Multiple addition theorem on arrangements of hyperplanes and a proof of the Shapiro-Steinberg-Kostant-Macdonald dual-partition formula

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The 1st Workshop of JSPS-MAE Sakura Program
“Geometry and Combinatorics of Hyperplane Arrangements and Related Problems”
Sapporo, Japan

2014.09.03
Credit

with

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Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)
Michael Cuntz (Leibniz Universität Hannover)
Torsten Hoge (Leibniz Universität Hannover)
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Free arrangements and the Addition Theorem (AT)
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Multiple Addition Theorem (MAT)

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1. Free arrangements and the Addition Theorem (AT)

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A free arrangement in an $\ell$-dimensional vector space $V$ is an arrangement of hyperplanes $A \subseteq V^*$ such that $\ker(\alpha_H) = H$ for all $H \in A$. The symmetric algebra $S(V^*)$ is defined. A graded $S$-module $D(A)$ is defined as the set of $R$-linear derivations $\theta$ with $\theta(\alpha_H) \in \alpha_H S$ for all $H \in A$. A free arrangement is one where $D(A)$ is a free $S$-module. When $A$ is free, there exist homogeneous bases $\theta_1, \theta_2, \ldots, \theta_\ell$ with degree $d_i = \deg \theta_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $A$. 

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\( \mathcal{A} \): an arrangement of hyperplanes in an \( \ell \)-dimensional vector space \( V \)
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
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- $S := S(V^*)$: the symmetric algebra of the dual space $V^*$
Free Arrangements and their Exponents

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- \( \alpha_H \in V^*: \ker(\alpha_H) = H \) for \( H \in \mathcal{A} \)
- \( S := S(V^*) \): the symmetric algebra of the dual space \( V^* \)
- Define a graded \( S \)-module

\[
D(\mathcal{A}) := \{ \theta | \theta \text{ is an } \mathbb{R}-\text{linear derivation with } \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}.
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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free $S$-module.
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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free $S$-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$. 

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Free Arrangements and their Exponents

Example.

(\text{the braid arrangement (Weyl arrangement of type }A_3)\]

$$A = \{ \ker(\lambda^i - \lambda^j) \mid 1 \leq i < j \leq 4 \}$$

The exponents are (0, 1, 2, 3) because ...
Free Arrangements and their Exponents

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$$\mathcal{A} := \{\ker(x_i - x_j) \mid 1 \leq i < j \leq 4\}$$

The $S$-module $D(\mathcal{A})$ is a free module with a basis

$$\begin{align*}
\theta_0 &= (\partial/\partial x_1) + (\partial/\partial x_2) + (\partial/\partial x_3) + (\partial/\partial x_4) \\
\theta_1 &= x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2) + x_3(\partial/\partial x_3) + x_4(\partial/\partial x_4) \\
\theta_2 &= x_1^2(\partial/\partial x_1) + x_2^2(\partial/\partial x_2) + x_3^2(\partial/\partial x_3) + x_4^2(\partial/\partial x_4) \\
\theta_3 &= x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2) + x_3^3(\partial/\partial x_3) + x_4^3(\partial/\partial x_4).
\end{align*}$$
Free Arrangements and their Exponents

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$$\theta_3 = x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2) + x_3^3(\partial/\partial x_3) + x_4^3(\partial/\partial x_4).$$

Thus the exponents are:

$$(\deg \theta_0, \deg \theta_1, \deg \theta_2, \deg \theta_3) = (0, 1, 2, 3).$$
A Triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\)

Fix \(H \in \mathcal{A}\).

Define a triple \((\mathcal{A}, \mathcal{A}', \mathcal{A}'')\) by

\[ \mathcal{A}' : = \mathcal{A} \setminus \{H\}, \]

\[ \mathcal{A}'' : = \{H \cap K \mid K \in \mathcal{A}'\} \] (an arrangement in \(H\)).

In this case we have:

\[ \exp(\mathcal{A}') = (0, 1, 2, 2), \]

\[ \exp(\mathcal{A}) = (0, 1, 2, 3), \]

\[ \exp(\mathcal{A}'') = (0, 1, 2). \]

This example is generalized into the Addition Theorem \((\mathcal{T})\).
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[Diagram showing braid arrangement $A_3$]
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In this case we have:

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This example is generalized into the Addition Theorem (AT) ....
Addition Theorem (AT)

Theorem

(H. Terao (1980)) For a triple $(A, A', A'')$, suppose that $A'$ is free with $\exp(A') = (d_1, d_2, \ldots, d_{\ell-1}, d_\ell)$ and $A''$ is free with $\exp(A'') = (d_1, d_2, \ldots, d_{\ell-1})$. Then $A$ is also free with $\exp(A) = (d_1, d_2, \ldots, d_\ell+1)$.

