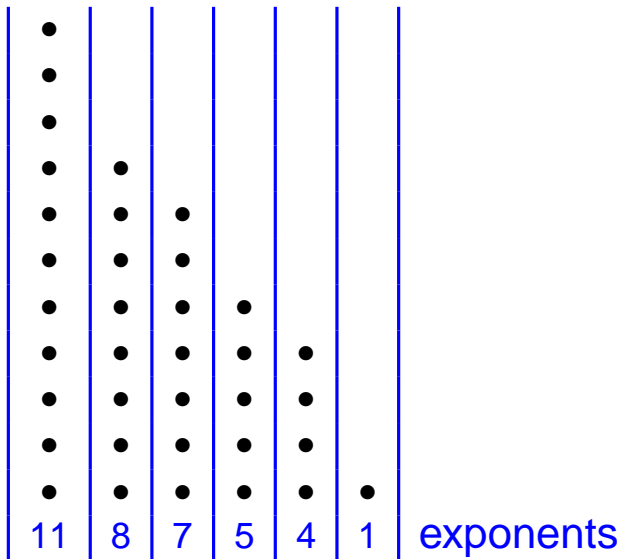
| | | | | | | |
|---------|-------|-----------------------|-----------------------|-----------------------|------------|-----------------------|
| | ht=11 | $\tilde{\alpha}$ | | | | |
| | ht=10 | • | | | | |
| | ht=9 | • | | | | |
| | ht=8 | • | • | | | |
| | ht=7 | • | • | • | | |
| | ht=6 | • | • | • | | |
| | ht=5 | • | • | • | • | |
| | ht=4 | • | • | • | • | • |
| | ht=3 | • | • | • | • | • |
| | ht=2 | $\alpha_1 + \alpha_3$ | $\alpha_2 + \alpha_4$ | $\alpha_3 + \alpha_4$ | • | • |
| | ht=1 | α_1 | α_2 | α_3 | α_4 | α_5 α_6 |
| heights | | | | | | |

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \quad \text{ht}(\tilde{\alpha}) = 11 \text{ (the highest root)}$$

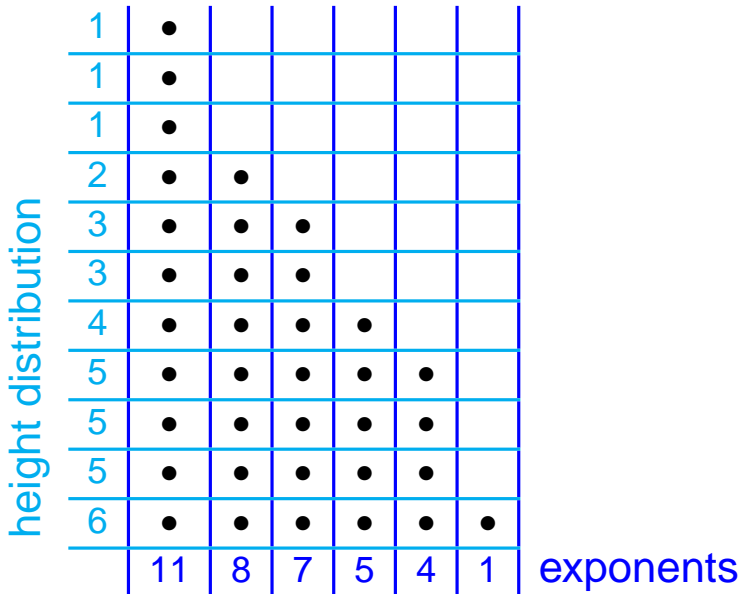
Height Distribution (E_6)



Exponents (E_6)



The Dual-Partition Formula (E_6)



History of the Dual-Partition Formula

History of the Dual-Partition Formula

THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*¹

By BERTRAM KOSTANT.

..... we shall presently describe, of “reading off” the exponents from the root structure of \mathfrak{g} was discovered by Arnold Shapiro. However, even though one verifies that the numbers produced by this procedure agree with the exponents the important question of proving that this “agreement” is more than just a coincidence remained open.

- (1959) A. Shapiro (empirical proof using the classification)
- (1959) R. Steinberg (empirical proof using the classification)
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 ↑ This is the Ideal Free Theorem (in the next Section).

- ① Free Arrangements and Exponents
- ② Shapiro-Steinberg-Kostant-Macdonald Dual-Partition Formula
- ③ **Ideal Free Theorem**
- ④ Affine Weyl Arrangement has a Saturated Free Filtration (SFF)

Weyl arrangements

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- Φ^+ : the set of **positive roots**

Weyl arrangements

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- $\mathcal{A} := \mathcal{A}(\Phi^+) := \{\ker(\alpha) \mid \alpha \in \Phi^+\}$: **the Weyl arrangement**

the root poset and ideals

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$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i.$$

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A lower-closed subset I of Φ^+ is called an *ideal*.

