On ideal subarrangements of Weyl arrangements

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Saõ Carlos, Brazil

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Credit

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with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

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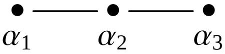
When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker\alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.

Examples of ideals/non-ideals of the root poset of A_3

A_3: 

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

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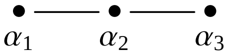
A_3: 

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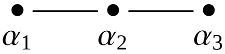

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Note that the entire set Φ^+ is always an ideal.

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- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

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$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$

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The exponents are

$$d_1 = \deg\theta_1 = 1, \quad d_2 = \deg\theta_2 = 3.$$

Factorization Theorem

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(H. T.(1981)) Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(\mathcal{A})$ splits as

$$\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

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Dynkin diagrams (root systems) and exponents

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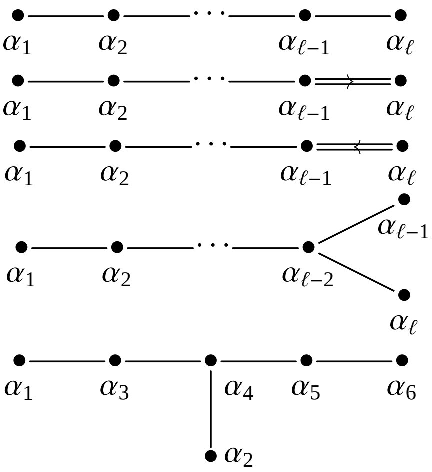


A_ℓ: $(1, 2, \ldots, \ell)$

B_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

C_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

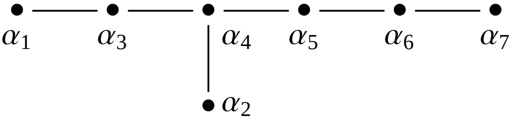
D_ℓ: $(1, 3, 5, \ldots, 2\ell - 3, \ell - 1)$

E_6: $(1, 4, 5, 7, 8, 11)$

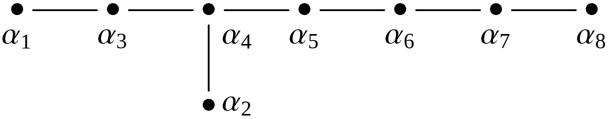
Exponents

Dynkin diagrams (root systems) and exponents

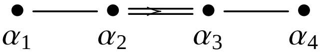
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

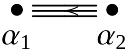


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

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This positively settles a conjecture by Sommers-Tymoczko (2006).

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The exponents of the entire Φ and the height distribution of the entire positive roots satisfy the dual-partition formula.

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- $\mathrm{ht}(\alpha) := \sum_{i=1}^{\ell} c_i$ (height) for a positive root $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ $(c_i \in \mathbb{Z}_{\geq 0})$
- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}|$ $(1 \leq j \leq m)$

Height of positive roots (E_6)

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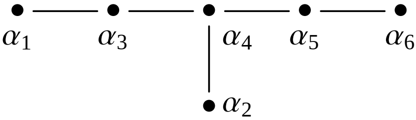
E_6:



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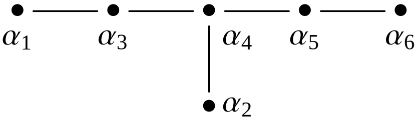

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height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

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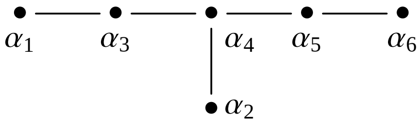

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$$\vdots$$

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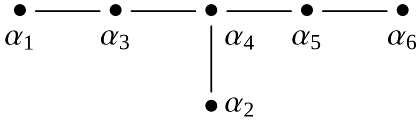

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height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