Remark. In the AT, $d_\ell$ is not necessarily the maximum exponent in $\exp(A')$. 

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Theorem

\textit{(H. T.(1980)) For a triple }\(\mathcal{A}, \mathcal{A}', \mathcal{A}''\)\textit{, suppose that }\(\mathcal{A}'\) \textit{is free with } \exp(\mathcal{A}') = (d_1, d_2, \ldots, d_{\ell-1}, d_\ell)\textit{ and }\mathcal{A}''\textit{ is free with } \exp(\mathcal{A}'') = (d_1, d_2, \ldots, d_{\ell-1})\textit{. Then }\mathcal{A}\textit{ is also free with } \exp(\mathcal{A}) = (d_1, d_2, \ldots, d_\ell + 1)\textit{.}
Addition Theorem (AT)

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Recall that, for the braid arrangement (the Weyl arrangement of type \(A_3\)),

\[\exp(\mathcal{A}') = (0, 1, 2, 2), \quad \exp(\mathcal{A}'') = (0, 1, 2), \quad \exp(\mathcal{A}) = (0, 1, 2, 3).\]
Theorem

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Recall that, for the braid arrangement (the Weyl arrangement of type \(A_3\)),

\[\exp(\mathcal{A}') = (0, 1, 2, 2), \quad \exp(\mathcal{A}'') = (0, 1, 2), \quad \exp(\mathcal{A}) = (0, 1, 2, 3).\]

Remark. In the AT, \(d_\ell\) is not necessarily the maximum exponent in \(\exp(\mathcal{A}')\).
1. Free arrangements and the Addition Theorem (A T)

2. Multiple Addition Theorem (MAT)

3. Shapiro-Steinberg-Kostant-Macdonald Dual-partition Formula

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Multiple Addition Theorem (MAT)

Theorem

Let $A'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d_\ell$.

Let $H_1, \ldots, H_q$ be (new) hyperplanes.

Define $A''_j := \{ H \cap H_j \mid H \in A' \}$ ($j = 1, \ldots, q$).

Assume

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X^* \cup H \in A'$,
3. $|A'\setminus |A''_j| = d_j$ ($j = 1, \ldots, q$) (Remark: $\leq$ always holds true).

Then

(a) $q \leq p$ and (b) $A := A' \cup \{ H_1, \ldots, H_q \}$ is free with exponents $(d_1, \ldots, d_\ell - q, (d_\ell + 1)q)$.

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Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$. 
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Let $H_1, \ldots, H_q$ be (new) hyperplanes.

Define $\mathcal{A}'' := \{H \cap H_j \mid H \in \mathcal{A}'\}$ ($j = 1, \ldots, q$).
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(1) $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
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(1) $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,

(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and
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Assume

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X \not\subset \bigcup_{H \in \mathcal{A}'} H$, and
3. $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($j = 1, \ldots, q$).
**Theorem**

\((ABCHT(2014?))\) Let \(\mathcal{A}'\) be a \textit{free} arrangement with \textit{exponents} \((d_1, \ldots, d_\ell)\) \((d_1 \leq \cdots \leq d_\ell)\) and \(1 \leq p \leq \ell\) the multiplicity of the maximum exponent \(d\).

Let \(H_1, \ldots, H_q\) be (new) hyperplanes.

Define \(\mathcal{A}'' := \{H \cap H_j \mid H \in \mathcal{A}'\}\) \((j = 1, \ldots, q)\).

Assume

1. \(X := H_1 \cap \cdots \cap H_q\) is \(q\)-\textit{codimensional},
2. \(X \not\subseteq \bigcup_{H \in \mathcal{A}'} H\), and
3. \(|\mathcal{A}'| - |\mathcal{A}_j''| = d\) \((j = 1, \ldots, q)\) \((\textbf{Remark:} \leq \text{always holds true})\).
Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$.

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(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and

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Then

(a) $q \leq p$
Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$.

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1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and
3. $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($j = 1, \ldots, q$) (Remark: $\leq$ always holds true).