In other words, I is an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$,

$$\beta_1 \geq \beta_2, \beta_1 \in I \Rightarrow \beta_2 \in I.$$

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Definition

When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker \alpha \mid \alpha \in I\}$ is called an *ideal subarrangement* of \mathcal{A} .

Examples of ideals/non-ideals of the root poset of A_3

$$A_3: \begin{array}{ccc} \bullet & \text{---} & \bullet & \text{---} & \bullet \\ \alpha_1 & & \alpha_2 & & \alpha_3 \end{array}$$

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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Thus $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ is an ideal, while $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not.

Note that the entire set Φ^+ is always an ideal.

Ideal-free Theorem

Theorem

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If

- Φ : an irreducible *root system* of rank ℓ

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(2) the *exponents* of $\mathcal{A}(I)$ and the *height distribution* of the positive roots in I are *dual partitions to each other*.

Ideal-free Theorem

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(1) $\mathcal{A}(I)$ is *free*, and

(2) the *exponents* of $\mathcal{A}(I)$ and the *height distribution* of the positive roots in I are *dual partitions to each other*.

This positively settles a conjecture by Sommers-Tymoczko (2006 TAMS).

Dual Partition Formula by Shapiro-Steinberg-Kostant-Macdonald

In particular, when the ideal I is equal to the entire Φ^+ , the ideal-free theorem yields:

Corollary

The *exponents* of the entire Φ and the *height distribution* of the entire positive roots are *dual partitions to each other*.

MAT (Multiple Addition Theorem - key to our proof -)

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Theorem

(ABCHT(2015)) Let \mathcal{A}' be a **free** arrangement with **exponents** (d_1, \dots, d_ℓ) ($d_1 \leq \dots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the **highest exponent** d . Let H_1, \dots, H_q be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \dots, q$. Define $\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}$ ($j = 1, \dots, q$). Assume that the following three conditions are satisfied:

- (1) $X := H_1 \cap \dots \cap H_q$ is q -codimensional,
- (2) $X \not\subseteq H$ ($\forall H \in \mathcal{A}'$),
- (3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($1 \leq j \leq q$).

Then $q \leq p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ is **free** with **exponents** $(d_1, \dots, d_{\ell-q}, (d+1)^q)$.

Ideal subarrangement has an SFF

Let I be an ideal of the root poset Φ^+ .

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Delete an element of **the maximum height** in I , and we get another ideal.

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Thus we have a **saturated filtration** of the ideal I :

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I.$$

Therefore we obtain

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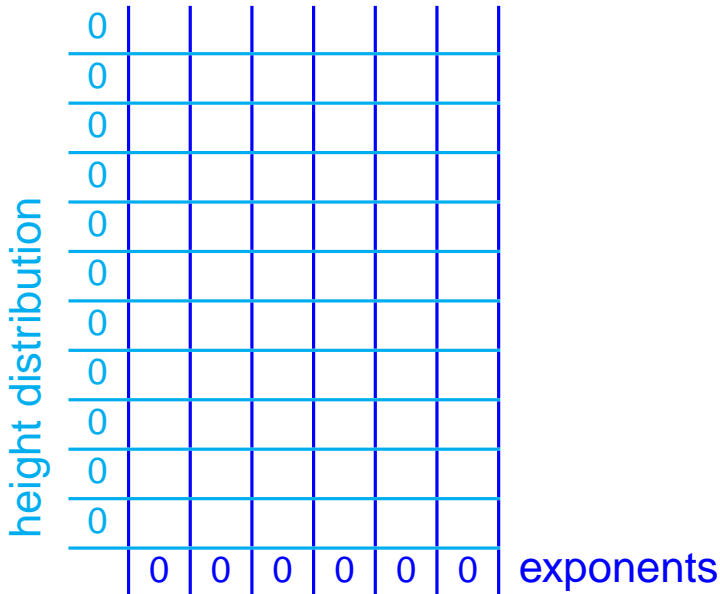
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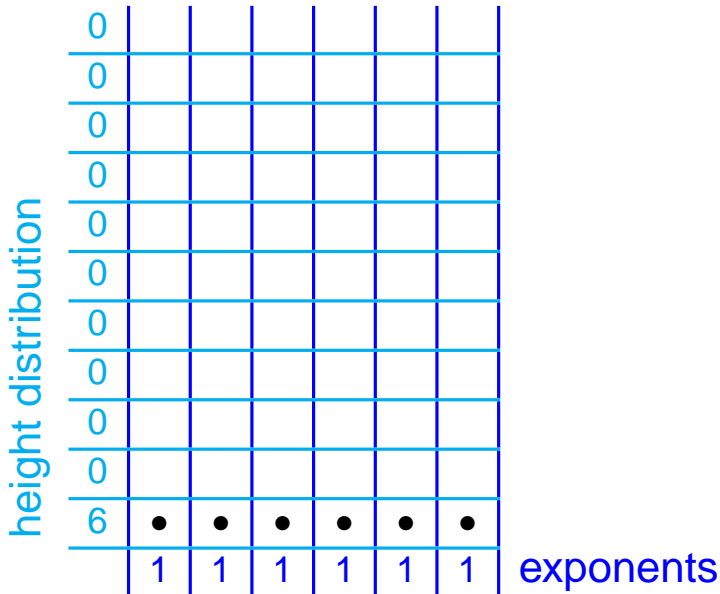
Every ideal subarrangement has an SFF.

