On ideal subarrangements of Weyl arrangements

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Credit

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with

Takuro Abe (Kyoto University)

Mohamed Barakat (Katholische Universität Eichstätt-Ingolstadt)

Michael Cuntz (Leibniz Universität Hannover)

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- Φ^+ : the set of positive roots
- $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$: the simple system (=the set of simple roots)

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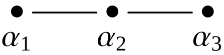
Definition

When I is an ideal of Φ^+ the arrangement $\mathcal{A}(I) := \{\ker\alpha \mid \alpha \in I\}$ is called an ideal subarrangement of $\mathcal{A}$.

Examples of ideals/non-ideals of the root poset of A_3

A_3:



$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$$

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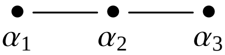

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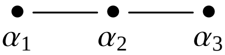
A_3: 

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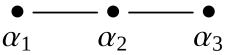

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Note that the entire set Φ^+ is always an ideal.

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- $\mathcal{A}$ is said to be a free arrangement if $D(\mathcal{A})$ is a free S-module.
- When $\mathcal{A}$ is free, then $\exists \theta_1, \theta_2, \ldots, \theta_\ell$: homogeneous basis with $\deg \theta_i = d_i$. The nonnegative integers $d_1, d_2, \ldots, d_\ell$ are called the exponents of $\mathcal{A}$.

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$\Phi^+ := \{\alpha_1 := x_1 - x_2, \alpha_2 := x_2, \alpha_1 + \alpha_2 = x_1, \alpha_1 + 2\alpha_2 = x_1 + x_2\}$

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The S-module $D(\mathcal{A}_G)$ is a free module with a basis

$$\theta_1 = x_1(\partial/\partial x_1) + x_2(\partial/\partial x_2), \ \ \theta_2 = x_1^3(\partial/\partial x_1) + x_2^3(\partial/\partial x_2),$$

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The exponents are

$$d_1 = \deg\theta_1 = 1, \ \ d_2 = \deg\theta_2 = 3.$$

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(H. T.(1981)) Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with exponents $(d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial (with its coefficients equal to the Betti numbers) of the topological space $M(\mathcal{A})$ splits as

$$\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

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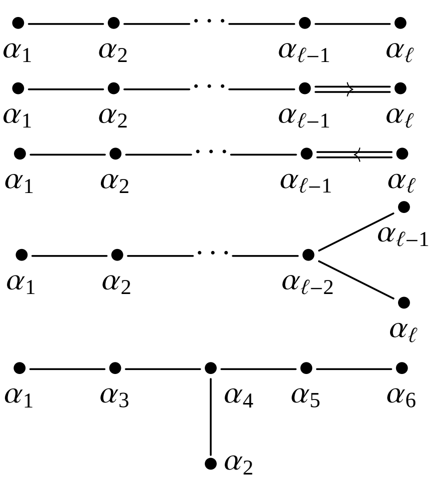

A_ℓ: $(1, 2, \ldots, \ell)$

B_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

C_ℓ: $(1, 3, 5, \ldots, 2\ell - 1)$

D_ℓ: $(1, 3, 5, \ldots, 2\ell - 3, \ell - 1)$

E_6: $(1, 4, 5, 7, 8, 11)$

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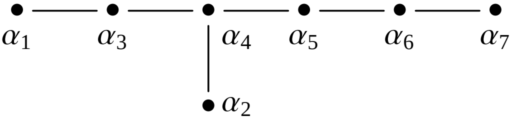
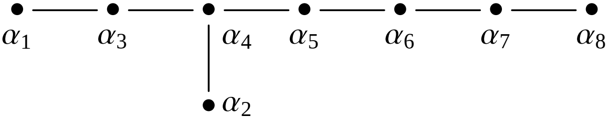
Dynkin diagrams (root systems) and exponents

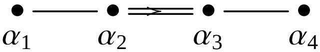
E_7: $(1, 5, 7, 9, 11, 13, 17)$



E_8: $(1, 7, 11, 13, 17, 19, 23, 29)$

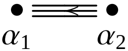


F_4:



$(1, 5, 7, 11)$

G_2:



$(1, 5)$

(from http://www.ms.u-tokyo.ac.jp/ abenori/tex/tex7.html)

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This positively settles a conjecture by Sommers-Tymoczko (2006).

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(The dual-partition formula by Shapiro, Steinberg, Kostant, Macdonald)
The exponents of the entire Φ and the height distribution of the entire positive roots satisfy the dual-partition formula.