Then (a) $q \leq p$ and (b) $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d + 1)^q)$. 

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Addition Theorem (AT)

\[ A : = A' \cup \{H_1, H_2\} \]

\[ \text{AT} \rightarrow \text{add} H_1 (0, 1, 1) \]

\[ \text{AT} \rightarrow \text{add} H_2 \]

• (0, 1, 1)

• (0, 1, 2)
Addition Theorem (AT)

\[ \mathcal{A}' \]

\( (0, 1, 1, 1) \)
Addition Theorem (AT)

\[ A' \]

\( (0, 1, 1, 1) \)
Addition Theorem (AT)

\[ \mathcal{A}' \]

\[ (0, 1, 1, 1) \]

\[ \mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\} \]

\[ (0, 1, 2, 2) \]
Addition Theorem (AT)

\[ \mathcal{A}' (0, 1, 1, 1) \xrightarrow{\text{add } H_1} (0, 1, 1, 2) \xrightarrow{\text{add } H_2} \mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\} (0, 1, 2, 2) \]
Multiple Addition Theorem (MAT)
Multiple Addition Theorem (MAT)

\[ \mathcal{A} : = \mathcal{A}' \cup \{ H_1, H_2 \} \]

\( (0, 1, 2, 2) \)

\[ |A'| - |A''_1| = 3 - 2 = 1 \]

\[ |A'| - |A''_2| = 3 - 2 = 1 \]
Multiple Addition Theorem (MAT)

Add 2 hyperplanes

\[ \mathcal{A}' = (0, 1, 1, 1) \]

\[ \mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\} \]

\[ (0, 1, 2, 2) \]
Multiple Addition Theorem (MAT)

\[ \mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\} \]

\[
\begin{align*}
\mathcal{A}' &= (0, 1, 1, 1) \\
X &= H_1 \cap H_2 \\
\mathcal{A} &= (0, 1, 2, 2)
\end{align*}
\]

Add 2 hyperplanes
Multiple Addition Theorem (MAT)

\[ \mathcal{A}' = (0, 1, 1, 1) \]

\[ \mathcal{A} := \mathcal{A}' \cup \{H_1, H_2\} \]

\[ (0, 1, 2, 2) \]

add 2 hyperplanes

\[ \mathcal{A}'' \]

\[ H_1 \cap H_2 = X \]

\[ \mathcal{A}_1 \]

\[ \mathcal{A}_2 \]
Multiple Addition Theorem (MAT)

\[ A' = (0, 1, 1, 1) \]

\[ A'' \]

add 2 hyperplanes \( \implies \) MAT

\[ \mathcal{A} := A' \cup \{H_1, H_2\} \]

\[ (0, 1, 2, 2) \]

\[ X = H_1 \cap H_2 \]

\[ |\mathcal{A}'| - |\mathcal{A}'_1| = 3 - 2 = 1 \]

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Multiple Addition Theorem (MAT)

$T$ and $A^\prime = A^\prime_1 \cup \left\{ H_1 \right\}$ is free with exponents $(1, 2, 2)$, $d = 2$ (the max exponent).

$A^\prime_1 = \left\{ H \cap H_1 | H \in A^\prime \right\}$, $|A^\prime_1| = 3$ and $|A^\prime| - |A^\prime_1| = 5 - 3 = 2$.

Thus $A = A^\prime \cup \left\{ H_1 \right\}$ with exponents $(1, 2, 3)$. 

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Multiple Addition Theorem (MAT)

\[ H_1 = A' \cup \{ H \} \]

\[ A' = \left\{ H \cap H_1 \mid H \in A' \right\} \]

\[ |A'| = 3 \] and \[ |A''| - |A'| = 2 \]

Thus \[ A = A' \cup \{ H_1 \} \] with exponents \((1, 2, 3)\).
Multiple Addition Theorem (MAT)

\[ A' \]

\[ A = A' \cup \{H_1\} \]
Multiple Addition Theorem (MAT)

\[ A' \]

\[ A = A' \cup \{H_1\} \]

\[ H_1 \]

\[ H_1' \]
\[ \mathcal{A}' \] is free with exponents \((1, 2, 2)\), \(d = 2\) (the max exponent).
Multiple Addition Theorem (MAT)

\( \mathcal{A}' \) is free with exponents \((1, 2, 2)\), \( d = 2 \) (the max exponent).