SFF of $\mathcal{A}(E_6)$

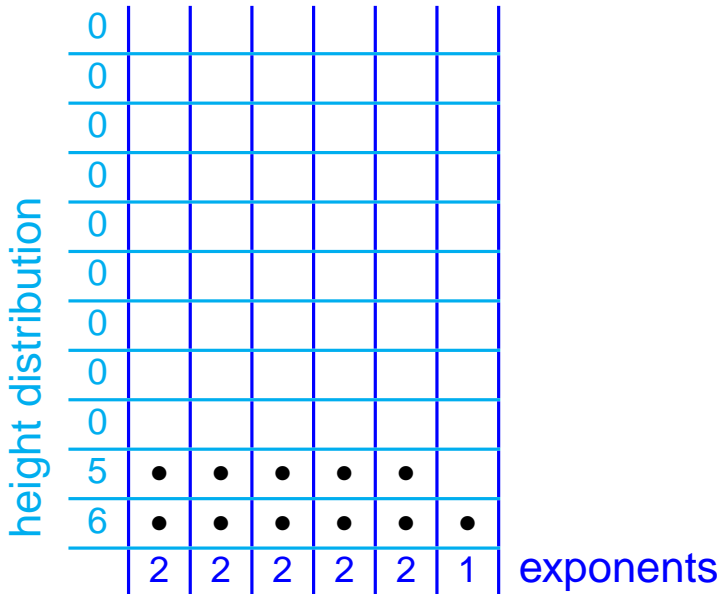
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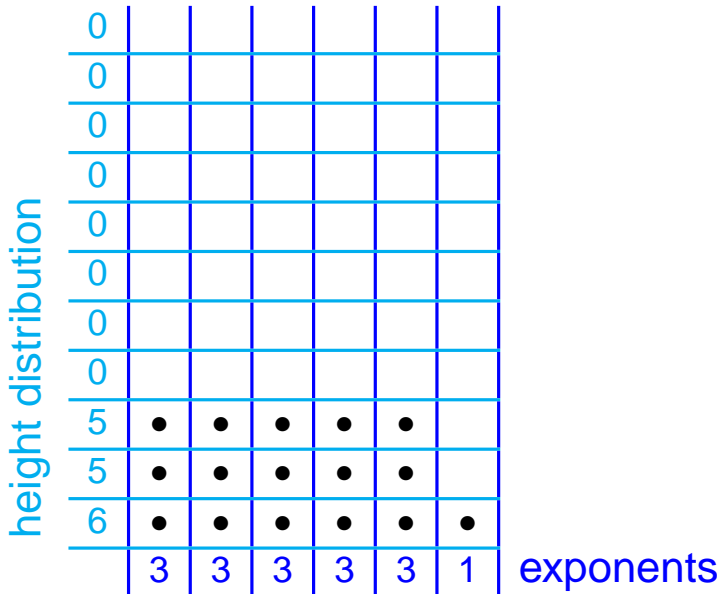
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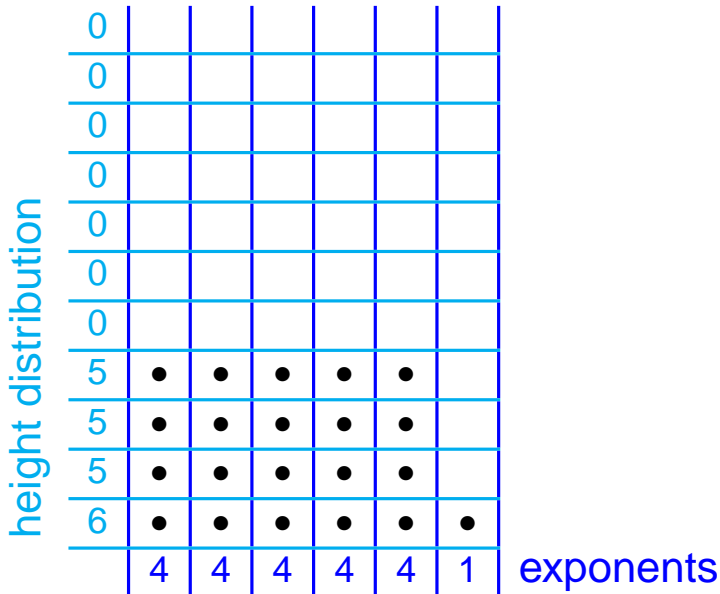
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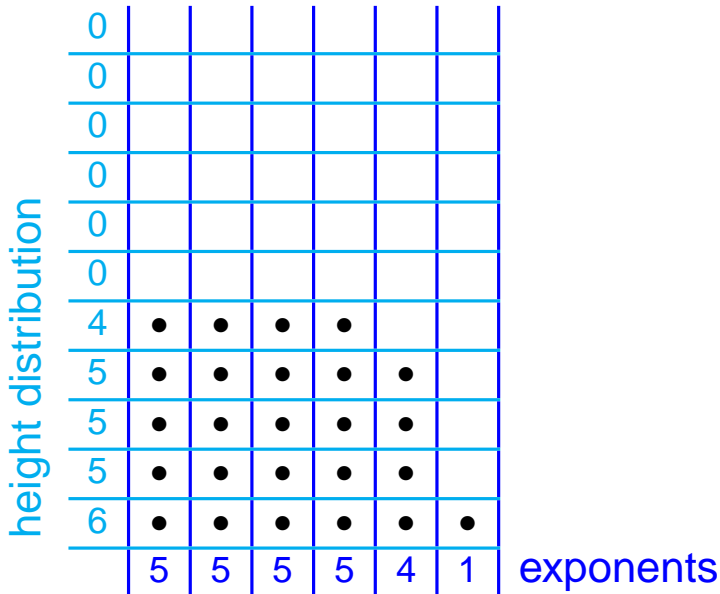