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- The height distribution in Φ^+ is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) = j\}|$ $(1 \leq j \leq m)$

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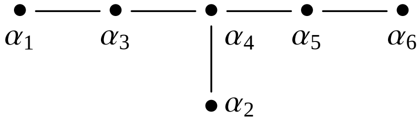

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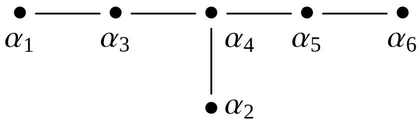

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height 1 : $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$

height 2 : $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$

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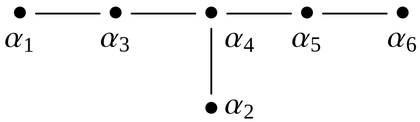

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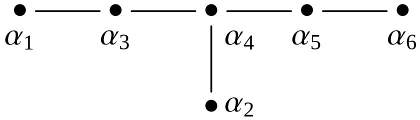

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height 11: $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ (the highest root)

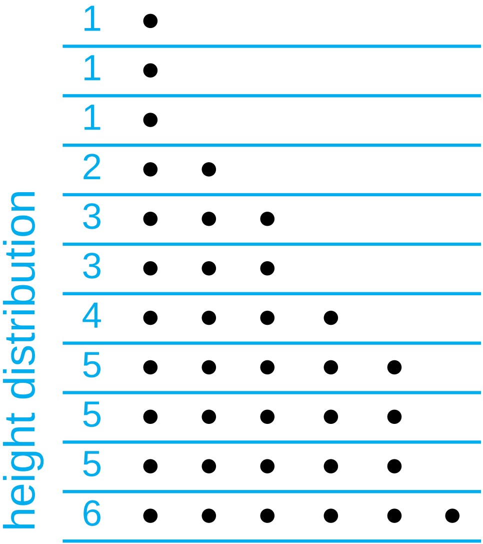
Height of positive roots (E_6)

heights

ht=11	$\tilde{\alpha}$					
ht=10	●					
ht=9	●					
ht=8	●	●				
ht=7	●	●	●			
ht=6	●	●	●			
ht=5	●	●	●	●		
ht=4	●	●	●	●	●	
ht=3	●	●	●	●	●	
ht=2	$\alpha_1 + \alpha_3$	$\alpha_2 + \alpha_4$	$\alpha_3 + \alpha_4$	●	●	
ht=1	α_1	α_2	α_3	α_4	α_5	α_6

$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, $\mathrm{ht}(\tilde{\alpha}) = 11$ (the highest root)

Height Distribution (E_6)



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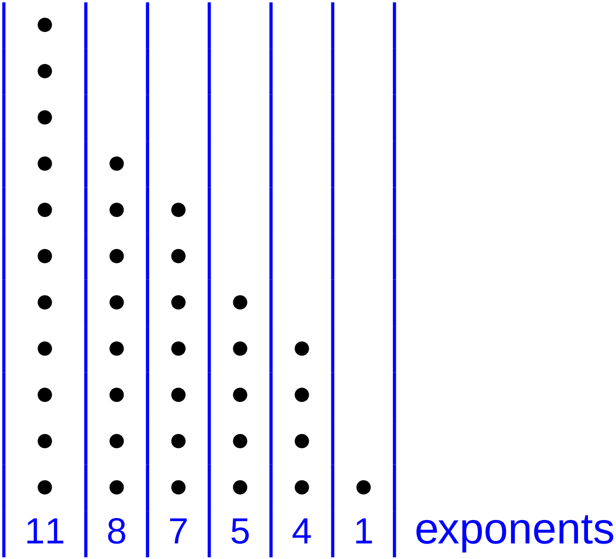

height distribution						
1	•					
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	11	8	7	5	4	1

History of the Dual-Partition Formula

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By Bertram Kostant.

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- (1972) I. G. Macdonald (2nd proof: generating functions)

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- (1959) R. Steinberg (empirical proof using the classification)
- (1959) B. Kostant (1st proof without using the classification)
- (1972) I. G. Macdonald (2nd proof: generating functions)
- (2014?) ABCHT (for ideal subarr.: using free arrangements)

MAT (Multiple Addition Theorem - key to our proof -)

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Theorem

(ABCHT(2014?)) Let $\mathcal{A}'$ be a free arrangement with exponents $(d_1, \ldots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$ and $1 \le p \le \ell$ the multiplicity of the highest exponent d. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \ldots, q$. Define $\mathcal{A}_j'' := \{H \cap H_j \mid H \in \mathcal{A}'\}$ $(j = 1, \ldots, q)$. Assume that the following three conditions are satisfied:

(1) $X := H_1 \cap \cdots \cap H_q$ is q-codimensional,

(2) $X \not\subseteq H \quad (\forall H \in \mathcal{A}')$,

(3) $|\mathcal{A}'| - |\mathcal{A}_j''| = d$ $(1 \le j \le q)$.

Then $q \le p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with exponents $(d_1, \ldots, d_{\ell-q}, (d+1)^q)$.

Main Theorem (again)

Theorem

If

- Φ : *an irreducible root system of rank* ℓ
- $\mathcal{A}$: *the Weyl arrangement (= the collection of hyperplanes orthogonal to the positive roots of* Φ*)*

Then

1. *any ideal subarrangement* $\mathcal{B}$ *of* $\mathcal{A}$ *is free,*
2. *its exponents and the height distribution of the positive roots satisfy the dual-partition formula.*

We prove the Main Theorem applying MAT inductively.

