Ideals of the roots posets and a new proof of the dual-partition formula by Shapiro-Steinberg-Kostant-Macdonlald

Hiroaki Terao

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at

The 2nd Franco-Japanese-Vietnamese Symposium on Singularities

Sapporo, Japan

2014.08.26

Credit

with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

Torsten Hoge (Leibniz Universität Hannover)

arXiv:1304.8033

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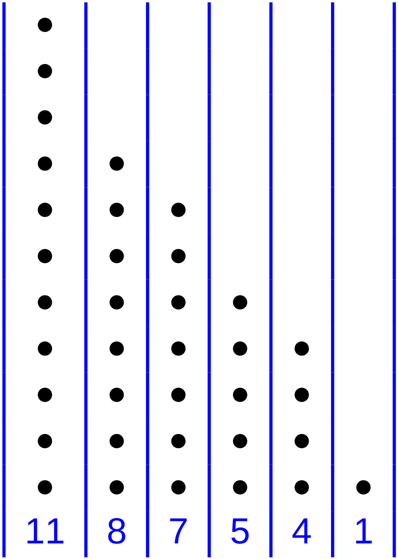
What are Dual Partitions?

$$36 = 1 + 4 + 5 + 7 + 8 + 11$$

$\updownarrow$ Dual Partitions

$$36 = 1 + 1 + 1 + 2 + 3 + 3 + 4 + 5 + 5 + 5 + 6$$

What are Dual Partitions?



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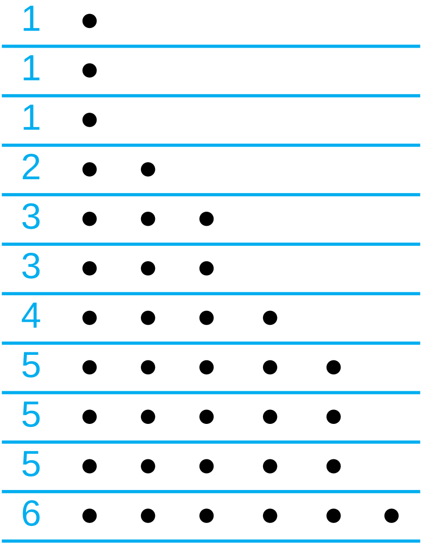

What are Dual Partitions?

1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
	11	8	7	5	4	1

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What are these numbers?

$(1, 4, 5, 7, 8, 11)$ is the exponents of the root system of the type E_6

$\updownarrow$ Dual Partitions

$(1, 1, 1, 2, 3, 3, 4, 5, 5, 5, 6)$ is the height distribution of the positive roots of the type E_6

the dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald

Theorem

(The dual-partition formula by Shapiro, Steinberg, Kostant (1959), Macdonald (1972))
The exponents of an irreducible root system and the height distribution of positive roots are dual partitions to each other.

Remark

(1) This theorem can be (was) regarded as a method to "reading off" the exponents from the root structure.
(2) The other methods to find the exponents include: (a) from the degrees of basic invariants, (b) from the eigenvalues of a Coxeter transformation, etc.

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Exponents

Dynkin diagrams (root systems) and exponents

A_ℓ: $(1, 2, \dots, \ell)$

B_ℓ: $(1, 3, 5, \dots, 2\ell - 1)$

C_ℓ: $(1, 3, 5, \dots, 2\ell - 1)$

D_ℓ: $(1, 3, 5, \dots, 2\ell - 3, \ell - 1)$

E_6: $(1, 4, 5, 7, 8, 11)$

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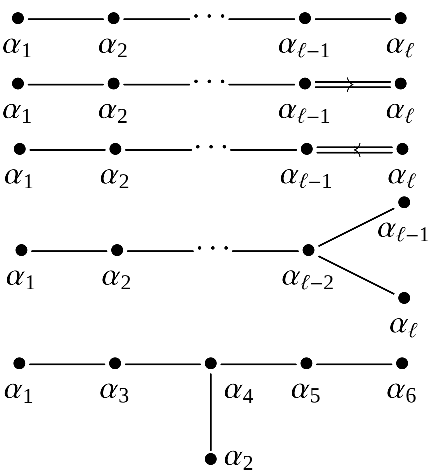

