On ideal subarrangements of Weyl arrangements

Hiroaki Terao

(Hokkaido University, Sapporo, Japan)

at

The 13th International Workshop in Real and Complex Singularities

Saõ Carlos, Brazil

2014.07.28
with

Takuro Abe (Kyoto University)
Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)
Michael Cuntz (Leibniz Universität Hannover)
Torsten Hoge (Leibniz Universität Hannover)
with

Takuro Abe (Kyoto University)
Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)
Michael Cuntz (Leibniz Universität Hannover)
Torsten Hoge (Leibniz Universität Hannover)

arXiv:1304.8033
with

Takuro Abe (Kyoto University)
Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)
Michael Cuntz (Leibniz Universität Hannover)
Torsten Hoge (Leibniz Universität Hannover)

arXiv:1304.8033
A Weyl arrangement is called a Weyl arrangement if there exists a set of roots such that $2(\alpha; \beta) = (\alpha; \sigma_\alpha \beta)$ for any $\alpha, \beta \in \Delta$. Then the lattice $\mathbb{Z}^n$ is stable under the Weyl group $W(\Gamma)$. 

The set of positive roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ forms the simple system (the set of simple roots).
A reflection arrangement $\mathcal{A}$ is called a Weyl arrangement if there exists a set $\Phi$ of roots such that

$$2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$$

for any $\alpha, \beta \in \Phi$. 
A reflection arrangement $\mathcal{A}$ is called a Weyl arrangement if there exists a set $\Phi$ of roots such that

$$2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$$

for any $\alpha, \beta \in \Phi$. Then the lattice $\mathbb{Z}\Phi$ is stable under the Weyl group $W(\mathcal{A})$. 
A reflection arrangement $\mathcal{A}$ is called a Weyl arrangement if there exists a set $\Phi$ of roots such that

$$2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$$

for any $\alpha, \beta \in \Phi$.

Then the lattice $\mathbb{Z}\Phi$ is stable under the Weyl group $W(\mathcal{A})$.

$\Phi^+ :$ the set of positive roots
A reflection arrangement $\mathcal{A}$ is called a Weyl arrangement if there exists a set $\Phi$ of roots such that

$$2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$$

for any $\alpha, \beta \in \Phi$. Then the lattice $\mathbb{Z}\Phi$ is stable under the Weyl group $W(\mathcal{A})$.

- $\Phi^+$: the set of positive roots
- $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$: the simple system (=the set of simple roots)
**Definition**

Introduce a partial order into the set $\mathcal{P}$ of positive roots by

$$1 \geq 2 \iff 1 - 2 = \sum_{i=1}^{\ell} z_i i,$$

The poset is called the (positive) root poset.

A subset $I$ of $\mathcal{P}$ is called an ideal if, for $f_1, f_2 \in I$,

$$f_1 \geq f_2.$$

**Definition**

When $I$ is an ideal of $\mathcal{P}$ the arrangement $A(I) = \ker j_{I_1} \cap \ker j_{I_2}$ is called an ideal subarrangement of $A$. 

H. Terao (Hokkaido University)
Introduce a partial order $\geq$ into the set $\Phi^+$ of positive roots by...
Definition

Introduce a partial order $\geq$ into the set $\Phi^+$ of positive roots by

$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i.$$

The poset is called the (positive) root poset.
**Definition**

Introduce a partial order \( \geq \) into the set \( \Phi^+ \) of positive roots by

\[
\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i.
\]

The poset is called the (positive) root poset.

A subset \( I \) of \( \Phi^+ \) is called an ideal if, for \( \{\beta_1, \beta_2\} \subset \Phi^+ \),

\[
\beta_1 \geq \beta_2, \ \beta_1 \in I \Rightarrow \beta_2 \in I.
\]
the root poset and ideals

**Definition**

Introduce a partial order $\geq$ into the set $\Phi^+$ of positive roots by

$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_2 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i.$$  

The poset is called the (positive) root poset.  
A subset $I$ of $\Phi^+$ is called an ideal if, for $\{\beta_1, \beta_2\} \subset \Phi^+$,

$$\beta_1 \geq \beta_2, \ \beta_1 \in I \implies \beta_2 \in I.$$  

**Definition**

When $I$ is an ideal of $\Phi^+$ the arrangement $\mathcal{A}(I) := \{\ker \alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.  