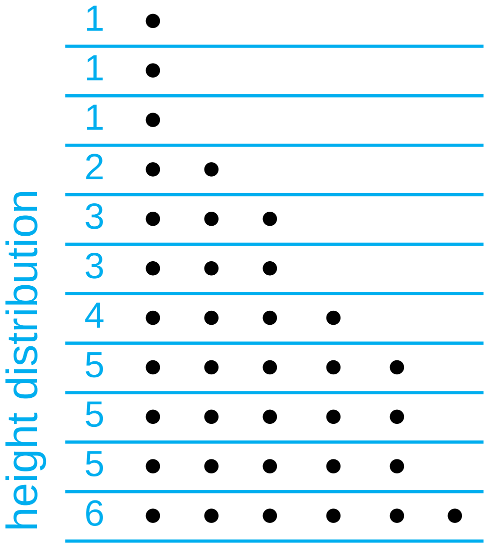
Height of positive roots (E_6)

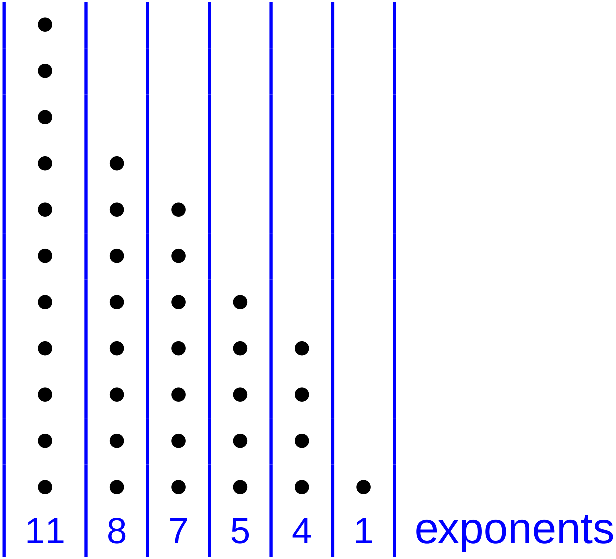
heights

ht=11	$\tilde{\alpha}$					
ht=10	•					
ht=9	•					
ht=8	•	•				
ht=7	•	•	•			
ht=6	•	•	•			
ht=5	•	•	•	•		
ht=4	•	•	•	•	•	
ht=3	•	•	•	•	•	
ht=2	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_4$	$\alpha_3 + \alpha_4$	•	•	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)




11
8
7
5
4
1
exponents

The Dual-Partition Formula (E_6)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

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THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.*[1]

By Bertram Kostant.

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- (2014?) ABCHT (for ideal subarr.: using free arrangements)

MAT (Multiple Addition Theorem - key to our proof -)

MAT (Multiple Addition Theorem - key to our proof -)

Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ *be a free arrangement with exponents* $(d_1, \dots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ *and* $1 \le p \le \ell$ *the multiplicity of the highest exponent* d*. Let* $H_1, \dots, H_q$ *be hyperplanes with* $H_i \notin \mathcal{A}'$ *for* $i = 1, \dots, q$*. Define* $\mathcal{A}''_j := \{H \cap H_j \mid H \in \mathcal{A}'\}$ $(j = 1, \dots, q)$*. Assume that the following three conditions are satisfied:*

(1) $X := H_1 \cap \cdots \cap H_q$ *is* q*-codimensional,*

(2) $X \not\subseteq H \quad (\forall H \in \mathcal{A}')$*,*

(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ $(1 \le j \le q)$*.*

Then $q \le p$ *and* $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ *is free with exponents* $(d_1, \dots, d_{\ell-q}, (d+1)^q)$*.*

Main Theorem (again)

Theorem

If

- Φ : *an irreducible root system of rank* ℓ
- $\mathcal{A}$: *the Weyl arrangement (= the collection of hyperplanes orthogonal to the positive roots of* Φ*)*

Then

1. *any ideal subarrangement* $\mathcal{B}$ *of* $\mathcal{A}$ *is free,*
2. *its exponents and the height distribution of the positive roots satisfy the dual-partition formula.*

We prove the Main Theorem applying MAT inductively.

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- To verify the three assumtions of MAT, we verify the corresponding combinatorial and geometric properties of the root system Φ. (A key Lemma is in the next page.)

Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then Φ_X is a root system of rank $\operatorname{codim} X$.