\( \mathcal{A}_{1}'' := \{ H \cap H_{1} \mid H \in \mathcal{A}' \} \), \( |\mathcal{A}_{1}''| = 3 \) and \( |\mathcal{A}'| - |\mathcal{A}_{1}''| = 5 - 3 = 2 \).
Multiple Addition Theorem (MAT)

\[ A' \] is free with exponents \((1, 2, 2)\), \(d = 2\) (the max exponent).

\[ A''_1 := \{ H \cap H_1 \mid H \in A' \}, |A''_1| = 3 \text{ and } |A'| - |A''_1| = 5 - 3 = 2. \]

Thus \( A = A' \cup \{ H_1 \} \) with exponents \((1, 2, 3)\)
Remark. The multiple addition theorem (MAT) does not generalize the addition theorem (AT). The MAT is a theorem which is applicable to a relatively narrow class of arrangements because the only maximum exponents can increase. So it is natural to ask the following question. Is there any significant application of MAT?
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The MAT is a theorem which is applicable to a relatively narrow class of arrangements because the only maximum exponents can increase.

So it is natural to ask the following

Question. Is there any significant application of MAT?
What are Dual Partitions?

$36 = 1 + 4 + 5 + 7 + 8 + 11$

\[\uparrow\text{Dual Partitions}\]

$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 5 + 6$
What are Dual Partitions?
What are Dual Partitions?
What are Dual Partitions?

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What are these numbers?

\[ 36 = 1 + 4 + 5 + 7 + 8 + 11 \]

\[ \uparrow \text{Dual Partitions} \]

\[ 36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6 \]
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(1, 4, 5, 7, 8, 11) is the exponents of the root system of the type $E_6$

$\uparrow$ Dual Partitions

(1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 5, 6) is the height distribution of the positive roots of the type $E_6$
the dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald

Theorem.

Remark.

Remark. (1) This theorem can be regarded as a method to "reading off" the exponents from the root structure.

Remark. (2) The other methods to find the exponents include: (a) from the degrees of basic invariants, (b) from the eigenvalues of a Coxeter transformation, etc. 

H. Terao (Hokkaido University)
Theorem

(The dual-partition formula by Shapiro, Steinberg, Kostant (1959), Macdonald (1972))
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The exponents of an irreducible root system and the height distribution of positive roots are dual partitions to each other.
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**Theorem**

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*The exponents of an irreducible root system and the height distribution of positive roots are dual partitions to each other.*

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Dynkin diagrams (root systems) and exponents

\[ A_\ell : \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{\ell-1} \cdot \alpha_{\ell} (1, 2, \ldots, \ell) \]

\[ B_\ell : \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{\ell-1} / \alpha_{\ell} (1, 3, 5, \ldots, 2\ell-1) \]

\[ C_\ell : \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{\ell-1} \cdot o \alpha_{\ell} (1, 3, 5, \ldots, 2\ell-1) \]

\[ D_\ell : \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_{\ell-2} \cdot \alpha_{\ell-1} \cdot \alpha_{\ell} (1, 3, 5, \ldots, 2\ell-3, \ell-1) \]

\[ E_6 : \alpha_1 \cdot \alpha_3 \cdot \alpha_4 \cdot \alpha_2 \cdot \alpha_5 \cdot \alpha_6 (1, 4, 5, 7, 8, 11) \]
Dynkin diagrams (root systems) and exponents
Dynkin diagrams (root systems) and exponents

\( \alpha_1 \alpha_2 \alpha_\ell \)

\( \alpha_1 \alpha_2 \alpha_{\ell-1} \alpha_\ell \)

\( \alpha_1 \alpha_2 \alpha_{\ell-1} \alpha_\ell \)

\( \alpha_1 \alpha_2 \alpha_{\ell-2} \alpha_{\ell-1} \alpha_\ell \)

\( \alpha_1 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \)

\( (1, 2, \ldots, \ell) \)

\( (1, 3, 5, \ldots, 2\ell - 1) \)

\( (1, 3, 5, \ldots, 2\ell - 1) \)

\( (1, 3, 5, \ldots, 2\ell - 3, \ell - 1) \)

\( (1, 4, 5, 7, 8, 11) \)
Dynkin diagrams (root systems) and exponents

$E_7$:  $(1, 5, 7, 9, 11, 13, 17)$ 

$E_8$:  $(1, 7, 11, 13, 17, 19, 23, 29)$

$F_4$:  $(1, 5, 7, 11)$

$G_2$:  $(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/abenori/tex/tex7.html)
Height of positive roots