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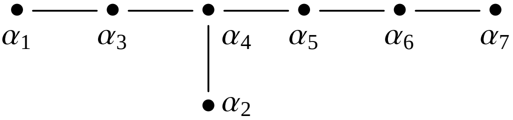
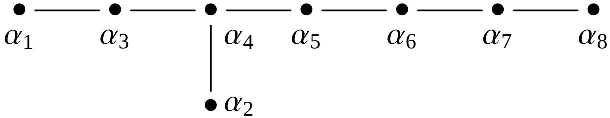
Dynkin diagrams (root systems) and exponents

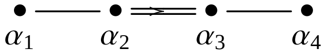
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

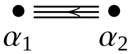


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

Height of positive roots

- Φ : an irreducible root system of rank ℓ
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of Φ
- Φ^+: the set of positive roots
- $\mathrm{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (height) for a positive root $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$ $(c_i \in \mathbb{Z}_{\geq 0})$
- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}|$ $(1 \leq j \leq m)$

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Height of positive roots (E_6)

E_6:

Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:

height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3 : $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

$\vdots$

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

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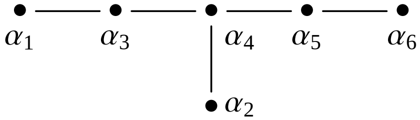

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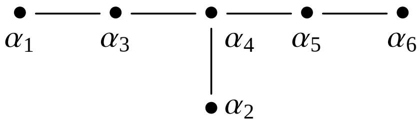

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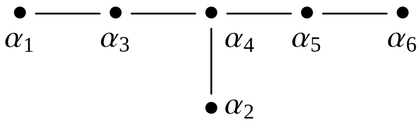

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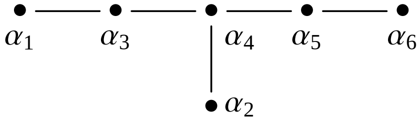

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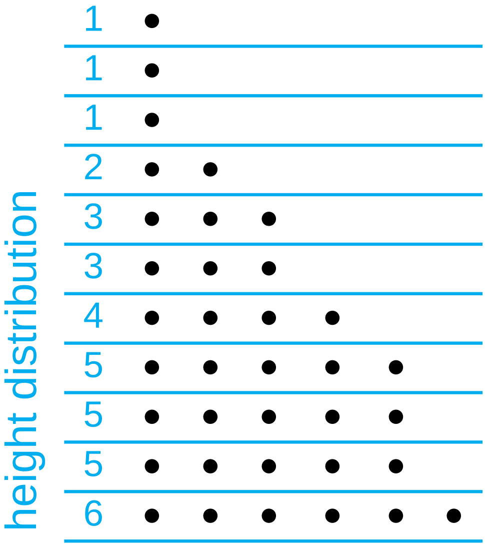
Height of positive roots (E_6)

heights

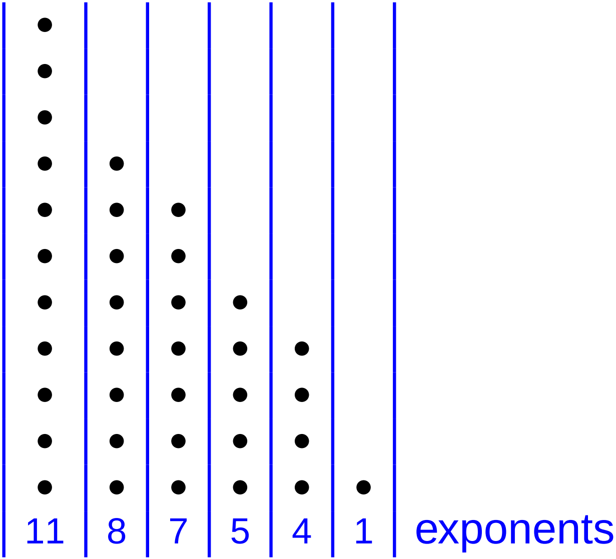
ht=11	$\tilde{\alpha}$					
ht=10	●					
ht=9	●					
ht=8	●	●				
ht=7	●	●	●			
ht=6	●	●	●			
ht=5	●	●	●	●		
ht=4	●	●	●	●	●	
ht=3	●	●	●	●	●	
ht=2	$\alpha_1+\alpha_3$	$\alpha_2+\alpha_4$	$\alpha_3+\alpha_4$	●	●	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)



Exponents (E_6)



The Dual-Partition Formula (E_6)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

History of the Dual-Partition Formula

THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

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. . . . . . we shall presently describe, of “reading off” the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. . . . . . . However, even though one verifies that the numbers produced by this procedure agree with the exponents . . . . . . the important question of proving that this “agreement” is more than just a coincidence remained open.