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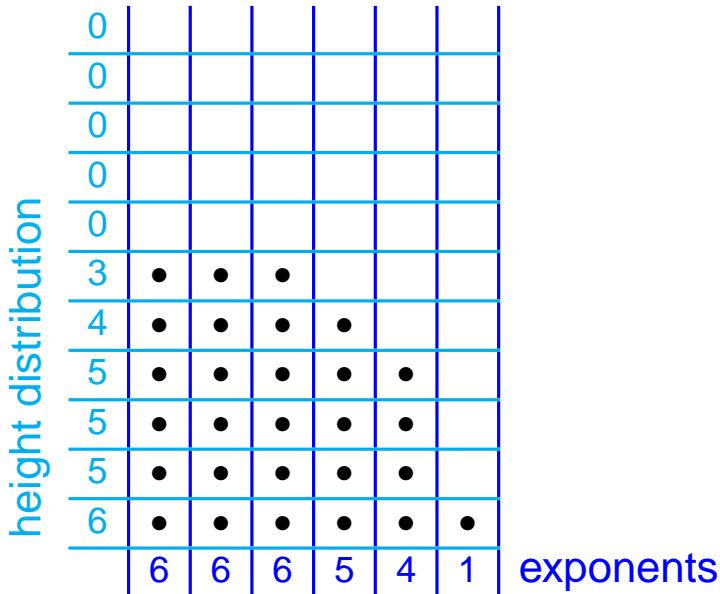
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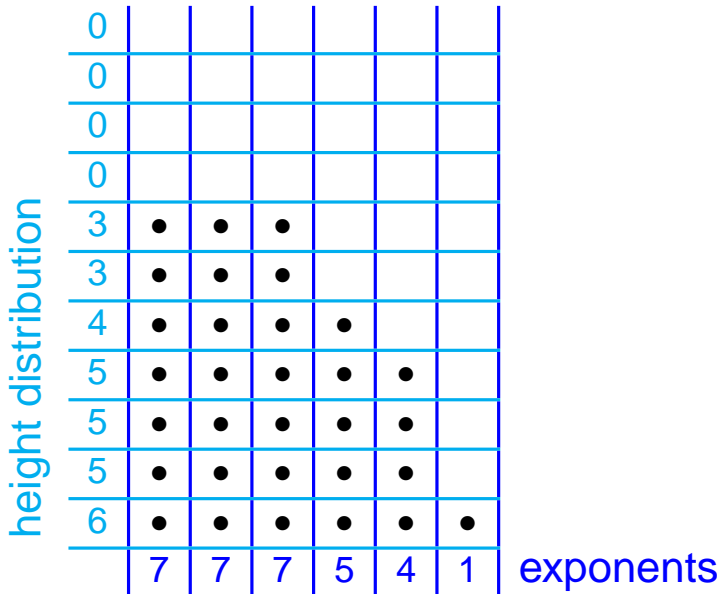
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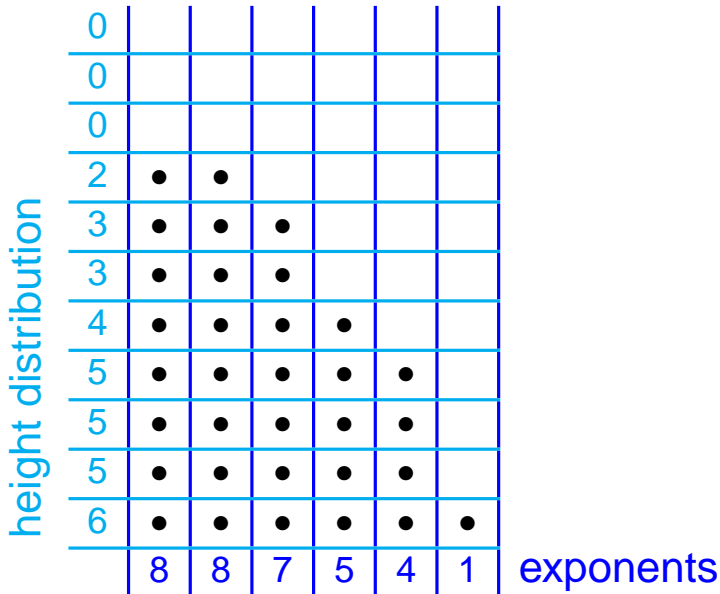