Φ: an irreducible root system of rank ℓ

∆ = {α₁, ..., αₙ}: a simple system of Φ

Φ⁺: the set of positive roots

ht(α) = ∑ₖ₌₁ⁿ cₖ(α) (cₖ ∈ Z ≥ 0)

The height distribution in Φ⁺ is a sequence of positive integers (i₁, i₂, ..., iₘ), where iₖ = |{α ∈ Φ⁺ | ht(α) = k}| (1 ≤ k ≤ m)

Φ : an irreducible root system of rank ℓ
Height of positive roots

- $\Phi$: an irreducible root system of rank $\ell$
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of $\Phi$
Height of positive roots

- $\Phi$: an irreducible root system of rank $\ell$
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of $\Phi$
- $\Phi^+$: the set of positive roots

$\text{ht}(\alpha) = \sum_{i=1}^{\ell} c_i \alpha_i$ (for a positive root $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ with $c_i \in \mathbb{Z}_{\geq 0}$)

The height distribution in $\Phi^+$ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j = |\{\alpha \in \Phi^+ | \text{ht}(\alpha) = j\}|$ (for $1 \leq j \leq m$).
Height of positive roots

- $\Phi$: an irreducible root system of rank $\ell$
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of $\Phi$
- $\Phi^+$: the set of positive roots
- $\text{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (height) for a positive root $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ ($c_i \in \mathbb{Z}_{\geq 0}$)
Height of positive roots

- \( \Phi \): an irreducible root system of rank \( \ell \)
- \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \): a simple system of \( \Phi \)
- \( \Phi^+ \): the set of positive roots
- \( \text{ht}(\alpha) := \sum_{i=1}^{\ell} c_i \) (height) for a positive root \( \alpha = \sum_{i=1}^{\ell} c_i \alpha_i \) (\( c_i \in \mathbb{Z}_{\geq 0} \))
- The height distribution in \( \Phi^+ \) is a sequence of positive integers \( (i_1, i_2, \ldots, i_m) \), where \( i_j := |\{ \alpha \in \Phi^+ \mid \text{ht}(\alpha) = j \}| \) \( (1 \leq j \leq m) \)
Height of positive roots ($E_6$)

• $\alpha_1$
• $\alpha_3$
• $\alpha_4$
• $\alpha_2$
• $\alpha_5$
• $\alpha_6$

Exponents:

(1, 4, 5, 7, 8, 11)

List of positive roots:

height 1:
$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2:
$\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3:
$\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

height 11:
$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)
Height of positive roots ($E_6$)

$E_6$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_2 \]

Exponents: (1, 4, 5, 7, 8, 11)

List of positive roots:
Height of positive roots ($E_6$)

$E_6$: \[ \bullet - - - - - - - - - \]

$\alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \bullet \quad \alpha_2$

**Exponents:** (1, 4, 5, 7, 8, 11)

**List of positive roots:**

- height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$
- height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$
- height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$
Height of positive roots ($E_6$)

$E_6$:  
\[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \]

\[ \bullet \alpha_2 \]

**Exponents:** (1, 4, 5, 7, 8, 11)

**List of positive roots:**
- height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$
- height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$
- height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$
  
  .
  .
  .
  .

\[ \sim \alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \]

(highest root)
Height of positive roots ($E_6$)

$E_6$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]
\[ \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \]
\[ \bullet \alpha_2 \]

Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:
height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$
height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$
height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

\[ \ldots \]

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)
### Height of positive roots ($E_6$)

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<td>$ht=2$</td>
<td>$\alpha_1 + \alpha_3$ $\alpha_2 + \alpha_4$ $\alpha_3 + \alpha_4$ $\bullet$ $\bullet$</td>
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<tr>
<td>$ht=1$</td>
<td>$\alpha_1$ $\alpha_2$ $\alpha_3$ $\alpha_4$ $\alpha_5$ $\alpha_6$</td>
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\[\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,\quad ht(\tilde{\alpha}) = 11\] (the highest root)
**Height Distribution** \((E_6)\)

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H. Terao (Hokkaido University)  
2014.09.03
## Exponents ($E_6$)

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- H. Terao (Hokkaido University)
- 2014.09.03
The Dual-Partition Formula ($E_6$)

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we shall presently describe, of "reading off" the exponents from the root structure of $g$ was discovered by Arnold Shapiro. However, even though one verifies that the numbers produced by this procedure agree with the exponents, the important question of proving that this "agreement" is more than just a coincidence remained open.