H. Terao (Hokkaido University)
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3$: \[ \bullet \quad \bullet \quad \bullet \]
\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \]

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3: \bullet \quad \bullet \quad \bullet$

$\alpha_1 \quad \alpha_2 \quad \alpha_3$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3$

$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3$

$\alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3$

$\alpha_3 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3$
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3$:  ⬤ —— ⬤ —— ⬤  
$\alpha_1$ $\alpha_2$ $\alpha_3$

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3,$

$\alpha_3 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3$

Thus $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ is an ideal, while $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not.
Examples of ideals/non-ideals of the root poset of $A_3$

$A_3$: \[ \bullet \quad \bullet \quad \bullet \]
\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \]

$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

\[ \alpha_1 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3, \]
\[ \alpha_2 \leq \alpha_1 + \alpha_2 \leq \alpha_1 + \alpha_2 + \alpha_3, \]
\[ \alpha_2 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3, \]
\[ \alpha_3 \leq \alpha_2 + \alpha_3 \leq \alpha_1 + \alpha_2 + \alpha_3 \]

Thus $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$ is an ideal, while $\{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is not.

Note that the entire set $\Phi^+$ is always an ideal.
Free Arrangements and their Exponents

A: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$.

$H_2(V) = \ker(H)$ for $H_2 \in A$:

$S(V) = \Sigma(V)$: the symmetric algebra of the dual space $V$.

Define a graded $S$-module $D(A)$:

$D(A)$ is an $R$-linear derivation such that $D(H) = d(H)$ for all $H \in A$.

$A$ is said to be a free arrangement if $D(A)$ is a free $S$-module.

When $A$ is free, then $x_1, x_2, \ldots, x_\ell$: homogeneous basis with $\deg x_i = d_i$.

The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $A$. 

H. Terao (Hokkaido University)
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
- $S := S(V^*)$: the symmetric algebra of the dual space $V^*$
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
- $S := S(V^*)$: the symmetric algebra of the dual space $V^*$
- Define a graded $S$-module

$$D(\mathcal{A}) := \{ \theta \mid \theta \text{ is an } \mathbb{R}\text{-linear derivation with} \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}.$$
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
- $S := S(V^*)$: the symmetric algebra of the dual space $V^*$
- Define a graded $S$-module
  
  \[ D(\mathcal{A}) := \{ \theta \mid \theta \text{ is an $\mathbb{R}$-linear derivation with} \]
  \[ \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}. \]

- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free $S$-module.
Free Arrangements and their Exponents

- $\mathcal{A}$: an arrangement of hyperplanes in an $\ell$-dimensional vector space $V$
- $\alpha_H \in V^*$: $\ker(\alpha_H) = H$ for $H \in \mathcal{A}$
- $S := S(V^*)$: the symmetric algebra of the dual space $V^*$
- Define a graded $S$-module

$$D(\mathcal{A}) := \{ \theta \mid \theta \text{ is an } \mathbb{R}\text{-linear derivation with } \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A} \}.$$ 

$\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free $S$-module.

When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$. 
Every Weyl arrangement is free

Theorem (K. Saito 1976 et al.) The Weyl arrangement $A$ is a free arrangement. The exponents of the Weyl arrangement $A$ coincide with the exponents of the corresponding root system.

Example. (Weyl arrangement of type $B_2$)

$+1 = x_1 + x_2;
2 = x_2 + x_1 + x_2$.
Every Weyl arrangement is free

Theorem

(K. Saito 1976 et al.) The Weyl arrangement $\mathcal{A}$ is a free arrangement. The exponents of the Weyl arrangement $\mathcal{A}$ coincide with the exponents of the corresponding root system.
Every Weyl arrangement is free

**Theorem**

(K. Saito 1976 et al.) The Weyl arrangement $\mathcal{A}$ is a free arrangement. The exponents of the Weyl arrangement $\mathcal{A}$ coincide with the exponents of the corresponding root system.

**Example.** (Weyl arrangement of type $B_2$)

$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$
Every Weyl arrangement is free

**Theorem**

(K. Saito 1976 et al.) The Weyl arrangement $\mathcal{A}$ is a free arrangement. The exponents of the Weyl arrangement $\mathcal{A}$ coincide with the exponents of the corresponding root system.

**Example.** (Weyl arrangement of type $B_2$)

$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$

The $S$-module $D(\mathcal{A}_G)$ is a free module with a basis

$$\theta_1 = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2), \quad \theta_2 = x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2),$$
Every Weyl arrangement is free

Theorem

(K. Saito 1976 et al.) The Weyl arrangement $\mathcal{A}$ is a free arrangement. The exponents of the Weyl arrangement $\mathcal{A}$ coincide with the exponents of the corresponding root system.