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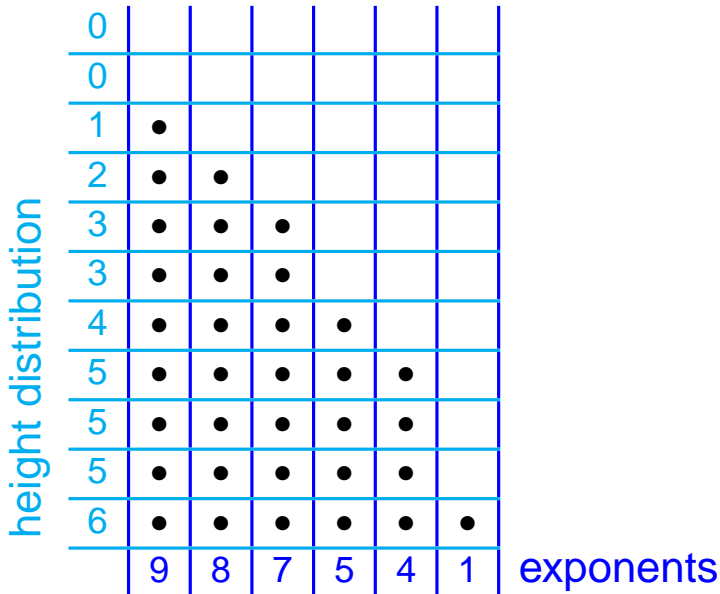
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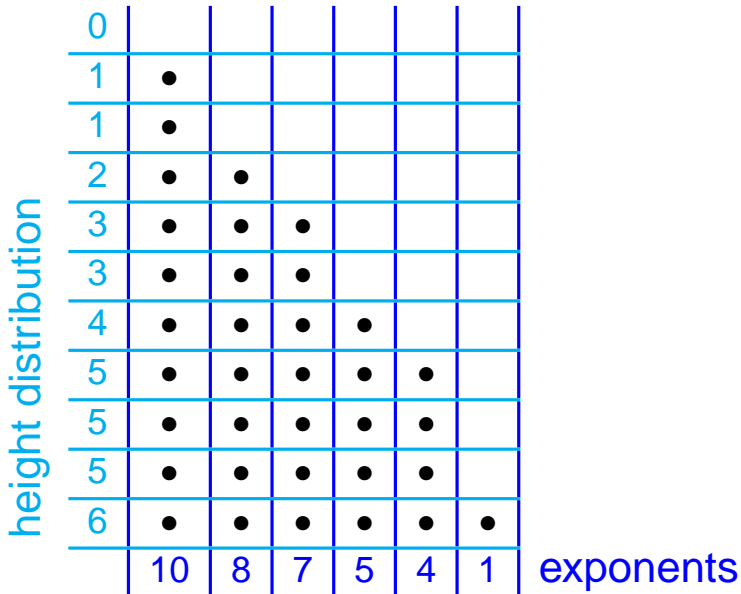
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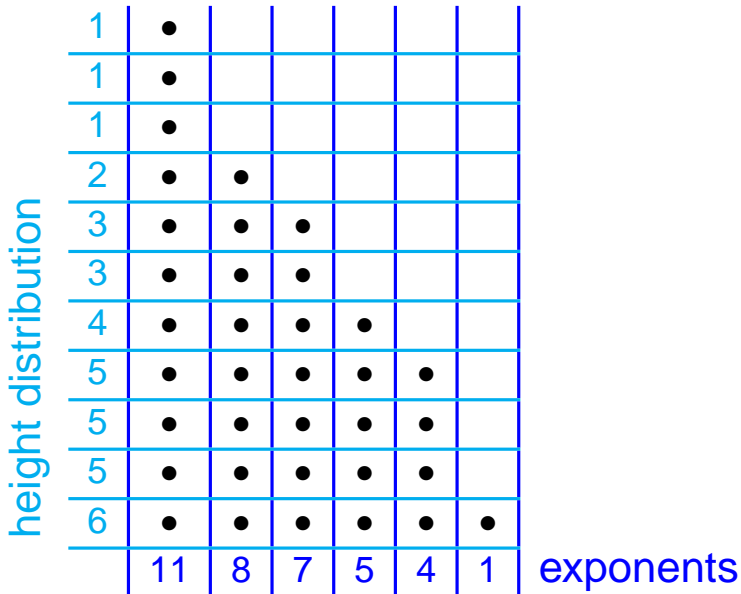
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Contents