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Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an ℓ-dimensional vector space V
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
- $S := S(V^*)$: the symmetric algebra of the dual space V^*
- Define a graded S-module

$$D(\mathcal{A}) := \{\theta \mid \theta \text{ is an } \mathbb{R}\text{-linear derivation with } \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free S-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

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- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an ℓ-dimensional vector space V
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Theorem

(K. Saito 1976 et al.) The Weyl arrangement $\mathcal{A}$ is a free arrangement. The exponents of the Weyl arrangement $\mathcal{A}$ coincide with the exponents of the corresponding root system.

Example. (Weyl arrangement of type B_2)
$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$
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$$\theta_1 = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2), \;\; \theta_2 = x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2),$$

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Factorization Theorem

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(H. T.(1981)) Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(\mathcal{A})$ splits as

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Definition

Introduce a partial order $\geq$ into the set Φ^+ of positive roots by

$$\beta_1 \geq \beta_2 \Longleftrightarrow \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i.$$

The poset is called the (positive) root poset.
A subset I of Φ^+ is called an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$,

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Examples of ideals/non-ideals of the root poset of A_3

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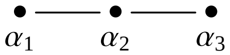

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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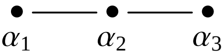

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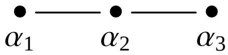

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If

- Φ *: an irreducible root system of rank* ℓ
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then

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(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ and $1 \le p \le \ell$ the multiplicity of the highest exponent d.
Let $H_1, \ldots, H_q$ be (new) hyperplanes.
Define $\mathcal{A}_j'' := \{H \cap H_j \mid H \in \mathcal{A}'\}$ $(1 \le j \le q)$.
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(1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional,
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The third condition is crucial.

MAT (Multiple Addition Theorem - key to our proof -)

Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ and $1 \le p \le \ell$ the multiplicity of the highest exponent d.
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Example of the third condition in the MAT: $|\mathcal{A}'| - |\mathcal{A}''_j| = d$

$\mathcal{A}'$ is free with exponents $(1, 2, 2)$, $d = 2$ (the max exponent).
$\mathcal{A}''_1 := \{H \cap H_1 \mid H \in \mathcal{A}'\}, |\mathcal{A}''_1| = 3$ and $|\mathcal{A}'| - |\mathcal{A}''_1| = 5 - 3 = 2$.
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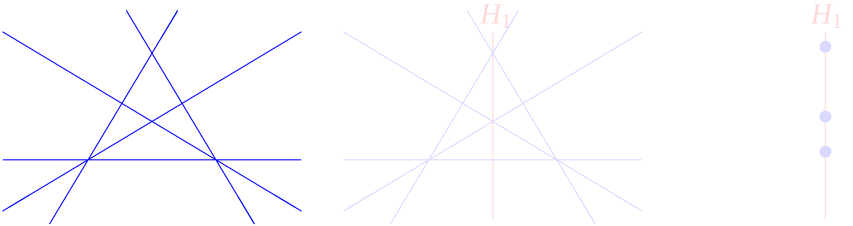

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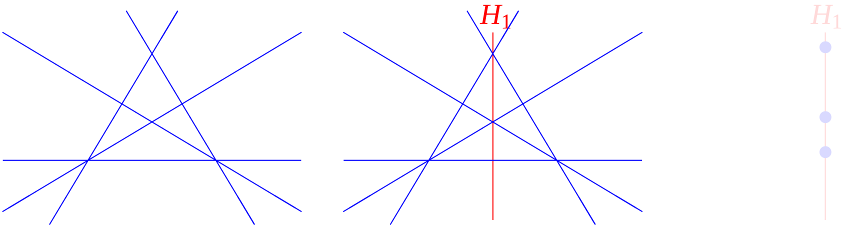