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Lemma (Local-global formula for heights)

For $\alpha \in \Phi^+$, we have

$$\operatorname{ht}_\Phi \alpha - 1 = \sum_{X \in \mathcal{A}^{\alpha}} (\operatorname{ht}_X \alpha - 1).$$

MAT (Revisited)

Theorem

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Then $q \leq p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.

Inductive use of MAT (E_6) : $I = \Phi_0^+$

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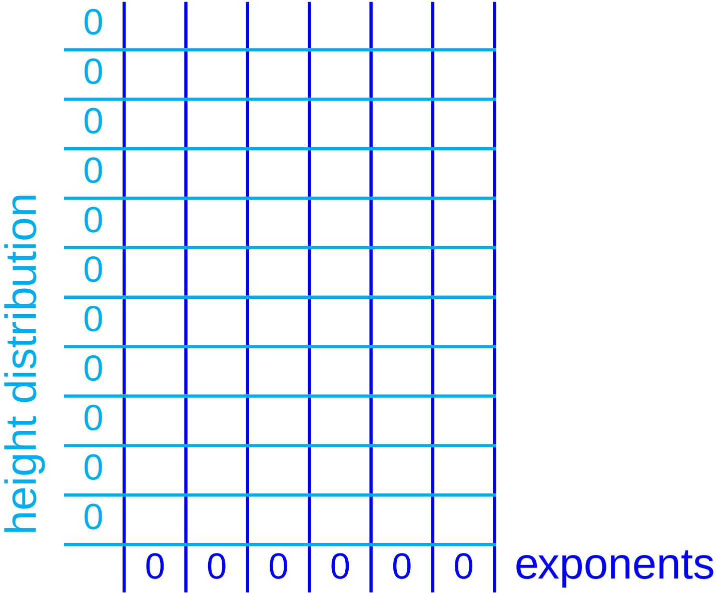
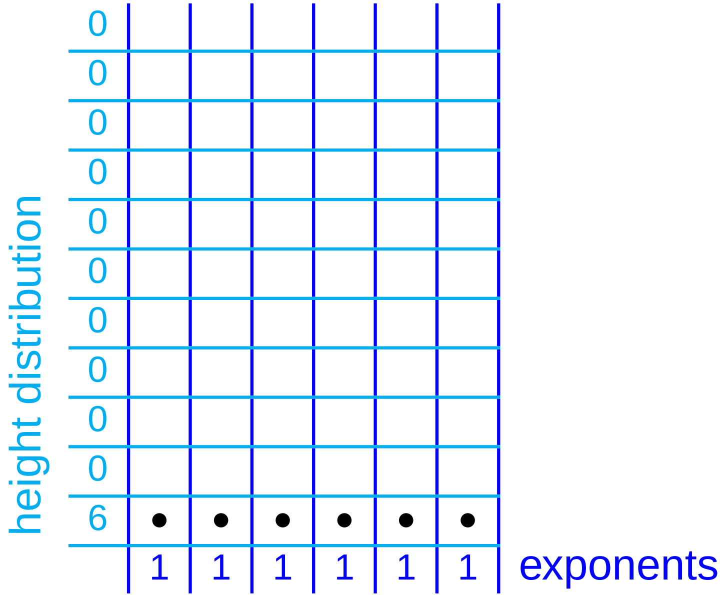
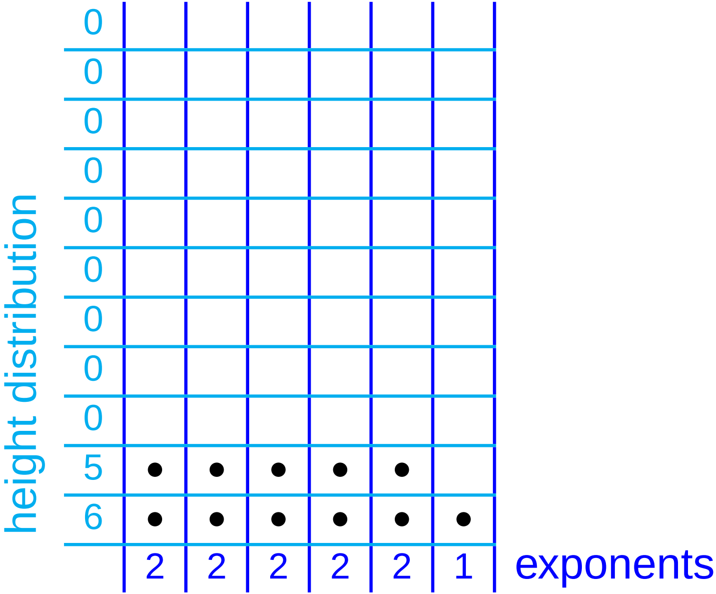