- 1 Free Arrangements and Exponents
- 2 Shapiro-Steinberg-Kostant-Macdonald
Dual-Partition Formula
- 3 Ideal Free Theorem
- 4 Affine Weyl Arrangement has a Saturated
Free Filtration (SFF)

Affine Weyl arrangement has an SFF

For a root system Φ , choose a linear order $(\alpha_1, \dots, \alpha_n)$ on the set Φ^+ of positive roots in such a way that $\{\alpha_i\}_{i=1}^k$ is an ideal of Φ^+ for any $1 \leq k \leq n$.

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$$\{ \{\alpha_i - j = 0\} \mid j \in \mathbb{Z}, 1 \leq i \leq n \}.$$

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Let $\mathcal{A}_\infty(\Phi)$ denote the cone of the affine Weyl arrangement:

$$\mathcal{A}_\infty(\Phi) := \{H_z\} \cup \{H_i^j \mid j \in \mathbb{Z}, 1 \leq i \leq n\}.$$

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Put

$$K_0 := H_z, \quad K_p := \begin{cases} H_r^{-q} & \text{if } 1 \leq r \leq n, \\ H_{2n+1-r}^{q+1} & \text{if } n+1 \leq r \leq 2n, \end{cases}$$

where $p \in \mathbb{Z}_{>0}$ with $p = 2nq + r$ ($1 \leq r \leq 2n$, $q \in \mathbb{Z}_{\geq 0}$).

Affine Weyl arrangement has an SFF

Then the filtration $\mathcal{A}_i := \{K_0 = H_z, K_1, K_2, \dots, K_i\}$ ($i \in \mathbb{Z}_{>0}$) is called the **standard filtration** of $\mathcal{A}_\infty(\Phi)$.

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The **standard filtration** $\mathcal{A}_1 \subseteq \dots \subseteq \mathcal{A}_i \subseteq \dots$ of $\mathcal{A}_\infty(\Phi)$ is an **SFF**.
Moreover, the exponents of \mathcal{A}_i is the **dual partition** of the pair

$$(\{z\} \cup \{\alpha - jz \mid \alpha \in \Phi^+, j \in \mathbb{Z}, \{\alpha - jz = 0\} \in \mathcal{A}_i\}, \widetilde{\text{ht}}).$$

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Here, for $\alpha \in \Phi^+$ and $j \in \mathbb{Z}$, define the **extended height function** $\widetilde{\text{ht}}$ by

$$\widetilde{\text{ht}}(\alpha - jz) = \begin{cases} -\text{ht}(\alpha) + jh + 1 & \text{if } j > 0, \\ \text{ht}(\alpha) - jh & \text{if } j \leq 0. \end{cases}$$

We also define $\widetilde{\text{ht}}(z) = 1$.

Shi arrangements and Catalan arrangements

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Then it is known (M. Yoshinaga (2004)) that

$$\exp(\text{Shi}^q(\Phi)) = (1, qh, qh, \dots, qh)$$

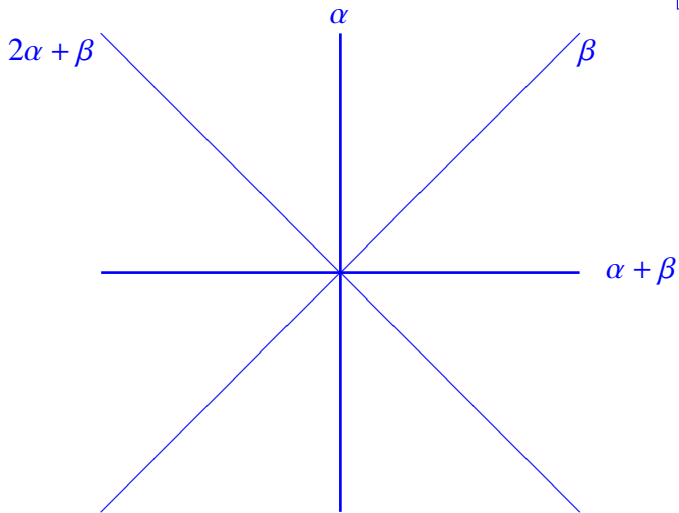
$$\exp(\text{Cat}^q(\Phi)) = (1, qh + m_1, qh + m_2, \dots, qh + m_\ell),$$