Example. (Weyl arrangement of type $B_2$)

$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$

The $S$-module $D(\mathcal{A}_G)$ is a free module with a basis

$$\theta_1 = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2), \quad \theta_2 = x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2),$$

The exponents are

$$d_1 = \deg \theta_1 = 1, \quad d_2 = \deg \theta_2 = 3.$$
Theorem. Assume that $A$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1; \ldots; d_\ell)$. Define the complement of $A$ by $M(A) = V \setminus \bigcup H_2 A H$. Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(A)$ splits as $Poin(M(A); t) = \prod_{i=1}^\ell (1 + d_i t)$.
Theorem

(H. T. (1981)) Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$ 

Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(\mathcal{A})$ splits as

$$\text{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^\ell (1 + d_i t).$$
Exponents

Dynkin diagrams (root systems) and exponents

A$^{\ell_1}$

B$^{\ell_1}/\ell_1$ 

C$^{\ell_1}$

D$^{\ell_1}$

E$^6$
Dynkin diagrams (root systems) and exponents
Dynkin diagrams (root systems) and exponents

\[ A_\ell: \quad \begin{array}{c}
\bullet \\
\alpha_1 \\
\alpha_2 \\
\alpha_{\ell-1} \\
\alpha_\ell
\end{array} \quad (1, 2, \ldots, \ell) \]

\[ B_\ell: \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \quad \bullet
\alpha_1 \\
\alpha_2 \\
\alpha_{\ell-1} \\
\alpha_\ell
\end{array} \quad (1, 3, 5, \ldots, 2\ell - 1) \]

\[ C_\ell: \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \quad \bullet
\alpha_1 \\
\alpha_2 \\
\alpha_{\ell-1} \\
\alpha_\ell
\end{array} \quad (1, 3, 5, \ldots, 2\ell - 1) \]

\[ D_\ell: \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \quad \bullet \quad \bullet
\alpha_1 \\
\alpha_2 \\
\alpha_{\ell-2} \\
\alpha_{\ell-1} \\
\alpha_\ell
\end{array} \quad (1, 3, 5, \ldots, 2\ell - 3, \ell - 1) \]

\[ E_6: \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\alpha_1 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_2
\end{array} \quad (1, 4, 5, 7, 8, 11) \]
Dynkin diagrams (root systems) and **exponents**

**E7:** \( (1, 5, 7, 9, 11, 13, 17) \)

\[
\begin{array}{cccccccc}
\alpha_1 & & & & & & & \\
& \alpha_3 & & & & & & \\
& & \alpha_4 & & & & & \\
& & & \alpha_5 & & & & \\
& & & & \alpha_6 & & & \\
& & & & & \alpha_7 & & \\
& & & & & & \alpha_2 & \\
\end{array}
\]

**E8:** \( (1, 7, 11, 13, 17, 19, 23, 29) \)

\[
\begin{array}{cccccccc}
\alpha_1 & & & & & & & \\
& \alpha_3 & & & & & & \\
& & \alpha_4 & & & & & \\
& & & \alpha_5 & & & & \\
& & & & \alpha_6 & & & \\
& & & & & \alpha_7 & & \\
& & & & & & \alpha_8 & \\
& & & & & & \alpha_2 & \\
\end{array}
\]

**F4:** \( (1, 5, 7, 11) \)

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

**G2:** \( (1, 5) \)

\[
\begin{array}{cc}
\alpha_1 & \alpha_2 \\
\end{array}
\]

(from http://www.ms.u-tokyo.ac.jp/abenori/tex/tex7.html)
Main Theorem

Theorem

If $I$: an irreducible root system of rank $\ell$

$I$: an ideal

$B$: the corresponding ideal subarrangement to $I$

Then

1. $B$ is free,
2. the exponents of $B$ and the height distribution of the positive roots in $I$ satisfy the dual-partition formula.

This positively settles a conjecture by Sommers-Tymoczko (2006).
Main Theorem

Theorem

If

\[ \Phi : an \text{ irreducible root system of rank } \ell \]
Theorem

If

- $\Phi$ : an irreducible root system of rank $\ell$
- $I$ : an ideal of $\Phi^+$

Then

1. $B$ is free,
2. the exponents of $B$ and the height distribution of the positive roots in $I$ satisfy the dual-partition formula.

This positively settles a conjecture by Sommers-Tymoczko (2006).
Main Theorem

**Theorem**

*If*

- \( \Phi \): an irreducible *root system* of rank \( \ell \)
- \( I \): an *ideal* of \( \Phi^+ \)
- \( \mathcal{B} \): the corresponding *ideal* subarrangement to \( I \)

Then

1. \( \mathcal{B} \) is *free*, and
2. the exponents of \( \mathcal{B} \) and the *height* distribution of the positive roots in \( I \) satisfy the dual-partition formula.