(1959) A. Shapiro (empirical proof using the classification)
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(1972) B. Kostant (1st proof without using the classification)
(2014?) I. G. Macdonald (2nd proof: generating functions)
(2014?) ABCHT (for ideal subarr.: using free arrangements)

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.* ¹

By Bertram Kostant.

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Contents

1. Free arrangements and the Addition Theorem (A T)

2. Multiple Addition Theorem (MA T)

3. Shapiro-Steinberg-Kostant-Macdonald Dual-partition Formula

4. Ideal Subarrangement Theorem
Weyl arrangements

\[ \Phi^+ = \{ \ker(\alpha) | \alpha \in \Phi^+ \} : \]

The Weyl arrangement.
\( \Phi^+ : \text{the set of positive roots} \)
Weyl arrangements

$\Phi^+$ : the set of positive roots

$\mathcal{A} := \mathcal{A}(\Phi^+) := \{\ker(\alpha) \mid \alpha \in \Phi^+\}$ : the Weyl arrangement
The root poset and ideals

Definition

Introduce a partial order $\geq$ into the set $\Phi^+$ of positive roots by

$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} Z \geq 0 \alpha_i.$$ 

The poset is called the (positive) root poset.

A subset $I$ of $\Phi^+$ is called an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$, $\beta_1 \geq \beta_2$, $\beta_1 \in I \Rightarrow \beta_2 \in I$.

Definition

When $I$ is an ideal of $\Phi^+$ the arrangement $A(I) := \{\ker \alpha | \alpha \in I\}$ is called an ideal subarrangement of $A$. 

H. Terao (Hokkaido University)
the root poset and ideals

**Definition**

*Introduce a partial order* \( \geq \) *into the set* \( \Phi^+ \) *of positive roots by*

\[
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\]

The poset is called the *(positive) root poset*.

A subset \( I \) of \( \Phi^+ \) is called an *ideal* if, for \( \{ \beta_1, \beta_2 \} \subset \Phi^+ \), \( \beta_1 \geq \beta_2 \), \( \beta_1 \in I \) \( \Rightarrow \) \( \beta_2 \in I \).

**Definition**

When \( I \) is an ideal of \( \Phi^+ \) the arrangement \( A(I) \) : \( \{ \ker \alpha \mid \alpha \in I \} \) is called an *(ideal subarrangement)* of \( A \).
the root poset and ideals

**Definition**

*Introduce a partial order* \(\geq\) *into the set* \(\Phi^+\) *of positive roots by*

\[
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$$\beta_1 \geq \beta_2, \ \beta_1 \in I \implies \beta_2 \in I.$$ 

**Definition**

When $I$ is an ideal of $\Phi^+$ the arrangement $\mathcal{A}(I) := \{\ker \alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$. 
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3$: $\bullet \quad \bullet \quad \bullet$

$\alpha_1$  $\alpha_2$  $\alpha_3$

$\Phi^+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \}$
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3: \bullet ——— \bullet ——— \bullet$

$\alpha_1 \quad \alpha_2 \quad \alpha_3$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_3 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3$
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3$: \[ \bullet - \bullet - \bullet \]

$\alpha_1 \quad \alpha_2 \quad \alpha_3$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

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$\alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_3 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3$

Thus $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ is an ideal, while $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not.
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3: \bullet \longrightarrow \bullet \longrightarrow \bullet$

$\alpha_1 \quad \alpha_2 \quad \alpha_3$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3,$

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Note that the entire set $\Phi^+$ is always an ideal.
Main Theorem

**Theorem**

If $\Phi$ is an irreducible root system of rank $\ell$, $I$ is an ideal of $\Phi^+$, then

1. $A(I)$ is free, and
2. the exponents of $A(I)$ and the height distribution of the positive roots in $I$ are dual partitions to each other.

This positively settles a conjecture by Sommers-Tyoczko (2006).
Main Theorem

Theorem

If

\( \Phi \) : an irreducible root system of rank \( \ell \)
Main Theorem

Theorem

If

- $\Phi$: an irreducible root system of rank $\ell$
- $I$: an ideal of $\Phi^+$,
Main Theorem

Theorem

If

- \( \Phi \): an irreducible root system of rank \( \ell \)
- \( I \): an ideal of \( \Phi^+ \),

then

(1) \( A(I) \) is free, and
(2) the exponents of \( A(I) \) and the height distribution of the positive roots in \( I \) are dual partitions to each other.