where h is the Coxeter number and

$$\exp(\mathcal{A}(\Phi)) = (m_1, m_2, \dots, m_\ell).$$

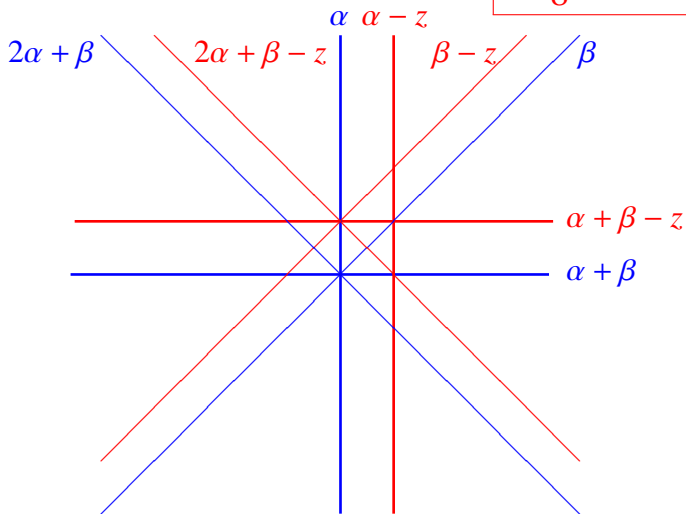
SFF of the Affine Weyl arrangement

$\mathcal{A}(B_2)$



SFF of the Affine Weyl arrangement

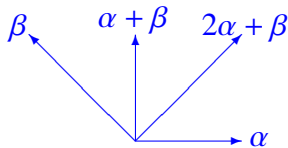
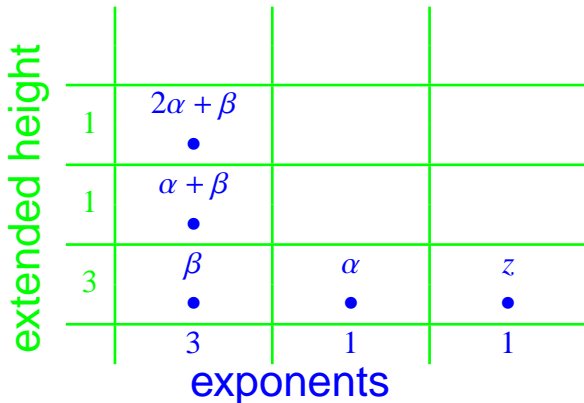
$$\mathcal{A}_8 = \text{Shi}^1(B_2)$$



SFF of the Affine Weyl arrangement

Dual-Partition Formula

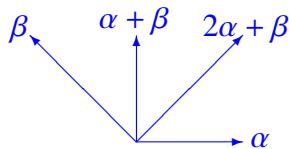
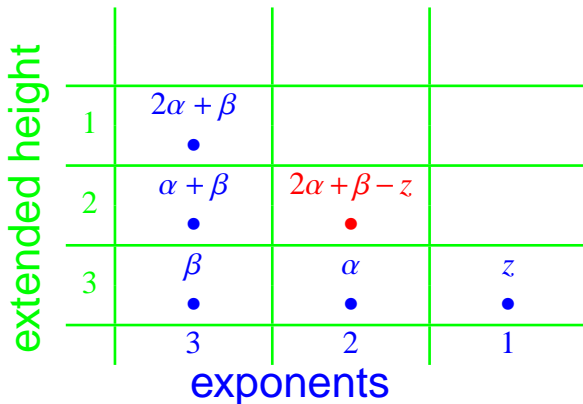
$$\mathcal{A}_4 = \mathcal{A}(B_2) \cup \{H_z\}$$