This positively settles a conjecture by Sommers-Tymoczko (2006).
Theorem

If

- $\Phi$: an irreducible root system of rank $\ell$
- $I$: an ideal of $\Phi^+$
- $\mathcal{B}$: the corresponding ideal subarrangement to $I$

Then

(1) $\mathcal{B}$ is free,
(2) the exponents of $\mathcal{B}$ and the height distribution of the positive roots in $I$ satisfy the dual-partition formula.
Main Theorem

Theorem

If

- \( \Phi \) : an irreducible root system of rank \( \ell \)
- \( I \) : an ideal of \( \Phi^+ \)
- \( \mathcal{B} \) : the corresponding ideal subarrangement to \( I \)

Then

(1) \( \mathcal{B} \) is free, and
Main Theorem

Theorem

If

- $\Phi$ : an irreducible root system of rank $\ell$
- $I$ : an ideal of $\Phi^+$
- $\mathcal{B}$ : the corresponding ideal subarrangement to $I$

Then

1. $\mathcal{B}$ is free, and
2. the exponents of $\mathcal{B}$ and the height distribution of the positive roots in $I$ satisfy the dual-partition formula.

This positively settles a conjecture by Sommers-Tymoczko (2006).
**Main Theorem**

**Theorem**

If

- $\Phi$ : an irreducible root system of rank $\ell$
- $I$ : an ideal of $\Phi^+$
- $\mathcal{B}$ : the corresponding ideal subarrangement to $I$

Then

(1) $\mathcal{B}$ is free, and (2) the exponents of $\mathcal{B}$ and the height distribution of the positive roots in $I$ satisfy the dual-partition formula.

This positively settles a conjecture by Sommers-Tymoczko (2006).
In particular, when the ideal $I$ is equal to the entire $\mathbb{R}^n$, our main theorem yields:

**Corollary.**

(The dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald)

The exponents of the entire and the height distribution of the entire positive roots satisfy the dual-partition formula.
In particular, when the ideal $I$ is equal to the entire $\Phi^+$, our main theorem yields:
In particular, when the ideal $I$ is equal to the entire $\Phi^+$, our main theorem yields:

**Corollary**

*(The dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald)*
In particular, when the ideal $I$ is equal to the entire $\Phi^+$, our main theorem yields:

**Corollary**

(Shapiro, Steinberg, Kostant, Macdonald)

The exponents of the entire $\Phi$ and the height distribution of the entire positive roots satisfy the dual-partition formula.
Height of positive roots

\[ \Delta = f_1; \ldots; f_\ell \]

\[ \Delta = g_1; \ldots; g_m \]

A simple system of positive roots

\[ \text{Height distribution} \in \mathbb{Z}^+ \]

\[ i_1, i_2, \ldots, i_m \]

\[ j_1, j_2, \ldots, j_m \]

\[ (c_i^2 \geq 0) \]

\[ H. Terao \ (Hokkaido \ University) \]

2014.07.28 13 / 29
Height of positive roots

\[ \Phi : \text{an irreducible root system of rank } \ell \]
Height of positive roots

- $\Phi$: an irreducible root system of rank $\ell$
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of $\Phi$
Height of positive roots

- $\Phi$: an irreducible root system of rank $\ell$
- $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$: a simple system of $\Phi$
- $\Phi^+$: the set of positive roots
Height of positive roots

• \( \Phi \): an irreducible root system of rank \( \ell \)
• \( \Delta = \{ \alpha_1, \ldots, \alpha_\ell \} \): a simple system of \( \Phi \)
• \( \Phi^+ \): the set of positive roots
• \( \text{ht}(\alpha) := \sum_{i=1}^{\ell} c_i \) (height) for a positive root \( \alpha = \sum_{i=1}^{\ell} c_i \alpha_i \) (\( c_i \in \mathbb{Z}_{\geq 0} \))
Height of positive roots

- \( \Phi \) : an irreducible root system of rank \( \ell \)
- \( \Delta = \{\alpha_1, \ldots, \alpha_\ell\} : a \) simple system of \( \Phi \)
- \( \Phi^+ : the \) set of positive roots
- \( ht(\alpha) := \sum_{i=1}^{\ell} c_i \) (height) for a positive root \( \alpha = \sum_{i=1}^{\ell} c_i \alpha_i \) (\( c_i \in \mathbb{Z}_{\geq 0} \))

The height distribution in \( \Phi^+ \) is a sequence of positive integers \( (i_1, i_2, \ldots, i_m) \), where
\[
i_j := \left| \{ \alpha \in \Phi^+ \mid ht(\alpha) = j \} \right| \quad (1 \leq j \leq m)
Height of positive roots ($E_6$)