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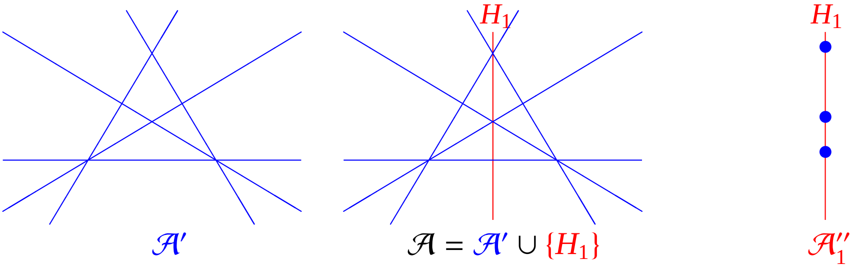

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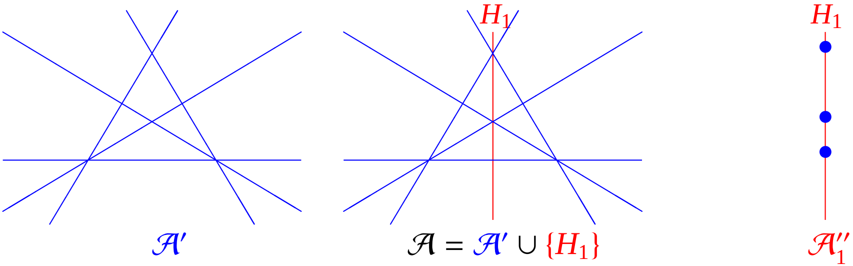

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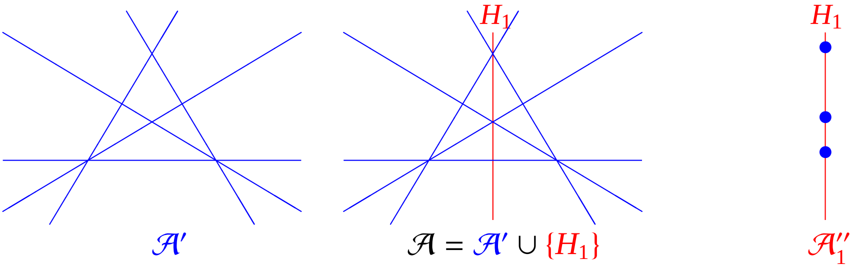

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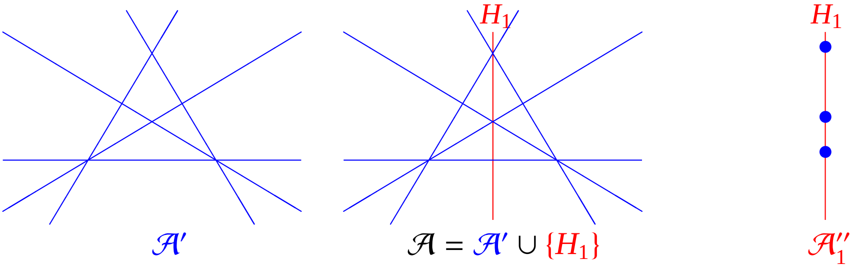

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Main Theorem (again)

Theorem

If

- Φ *: an irreducible root system of rank* ℓ
- $\mathcal{A}$ *: the Weyl arrangement (= the collection of hyperplanes orthogonal to the positive roots of* Φ*)*

Then

1. *any ideal subarrangement* $\mathcal{B}$ *of* $\mathcal{A}$ *is free,*
2. *its exponents and the height distribution of the positive roots satisfy the dual-partition formula.*

We prove the Main Theorem applying MAT inductively.

Proof of Main Theorem (just an outline)

Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system Φ.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{H \in \mathcal{B} \mid \mathrm{ht}(\alpha_H) \leq k\}$.
- We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$.
- We may apply MAT for $\mathcal{A}' := \mathcal{B}_{\leq k}$ and $\mathcal{A} := \mathcal{B}_{\leq k+1}$.
- To verify the three assumtions of MAT, we verify the corresponding combinatorial and geometric properties of the root system Φ. (A key Lemma is in the next page.)