This positively settles a conjecture by Sommers-Tymoczko (2006).
If

\( \Phi : \) an irreducible root system of rank \( \ell \)

\( I : \) an ideal of \( \Phi^+ \),

then

(1) \( \mathcal{A}(I) \) is free, and
Main Theorem

**Theorem**

If

- $\Phi$: an irreducible root system of rank $\ell$
- $I$: an ideal of $\Phi^+$,

then

1. $\mathcal{A}(I)$ is free, and
2. the exponents of $\mathcal{A}(I)$ and the height distribution of the positive roots in $I$ are dual partitions to each other.

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If

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(2) the exponents of \( \mathcal{A}(I) \) and the height distribution of the positive roots in \( I \) are dual partitions to each other.

This positively settles a conjecture by Sommers-Tymoczko (2006).
Main Corollary

In particular, when the ideal $I$ is equal to the entire $\Phi +$, our main theorem yields:

\[ \text{(The dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald)} \]

The exponents of the entire $\Phi$ and the height distribution of the entire positive roots are dual partitions to each other.
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**Corollary**

*(The dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald)*

The exponents of the entire $\Phi$ and the height distribution of the entire positive roots are dual partitions to each other.
Main Theorem (revisited)

Theorem

If $\Phi$: an irreducible root system
then $I$: an ideal of $\Phi$,
then

1. $A(I)$ is free,
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This positively settles a conjecture by Sommers-Tymoczko (2006).
Main Theorem (revisited)

Theorem

If

1. $\Phi :$ an irreducible root system of rank $\ell$

This positively settles a conjecture by Sommers-Tymoczko (2006).
Main Theorem (revisited)

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If

- $\Phi$: an irreducible root system of rank $\ell$
- $I$: an ideal of $\Phi^+$,
Main Theorem (revisited)

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If

- $\Phi$ : an irreducible root system of rank $\ell$
- $I$ : an ideal of $\Phi^+$,

then

(1) $\mathcal{A}(I)$ is free, and
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Proof of Main Theorem (just an outline)

We prove the Main Theorem applying MAT inductively.

Proof.
Let \( B \) be an ideal subarrangement of the Weyl arrangement \( A \) of a root system \( \Phi \).

For \( k \in \mathbb{Z}_{>0} \), define \( B \leq k = \{ H \in B \mid ht(\alpha_H) \leq k \} \).

We may easily verify \( B \leq 1 \) is a free arrangement with exponents \((0, 0, \ldots, 0, 1, 1, \ldots, 1)\).

We may apply MAT for \( A' = B \leq k \) and \( A = B \leq k + 1 \).

To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system \( \Phi \).

(A key Lemma is in the next page.)
We prove the Main Theorem applying MAT inductively.
Proof of Main Theorem (just an outline)

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Proof.
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Proof of Main Theorem (just an outline)

We prove the Main Theorem applying MAT inductively.

Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{ H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k \}$. 
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- We may apply MAT for $\mathcal{A}' := \mathcal{B}_{\leq k}$ and $\mathcal{A} := \mathcal{B}_{\leq k+1}$.
- To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Phi$. (A key Lemma is in the next page.)
Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim } X$. 
Local-global formula for heights *(A key Lemma)*

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim} X$.

The height of $\alpha$ in $\Phi_X$ is called the *local height* and is denoted by $\text{ht}_X \alpha$. 
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For $\alpha \in \Phi^+$, let

$$\mathcal{A}^\alpha := \mathcal{A}^{H_\alpha} = \{ K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{ H_\alpha \} \}.$$
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Lemma (Local-global formula for heights)

For $\alpha \in \Phi^+$, we have

$$\operatorname{ht}_\Phi \alpha - 1 = \sum_{X \in \mathcal{A}^\alpha} (\operatorname{ht}_X \alpha - 1).$$
Theorem. Let $A'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent.

Let $H_1, \ldots, H_q$ be (new) hyperplanes. Define $A''_j := \{ H \cap H_j | H \in A' \}$ ($j = 1, \ldots, q$).

Assume

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X^* \cup H \in A'$, and
3. $|A'| - |A''_j| = d_j$ ($j = 1, \ldots, q$) (Remark: $\leq$ always holds true).