Exponents: $(1; 4; 5; 7; 8; 11)$

List of positive roots:

height 1: $1; 2; 3; 4; 5; 6$

height 2: $1+3; 2+4; 3+4; 4+5; 5+6$

height 3: $1+3+4; 2+3+4; \ldots$

height 11: $\sim = 1+2+2+2+2+2+2+2+2+2+6$ (the highest root)
Height of positive roots ($E_6$)

$E_6$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

Exponents: $(1, 4, 5, 7, 8, 11)$

List of positive roots:
Height of positive roots ($E_6$)

$E_6$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

\[ \alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \]

\[ \bullet \alpha_2 \]

**Exponents:** (1, 4, 5, 7, 8, 11)

**List of positive roots:**

height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$
Height of positive roots ($E_6$)

$E_6$: $\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet$

$\alpha_1 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6

\bullet \alpha_2$

Exponents: (1, 4, 5, 7, 8, 11)

List of positive roots:
height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$
height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$
height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

$\ldots$

$\ldots$
Height of positive roots ($E_6$)

$E_6$:  
\[ \alpha_1 \quad \cdots \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \]

\[ \bullet \alpha_2 \]

**Exponents:** (1, 4, 5, 7, 8, 11)

**List of positive roots:**

height 1: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2: $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

height 3: $\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \ldots$

. . .

height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)
### Height of positive roots ($E_6$)

<table>
<thead>
<tr>
<th>Height (ht)</th>
<th>Root Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>$\tilde{\alpha}$</td>
</tr>
<tr>
<td>10</td>
<td>•</td>
</tr>
<tr>
<td>9</td>
<td>•</td>
</tr>
<tr>
<td>8</td>
<td>• •</td>
</tr>
<tr>
<td>7</td>
<td>• • •</td>
</tr>
<tr>
<td>6</td>
<td>• • • •</td>
</tr>
<tr>
<td>5</td>
<td>• • • • •</td>
</tr>
<tr>
<td>4</td>
<td>• • • • • •</td>
</tr>
<tr>
<td>3</td>
<td>• • • • • •</td>
</tr>
<tr>
<td>2</td>
<td>$\alpha_1 + \alpha_3$  $\alpha_2 + \alpha_4$  $\alpha_3 + \alpha_4$  •  •</td>
</tr>
<tr>
<td>1</td>
<td>$\alpha_1$  $\alpha_2$  $\alpha_3$  $\alpha_4$  $\alpha_5$  $\alpha_6$</td>
</tr>
</tbody>
</table>

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \quad ht(\tilde{\alpha}) = 11$ (the highest root)
## Height Distribution ($E_6$)

<table>
<thead>
<tr>
<th>Height</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
</tbody>
</table>

Exponents

H. Terao (Hokkaido University)

2014.07.28
Exponents ($E_6$)

11 8 7 5 4 1  exponents
The Dual-Partition Formula ($E_6$)

<table>
<thead>
<tr>
<th>Height Distribution</th>
<th>Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

H. Terao (Hokkaido University) 2014.07.28
History of the Dual-Partition Formula

we shall presently describe, of “reading off” the exponents from the root structure of $g$ was discovered by Arnold Shapiro. However, even though one verifies that the numbers produced by this procedure agree with the exponents, the important question of proving that this “agreement” is more than just a coincidence remained open.

(1959) A. Shapiro (empirical proof using the classification)
(1959) R. Steinberg (empirical proof using the classification)
(1972) B. Kostant (1st proof without using the classification)
(2014?) I. G. Macdonald (2nd proof: generating functions)
(2014?) ABCHT (for ideal subarr.: using free arrangements)
History of the Dual-Partition Formula

THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.* 1

By Bertram Kostant.

......we shall presently describe, of “reading off” the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. ...... However, even though one verifies that the numbers produced by this procedure agree with the exponents ...... the important question of proving that this “agreement” is more than just a coincidence remained open.

- (1959) A. Shapiro (empirical proof using the classification)
- (1959) R. Steinberg (empirical proof using the classification)
- (1959) B. Kostant (1st proof without using the classification)
- (1972) I. G. Macdonald (2nd proof: generating functions)
History of the Dual-Partition Formula

THE PRINCIPAL THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP.* ¹

By Bertram Kostant.

......we shall presently describe, of “reading off” the exponents from the root structure of $\mathfrak{g}$ was discovered by Arnold Shapiro. ......However, even though one verifies that the numbers produced by this procedure agree with the exponents ...... the important question of proving that this “agreement” is more than just a coincidence remained open.

- (1959) A. Shapiro (empirical proof using the classification)
- (1959) R. Steinberg (empirical proof using the classification)
- (1959) B. Kostant (1st proof without using the classification)
- (1972) I. G. Macdonald (2nd proof: generating functions)
- (2014?) ABCHT (for ideal subarr.: using free arrangements)
MAT (Multiple Addition Theorem - key to our proof -)

Theorem.