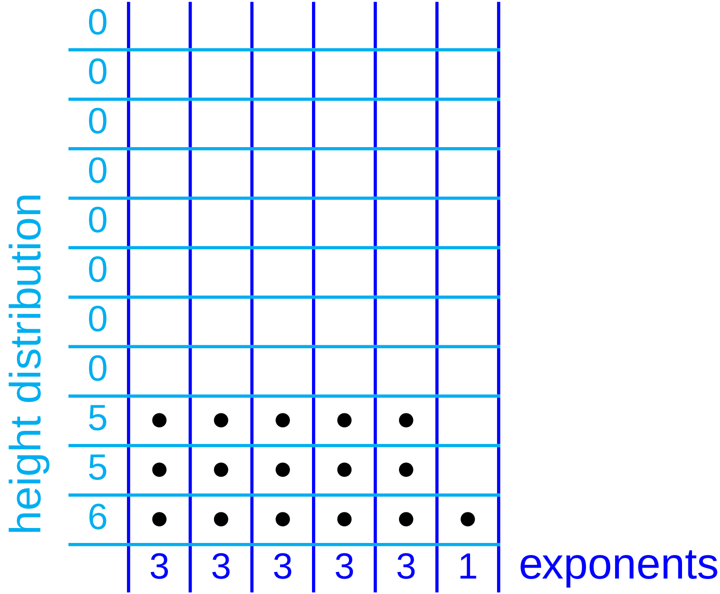
Inductive use of MAT (E_6) : $I = \Phi_1^+$



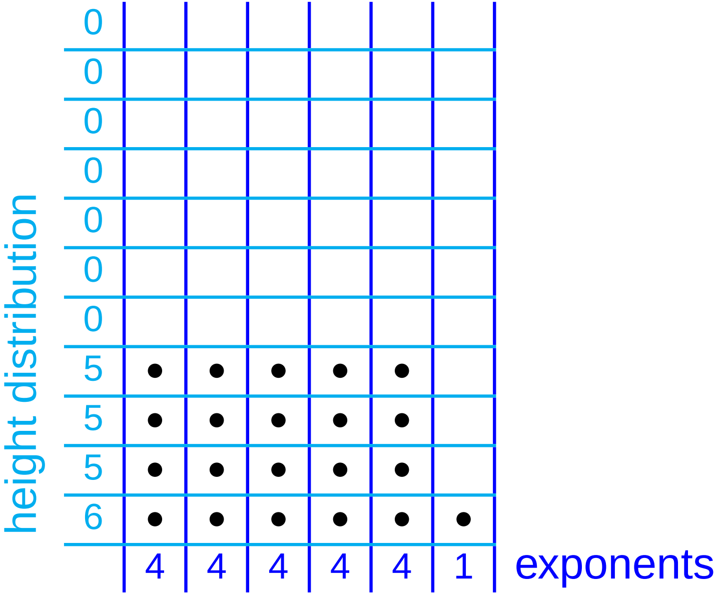
Inductive use of MAT (E_6) : $I = \Phi_2^+$