Then

(a) $q \leq p$ and
(b) $A' := A' \cup \{ H_1, \ldots, H_q \}$ is free with exponents $(d_1, \ldots, d_\ell-q, (d+1)q)$.
Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$. 
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Multiple Addition Theorem (MAT) (Revisited)

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Theorem

\((ABCHT(2014?))\) Let \(\mathcal{A}'\) be a free arrangement with exponents \((d_1, \ldots, d_\ell)\) \((d_1 \leq \cdots \leq d_\ell)\) and \(1 \leq p \leq \ell\) the multiplicity of the maximum exponent \(d\).

Let \(H_1, \ldots, H_q\) be (new) hyperplanes.

Define \(\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}\) \((j = 1, \ldots, q)\).

Assume

(1) \(X := H_1 \cap \cdots \cap H_q\) is \(q\)-codimensional,
(2) \(X \nsubseteq \bigcup_{H \in \mathcal{A}'} H\), and
Multiple Addition Theorem (MAT) (Revisited)

Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$.

Let $H_1, \ldots, H_q$ be (new) hyperplanes.

Define $\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}$ ($j = 1, \ldots, q$).

Assume

(1) $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and
(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($j = 1, \ldots, q$)
Theorem

(ABCHT(2014?)) Let \( \mathcal{A}' \) be a free arrangement with exponents \((d_1, \ldots, d_\ell)\) \((d_1 \leq \cdots \leq d_\ell)\) and \(1 \leq p \leq \ell\) the multiplicity of the maximum exponent \(d\).

Let \( H_1, \ldots, H_q \) be (new) hyperplanes.

Define \( \mathcal{A}''_j : = \{ H \cap H_j \mid H \in \mathcal{A}' \}\) \((j = 1, \ldots, q)\).

Assume

1. \( X : = H_1 \cap \cdots \cap H_q \) is \( q \)-codimensional,
2. \( X \not\subseteq \bigcup_{H \in \mathcal{A}'} H\), and
3. \( |\mathcal{A}'| - |\mathcal{A}''_j| = d \) \((j = 1, \ldots, q)\) (Remark: \( \leq \) always holds true).
Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$.

Let $H_1, \ldots, H_q$ be (new) hyperplanes.

Define $\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}$ ($j = 1, \ldots, q$).

Assume

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X \notin \bigcup_{H \in \mathcal{A}'} H$, and
3. $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($j = 1, \ldots, q$) \textbf{(Remark: $\leq$ always holds true)}.

Then (a) $q \leq p$.
Multiple Addition Theorem (MAT) (Revisited)

Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the maximum exponent $d$.

Let $H_1, \ldots, H_q$ be (new) hyperplanes.

Define $\mathcal{A}''_j := \{ H \cap H_j \mid H \in \mathcal{A}' \}$ ($j = 1, \ldots, q$).

Assume

(1) $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$, and
(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ ($j = 1, \ldots, q$) (Remark: $\leq$ always holds true).

Then (a) $q \leq p$ and (b) $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.
Inductive use of MAT ($E_6$) : $I = \Phi_0^+$
Inductive use of MAT ($E_6$): $I = \Phi_0^+$
Inductive use of MAT ($E_6$): $I = \Phi_1^+$
### Inductive use of MAT ($E_6$): $I = \Phi_2^+$

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Inductive use of MAT ($E_6$) : $I = \Phi_3^+$
Inductive use of MAT ($E_6$) : $I = \Phi^+_4$

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Inductive use of MAT ($E_6$) : $I = \Phi^+_5$

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H. Terao (Hokkaido University)
Inductive use of MAT ($E_6$) : $I = \Phi^+_6$

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H. Terao (Hokkaido University)
Inductive use of MAT ($E_6$): $I = \Phi_7^+$

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Inductive use of MAT ($E_6$) : $I = \Phi_8^+$

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Inductive use of MAT ($E_6$): $I = \Phi_9^+$

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exponents
**Inductive use of MAT ($E_6$):** $I = \Phi_{10}^+$

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**exponents**
The Dual-Partition Formula \((E_6)\) (again)

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H. Terao (Hokkaido University)
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Although the MAT is similar to the old addition theorem (AT) (1980), it does not generalize the AT.

As an application of the MAT, we may give a new classification-free proof of the celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald.

Moreover, we have the dual-partition formula for any ideal subarrangements.
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I stop here.
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Thanks for your attention!