Let $A'$ be a free arrangement with exponents $(d_1; \ldots; d_\ell)$ and $1 \leq p \leq \ell$ the multiplicity of the highest exponent $d_i$. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \not< A'$ for $i = 1, \ldots, q$. Define $A''_j := H_i \setminus H_j |_{H_i \not< A'}$ ($j = 1, \ldots, q$).

Assume that the following three conditions are satisfied:

1. $X := H_1 \setminus \bigcup H_q$ is $q$-codimensional,
2. $X \not\subseteq H_i (8 H_i \not< A')$,
3. $|A'| |_j A''_j | = d_1 \ldots d_\ell q$.

Then $q < p$ and $A : = A' \setminus \{H_1, \ldots, H_q\}$ is free with exponents $(d_1; \ldots; d_\ell q; (d_1+1) q)$. 

H. Terao (Hokkaido University)
(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the highest exponent $d$. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \ldots, q$. Define $\mathcal{A}'' := \{H \cap H_j \mid H \in \mathcal{A}'\}$ ($j = 1, \ldots, q$). Assume that the following three conditions are satisfied:

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X \notin H$ ($\forall H \in \mathcal{A}'$),
3. $|\mathcal{A}'| - |\mathcal{A}''| = d$ ($1 \leq j \leq q$).

Then $q \leq p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d + 1)^q)$. 
Main Theorem (again)

Theorem

If

1. \( \Phi : \) an irreducible root system of rank \( \ell \)
2. \( \mathcal{A} : \) the Weyl arrangement (= the collection of hyperplanes orthogonal to the positive roots of \( \Phi \))

Then

1. any ideal subarrangement \( \mathcal{B} \) of \( \mathcal{A} \) is free,
2. its exponents and the height distribution of the positive roots satisfy the dual-partition formula.

We prove the Main Theorem applying MAT inductively.
Proof of Main Theorem (just an outline)

Let $B$ be an ideal subarrangement of the Weyl arrangement $A$ of a root system $\Delta$:

For $k \geq 0$, define $B_k = \{H \in B | h_t(H) = k\}$.

We may easily verify $B_1$ is a free arrangement with exponents $(0; 0; \ldots; 0; 1; 1; \ldots)$.

We may apply MAT for $A' = B_k$ and $A = B_k + 1$.

To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Delta$.

(A key Lemma is in the next page.)
Proof.

Let $B$ be an ideal subarrangement of the Weyl arrangement $A$ of a root system $\Phi$.

For $k \geq 1$, define $B_k = \{H \in B \mid ht(H) = k\}$.

We may easily verify $B_1$ is a free arrangement with exponents $(0; 0; \ldots; 0; 1; 1; \ldots)$.

We may apply MAT for $A' = B_k$ and $A = B_k + 1$.

To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Phi$.

(A key Lemma is in the next page.)
Proof.

Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$. 
Proof of Main Theorem (just an outline)

Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k\}$.

We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0; 0; \ldots; 0; 1; 1; \ldots)$. We may apply MAT for $\mathcal{A}' = \mathcal{B}_k$ and $\mathcal{A} = \mathcal{B}_{k+1}$.

To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Phi$. (A key Lemma is in the next page.)
Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{\geq 0}$, define $\mathcal{B}_{\leq k} := \{H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k\}$.
- We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$. 
Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k\}$.
- We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$.
- We may apply MAT for $\mathcal{A}' := \mathcal{B}_{\leq k}$ and $\mathcal{A} := \mathcal{B}_{\leq k+1}$.
Proof of Main Theorem (just an outline)

Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{ H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k \}$.
- We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$.
- We may apply MAT for $\mathcal{A}' := \mathcal{B}_{\leq k}$ and $\mathcal{A} := \mathcal{B}_{\leq k+1}$.
- To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Phi$. 
Proof.

- Let $\mathcal{B}$ be an ideal subarrangement of the Weyl arrangement $\mathcal{A}$ of a root system $\Phi$.
- For $k \in \mathbb{Z}_{>0}$, define $\mathcal{B}_{\leq k} := \{H \in \mathcal{B} \mid \text{ht}(\alpha_H) \leq k\}$.
- We may easily verify $\mathcal{B}_{\leq 1}$ is a free arrangement with exponents $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$.
- We may apply MAT for $\mathcal{A}' := \mathcal{B}_{\leq k}$ and $\mathcal{A} := \mathcal{B}_{\leq k+1}$.
- To verify the three assumptions of MAT, we verify the corresponding combinatorial and geometric properties of the root system $\Phi$. (A key Lemma is in the next page.)
Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim} \ X$. 
For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim} \ X$.