SFF of the Affine Weyl arrangement

Dual-Partition Formula

\mathcal{A}_5



SFF of the Affine Weyl arrangement

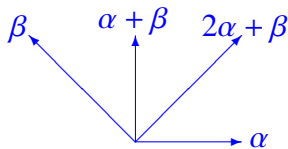
Dual-Partition Formula

\mathcal{A}_6

extended height

| | | | |
|---|------------------------|----------------------------|----------|
| 2 | $2\alpha + \beta$ • | $\alpha + \beta - z$ • | |
| 2 | $\alpha + \beta$ • | $2\alpha + \beta - z$ • | |
| 3 | β • | α • | z • |
| | 3 | 3 | 1 |

exponents

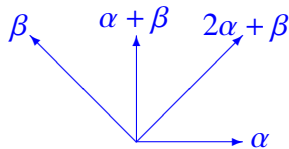


SFF of the Affine Weyl arrangement

Dual-Partition Formula

\mathcal{A}_7

| | | | | |
|-----------------|-----------|-------------------|-----------------------|--|
| extended height | 1 | | $\beta - z$ | |
| | | | • | |
| | 2 | $2\alpha + \beta$ | $\alpha + \beta - z$ | |
| | | • | • | |
| | 2 | $\alpha + \beta$ | $2\alpha + \beta - z$ | |
| | • | • | | |
| 3 | β | α | z | |
| | • | • | • | |
| | 3 | 4 | 1 | |
| | exponents | | | |

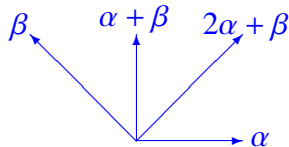


SFF of the Affine Weyl arrangement

Dual-Partition Formula

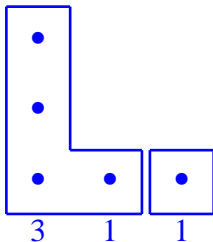
$$\mathcal{A}_8 = \text{Shi}^1(B_2)$$

| | | | | |
|-----------------|-----------|------------------------|----------------------------|----------|
| extended height | 2 | $\alpha - z$ • | $\beta - z$ • | |
| | 2 | $2\alpha + \beta$ • | $\alpha + \beta - z$ • | |
| | 2 | $\alpha + \beta$ • | $2\alpha + \beta - z$ • | |
| | 3 | β • | α • | z • |
| | | 4 | 4 | 1 |
| | exponents | | | |



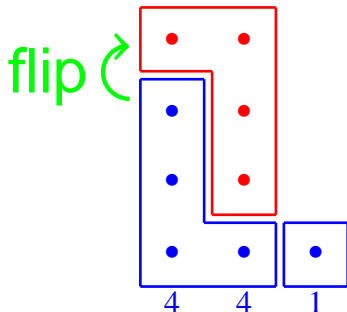
SFF of the Affine Weyl arrangement

$$\mathcal{A}_4 = \mathcal{A}(B_2)$$



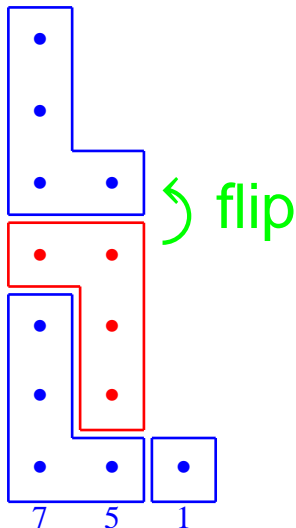
SFF of the Affine Weyl arrangement

$$\mathcal{A}_8 = \text{Shi}^1(B_2)$$

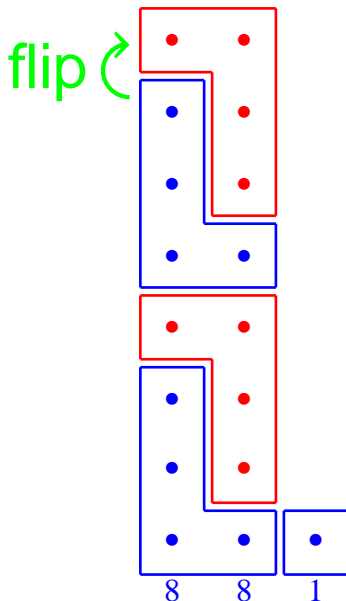


SFF of the Affine Weyl arrangement

$$\mathcal{A}_{12} = \text{Cat}^1(B_2)$$



SFF of the Affine Weyl arrangement



$$\mathcal{A}_{16} = \text{Shi}^2(B_2)$$

Summary

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Summary

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- ③ Each ideal subarrangement of the Weyl arrangements has an **SFF**. Each filter satisfies the celebrated **dual-partition formula** for a root system by **Shapiro-Steinberg-Kostant-Macdonald**.
- ④ Each affine Weyl arrangement has an **SFF**. Each filter satisfies the **dual-partition formula** with respect to the extended height function.

Two Questions

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- 1 Which subarrangement of a Weyl arrangement has an SFF?

Two Questions

- ① Which subarrangement of a Weyl arrangement has an SFF?
- ② Which subarrangement of the (cone of) an affine Weyl arrangement has an SFF?

I stop here.

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Thanks for your attention!