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Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^{\perp}$. Then Φ_X is a root system of rank $\operatorname{codim} X$.

The height of α in Φ_X is called the local height and is denoted by $\mathrm{ht}_X\, \alpha$.

For $\alpha \in \Phi^+$, let

$$\mathcal{A}^{\alpha} := \mathcal{A}^{H_\alpha} = \{K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{H_\alpha\}\}.$$

Lemma (Local-global formula for heights)

For $\alpha \in \Phi^+$, we have

$$\mathrm{ht}_\Phi\, \alpha - 1 = \sum_{X \in \mathcal{A}^\alpha} (\mathrm{ht}_X\, \alpha - 1).$$

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Inductive use of MAT (E_6) : $I = \Phi_0^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
0						
0						
0						
0						
exponents	0	0	0	0	0	0

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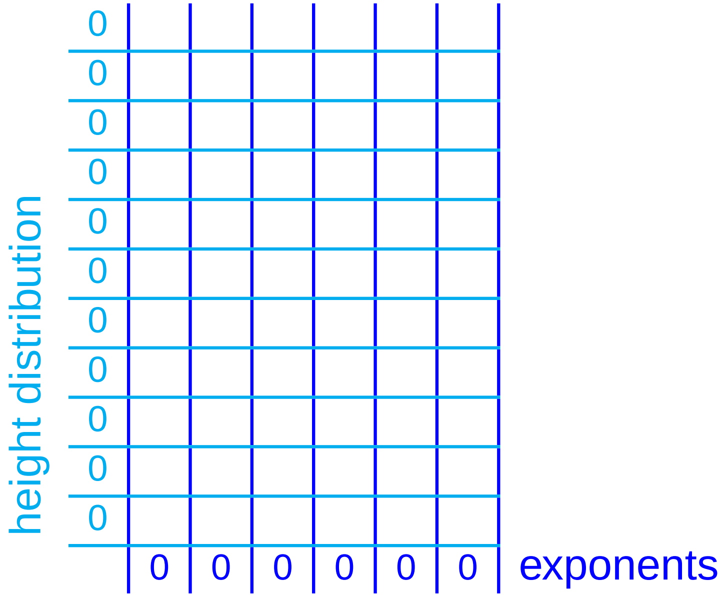

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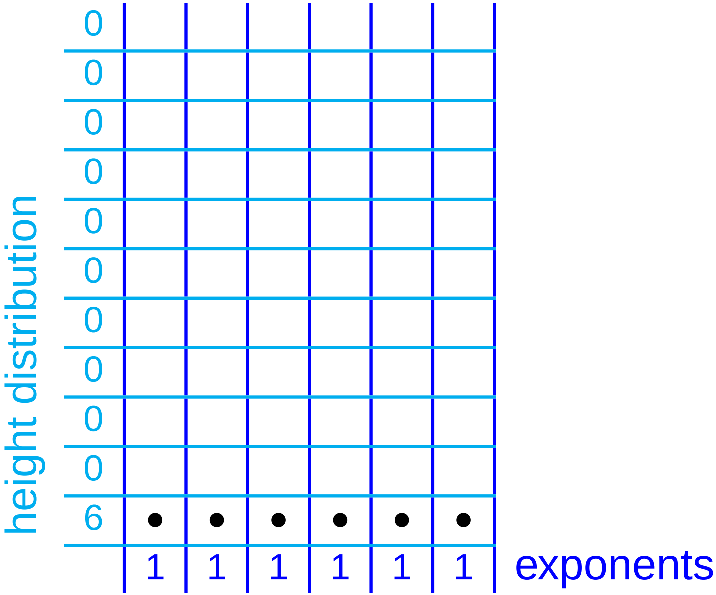
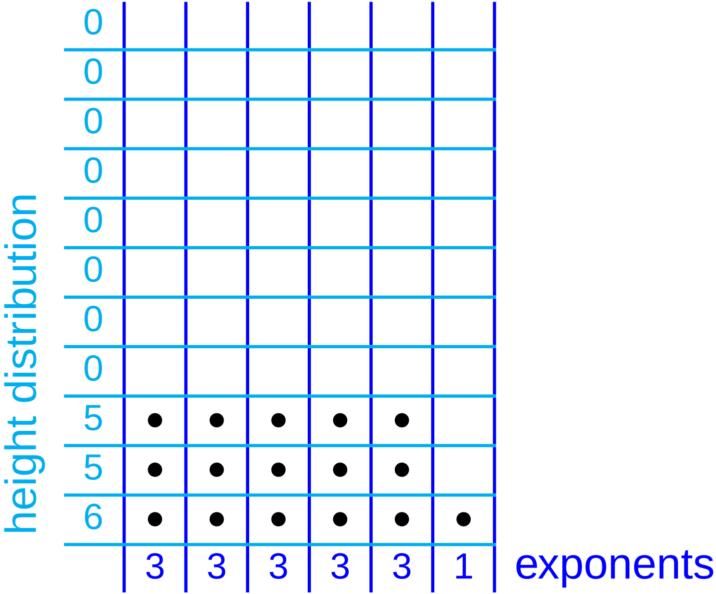