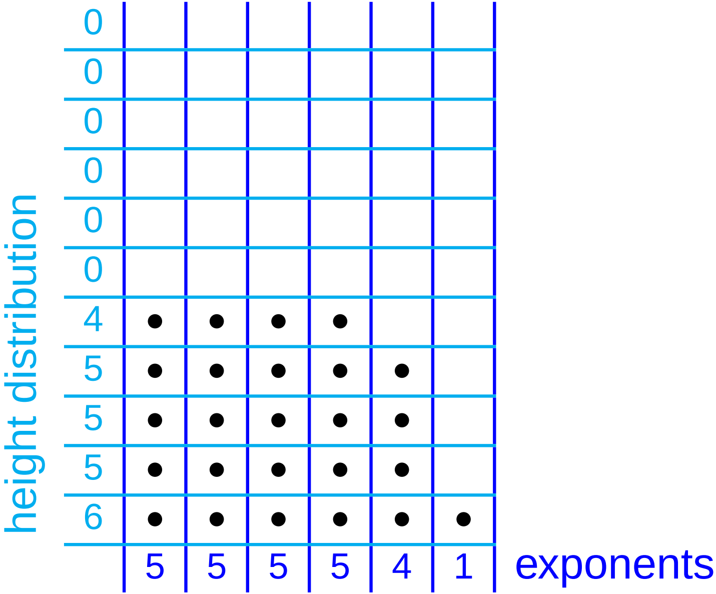
Inductive use of MAT (E_6) : $I = \Phi_3^+$



Inductive use of MAT (E_6) : $I = \Phi_4^+$



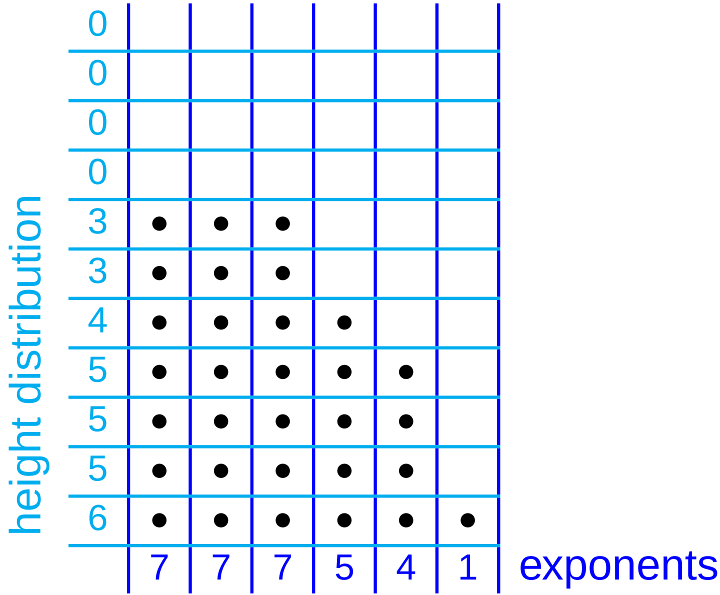
Inductive use of MAT (E_6) : $I = \Phi_5^+$



Inductive use of MAT (E_6) : $I = \Phi_6^+$

height distribution						
0						
0						
0						
0						
0						
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	6	6	6	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_7^+$



Inductive use of MAT (E_6) : $I = \Phi_8^+$

height distribution						
0						
0						
0						
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	8	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_9^+$

height distribution						
0						
0						
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	9	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi^+_{10}$

height distribution						
0						
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	10	8	7	5	4	1

The Dual-Partition Formula (E_6) (again)

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

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1. The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.

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1. The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.
2. The theory of free arrangements (the multiple addition theorem (MAT)) provides a base of our proof.
3. Our proof needs combinatorial and geometric properties concerning the height of positive roots.

A Natural Question

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- Is there any nice characterization of the set

$$\mathcal{F} := \{F \subseteq \Phi^+ \mid \mathcal{A}(F) \text{ is free}\},$$

  where $\mathcal{A}(F)$ is the corresponding subarrangement to F?

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Remark. At least we know

$$\mathcal{F} \supseteq \{wI \mid w \in W,\ I \text{ is an ideal}\}.$$

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of the Weyl arrangement $\mathcal{A} = \mathcal{A}(\Phi^+)$.

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Theorem

(W. Slofstra (2014)) The element w is rationally smooth ($\fallingdotseq$ the Schubert cell $X(w)$ is smooth) if and only if the inverse subarrangement $\mathcal{A}(w)$ is a free arrangement and its number of chambers is equal to the size of the Bruhat interval $[1, w]$.

I stop here.

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Thanks for your attention!