The height of $\alpha$ in $\Phi_X$ is called the local height and is denoted by $ht_X \alpha$. 
Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim} \ X$.

The height of $\alpha$ in $\Phi_X$ is called the local height and is denoted by $\text{ht}_X \alpha$.

For $\alpha \in \Phi^+$, let

$$
\mathcal{A}^\alpha := \mathcal{A}^{H_\alpha} = \{K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{H_\alpha\}\}.
$$
Local-global formula for heights (A key Lemma)

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then $\Phi_X$ is a root system of rank $\text{codim} \ X$.

The height of $\alpha$ in $\Phi_X$ is called the local height and is denoted by $ht_X \alpha$.

For $\alpha \in \Phi^+$, let

$$\mathcal{A}^\alpha := \mathcal{A}^{H_\alpha} = \{K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{H_\alpha\}\}.$$

Lemma (Local-global formula for heights)

For $\alpha \in \Phi^+$, we have

$$ht_\Phi \alpha - 1 = \sum_{X \in \mathcal{A}^\alpha} (ht_X \alpha - 1).$$
Theorem. Let $A'$ be a free arrangement with exponents $(d_1; \ldots; d_\ell)$ and $1 \leq p \leq \ell$ the multiplicity of the highest exponent $d$. Let $H_1; \ldots; H_q$ be hyperplanes with $H_i < A'$ for $i = 1; \ldots; q$. Define $A''_j := \{H_i \cap H_j \mid H_i \subset A' \}$ ($j = 1; \ldots; q$).

Assume that the following three conditions are satisfied:

1. $X := \bigcap_{i=1}^q H_i$ is $q$-codimensional,
2. $X \not\subset H_i (8 H_i \subset A')$,
3. $|A'| |A''_j| = d(1 \leq j \leq q)$.

Then $q \leq p$ and $A'' := A' \setminus \{H_1; \ldots; H_q\}$ is free with exponents $(d_1; \ldots; d_\ell q; (d+1)q)$. 

H. Terao (Hokkaido University)
Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ ($d_1 \leq \cdots \leq d_\ell$) and $1 \leq p \leq \ell$ the multiplicity of the highest exponent $d$. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \ldots, q$. Define $\mathcal{A}'' := \{ H \cap H_j \mid H \in \mathcal{A}' \}$ ($j = 1, \ldots, q$). Assume that the following three conditions are satisfied:

1. $X := H_1 \cap \cdots \cap H_q$ is $q$-codimensional,
2. $X \notin H$ ($\forall H \in \mathcal{A}'$),
3. $|\mathcal{A}'| - |\mathcal{A}''| = d$ ($1 \leq j \leq q$).

Then $q \leq p$ and $\mathcal{A} := \mathcal{A}' \cup \{ H_1, \ldots, H_q \}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d + 1)^q)$. 
Inductive use of MAT ($E_6$) : $I = \Phi_0^+$
**Inductive use of MAT ($E_6$): $I = \Phi_0^+$**

<table>
<thead>
<tr>
<th>height distribution</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$): $I = \Phi_1^+$

<table>
<thead>
<tr>
<th>Height Distribution</th>
<th>Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>\bullet\bullet\bullet\bullet\bullet\bullet\bullet</td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$) : $I = \Phi_2^+$

<table>
<thead>
<tr>
<th>height distribution</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>\bullet</td>
</tr>
<tr>
<td>6</td>
<td>\bullet</td>
</tr>
<tr>
<td></td>
<td>\bullet</td>
</tr>
<tr>
<td></td>
<td>\bullet</td>
</tr>
<tr>
<td></td>
<td>\bullet</td>
</tr>
<tr>
<td></td>
<td>\bullet</td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$) : $I = \Phi_3^+$

<table>
<thead>
<tr>
<th>Height Distribution</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>5</th>
<th>5</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$) \( I = \Phi_4^+ \)

\[
\begin{array}{ccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
0 & & & & & & & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
5 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
6 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\
4 & 4 & 4 & 4 & 4 & 4 & 1 & \\
\end{array}
\]

height distribution

exponents
Inductive use of MAT ($E_6$) : $I = \Phi_5^+$
**Inductive use of MAT ($E_6$) : $I = \Phi_6^+$**

<table>
<thead>
<tr>
<th>height distribution</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$): $I = \Phi_7^+$
Inductive use of MAT ($E_6$) : $I = \Phi_8^+$