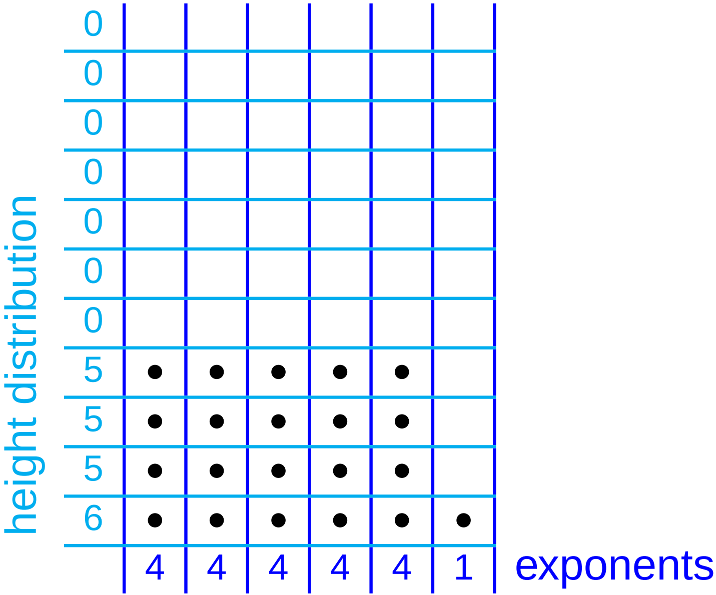
Inductive use of MAT (E_6) : $I = \Phi_2^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
0						
0						
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	2	2	2	2	2	1

Inductive use of MAT (E_6) : $I = \Phi_3^+$



Inductive use of MAT (E_6) : $I = \Phi_4^+$



Inductive use of MAT (E_6) : $I = \Phi_5^+$

height distribution						
0						
0						
0						
0						
0						
0						
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	5	5	5	5	4	1