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>⋅ ⋅</td>
</tr>
<tr>
<td>3</td>
<td>⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>3</td>
<td>⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>4</td>
<td>⋅ ⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>5</td>
<td>⋅ ⋅ ⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>5</td>
<td>⋅ ⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>5</td>
<td>⋅ ⋅ ⋅ ⋅</td>
</tr>
<tr>
<td>6</td>
<td>⋅ ⋅ ⋅ ⋅ ⋅</td>
</tr>
</tbody>
</table>

H. Terao (Hokkaido University)
Inductive use of MAT ($E_6$) : $I = \Phi_9^+$

<table>
<thead>
<tr>
<th>height distribution</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9 8 7 5 4 1</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
Inductive use of MAT ($E_6$) : $I = \Phi_{10}^+$

<table>
<thead>
<tr>
<th>height distribution</th>
<th>0</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>5</th>
<th>5</th>
<th>6</th>
<th>exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10</td>
<td>8 7 5 4 1</td>
</tr>
</tbody>
</table>
The Dual-Partition Formula ($E_6$) (again)

<table>
<thead>
<tr>
<th>height distribution</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11</td>
<td>8</td>
<td>7</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

H. Terao (Hokkaido University)

2014.07.28
The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.

The theory of free arrangements (the multiple addition theorem (MA T)) provides a base of our proof.

Our proof needs combinatorial and geometric properties concerning the height of positive roots.
The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.
1. The celebrated \textbf{dual-partition formula} for a root system by Shapiro-Steinberg-Kostant-Macdonald is \textbf{generalized} to the class of ideals.

2. The theory of \textbf{free arrangements} (the multiple addition theorem (\textbf{MAT})) provides a base of our proof.
The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.

The theory of free arrangements (the multiple addition theorem (MAT)) provides a base of our proof.

Our proof needs combinatorial and geometric properties concerning the height of positive roots.
A Natural Question

Is there any nice characterization of the set $F := \sum_{w \in I} w A(F)$ is free $g$; where $A(F)$ is the corresponding subarrangement to $F$?

Remark. At least we know $F \not\subseteq \bigcup_{w \in I} w W$; $I$ is an ideal $g$: 

H. Terao (Hokkaido University)
2014.07.28 27 / 29
Is there any nice characterization of the set

\[ \mathcal{F} := \{ F \subseteq \Phi^+ \mid \mathcal{A}(F) \text{ is free} \}, \]

where \( \mathcal{A}(F) \) is the corresponding subarrangement to \( F \)?
Is there any nice characterization of the set

\[ \mathcal{F} := \{ F \subseteq \Phi^+ \mid \mathcal{A}(F) \text{ is free} \}, \]

where \( \mathcal{A}(F) \) is the corresponding subarrangement to \( F \)?

**Remark.** At least we know

\[ \mathcal{F} \supseteq \{ wI \mid w \in W, I \text{ is an ideal} \}. \]
A related theorem by W. Slofstra

For $w \in W$, define its inverse set $I(w) = \{f, j, w + g\}$. Consider the inverse subarrangement $A(w) = A(I(w)) = \ker(j_{I(w)})$. The element $w$ is rationally smooth (the Schubert cell $X(w)$ is smooth) if and only if the inverse subarrangement $A(w)$ is a free arrangement and its number of chambers is equal to the size of the Bruhat interval $[1, w]$. 

H. Terao (Hokkaido University)
For $w \in W$, define its inverse set $\mathcal{I}(w) := \{ \alpha \in \Phi^+ \mid -w\alpha \in \Phi^+ \}$. 
A related theorem by W. Slofstra

- For $w \in W$, define its inverse set $I(w) := \{ \alpha \in \Phi^+ \mid -w\alpha \in \Phi^+ \}$. 
- Consider the inverse subarrangement

$$A(w) := A(I(w)) := \{ \ker(\alpha) \mid \alpha \in I(w) \}$$

of the Weyl arrangement $A = A(\Phi^+)$. 
A related theorem by W. Slofstra

- For \( w \in W \), define its inverse set \( \mathcal{I}(w) := \{ \alpha \in \Phi^+ \mid -w\alpha \in \Phi^+ \} \).
- Consider the inverse subarrangement

\[
\mathcal{A}(w) := \mathcal{A}(\mathcal{I}(w)) := \{ \ker(\alpha) \mid \alpha \in \mathcal{I}(w) \}
\]

of the Weyl arrangement \( \mathcal{A} = \mathcal{A}(\Phi^+) \).

**Theorem**

*(W. Slofstra (2014))* The element \( w \) is rationally smooth (\( \equiv \) the Schubert cell \( X(w) \) is smooth) if and only if the inverse subarrangement \( \mathcal{A}(w) \) is a free arrangement and its number of chambers is equal to the size of the Bruhat interval \([1, w]\).*
I stop here.
I stop here.

Thanks for your attention!