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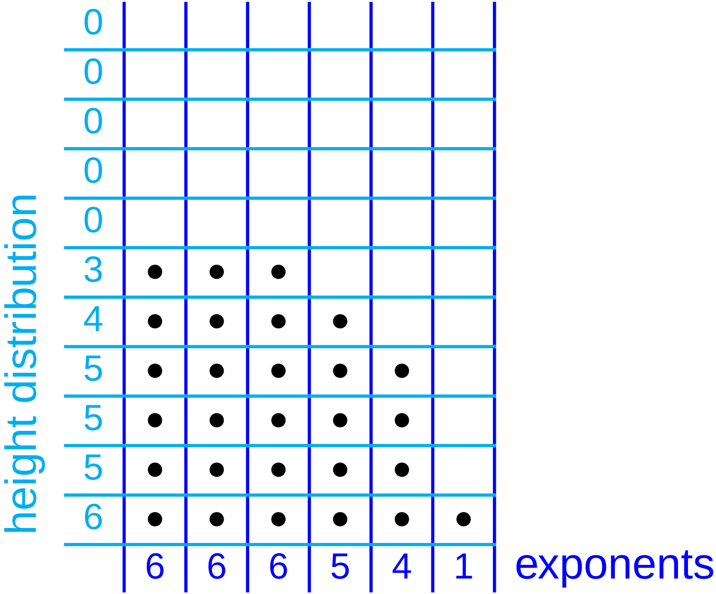
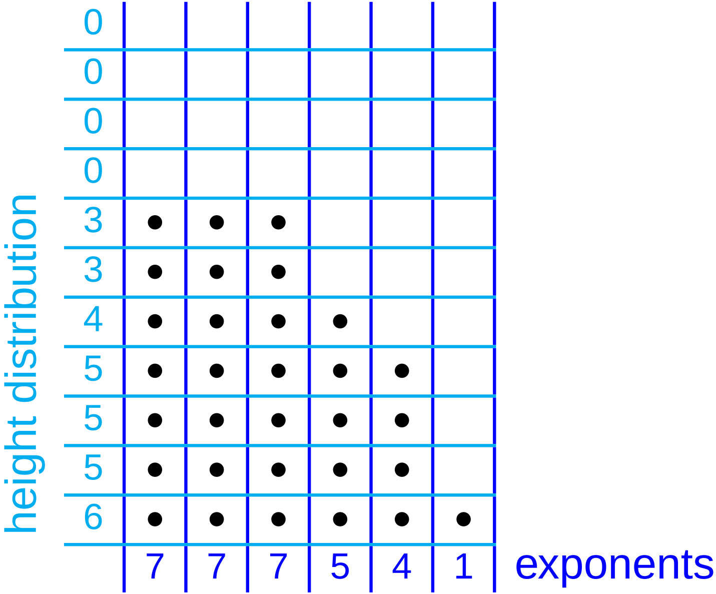


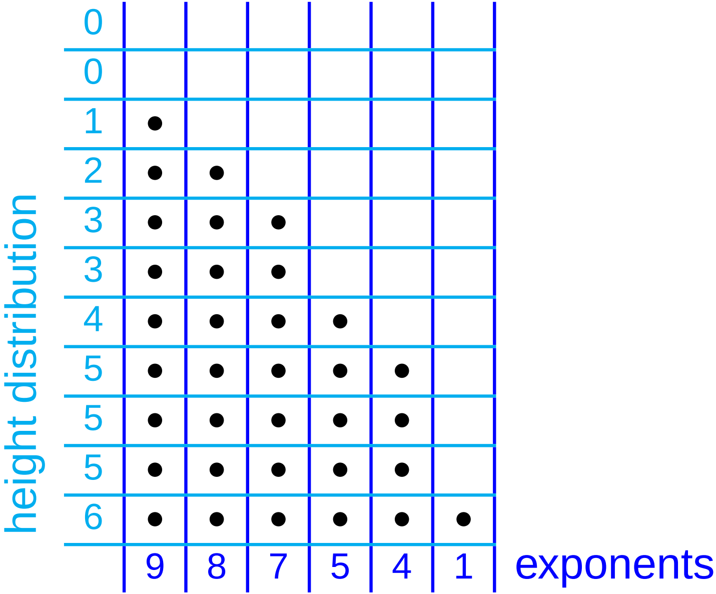
Inductive use of MAT (E_6) : $I = \Phi_7^+$



Inductive use of MAT (E_6) : $I = \Phi_8^+$

height distribution						
0						
0						
0						
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	8	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_9^+$



Inductive use of MAT (E_6) : $I = \Phi^+_{10}$

height distribution						
0						
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	10	8	7	5	4	1

The Dual-Partition Formula (E_6) (again)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

Summary

1. The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.
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- Is there any nice characterization of the set

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- For $w \in W$, define its inverse set $\mathcal{I}(w) := \{\alpha \in \Phi^+ \mid -w\alpha \in \Phi^+\}$.
- Consider the inverse subarrangement

$$\mathcal{A}(w) := \mathcal{A}(\mathcal{I}(w)) := \{\ker(\alpha) \mid \alpha \in \mathcal{I}(w)\}$$

of the Weyl arrangement $\mathcal{A} = \mathcal{A}(\Phi^+)$.

Theorem

(W. Slofstra (2014)) The element w is rationally smooth ($\fallingdotseq$ the Schubert cell $X(w)$ is smooth) if and only if
(1) the inverse subarrangement $\mathcal{A}(w)$ is a free arrangement and
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