Proof of Main Theorem (just an outline)

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- To verify the three assumtions of MAT, we verify the corresponding combinatorial and geometric properties of the root system Φ. (A key Lemma is in the next page.)

Local-global formula for heights (A key Lemma)

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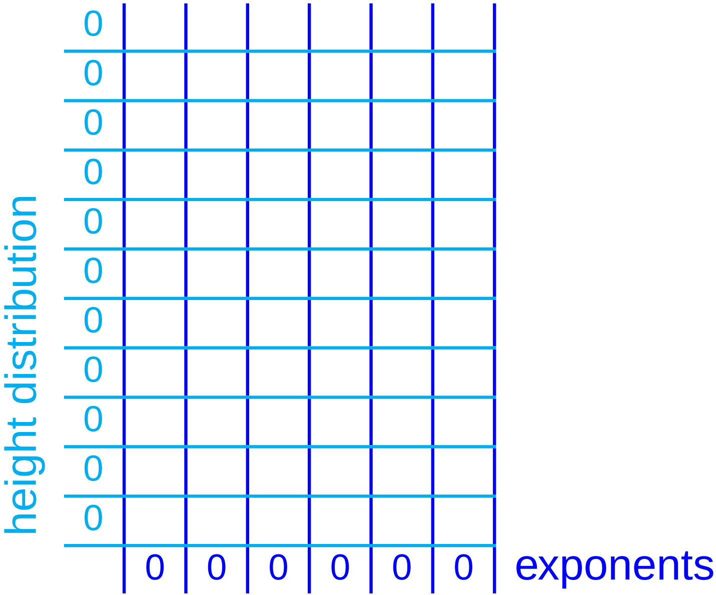
Lemma (Local-global formula for heights)

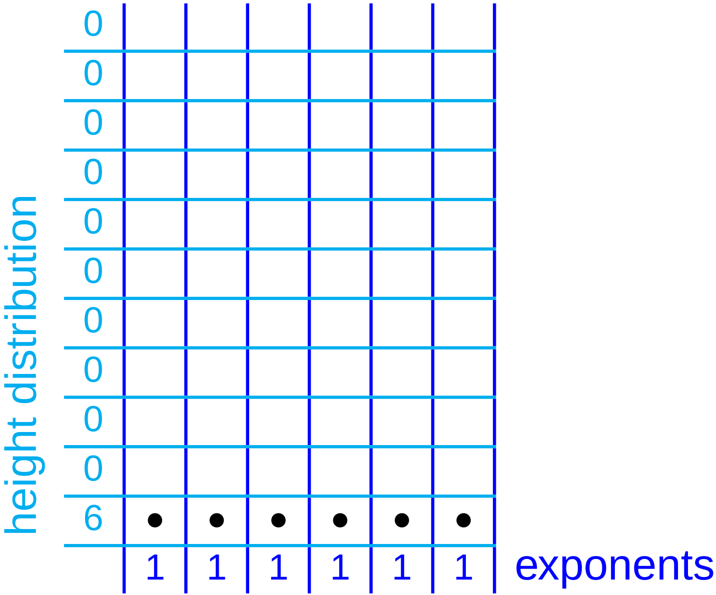
For $\alpha \in \Phi^+$, we have

$$\mathrm{ht}_{\Phi}\, \alpha - 1 = \sum_{X \in \mathcal{A}^{\alpha}} (\mathrm{ht}_X\, \alpha - 1).$$

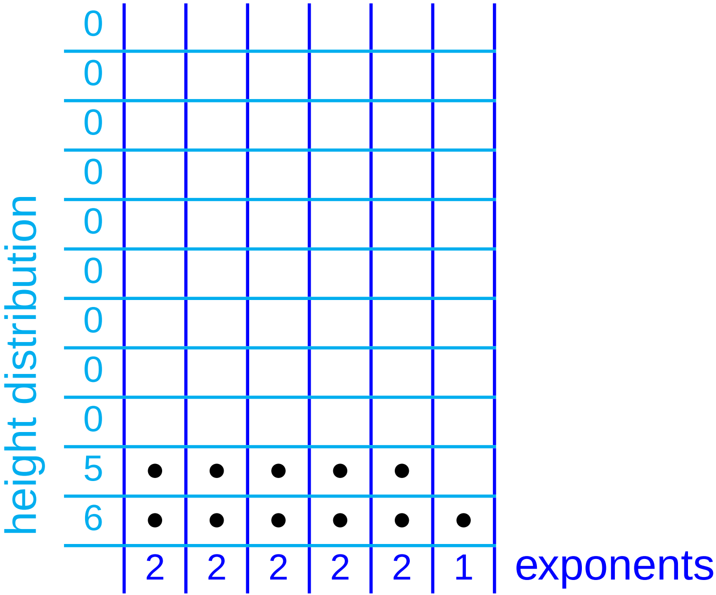
Inductive use of MAT (E_6) : $I = \Phi_0^+$


height distribution
0
0
0
0
0
0
0
0
0
0
0
0 0 0 0 0 0
exponents

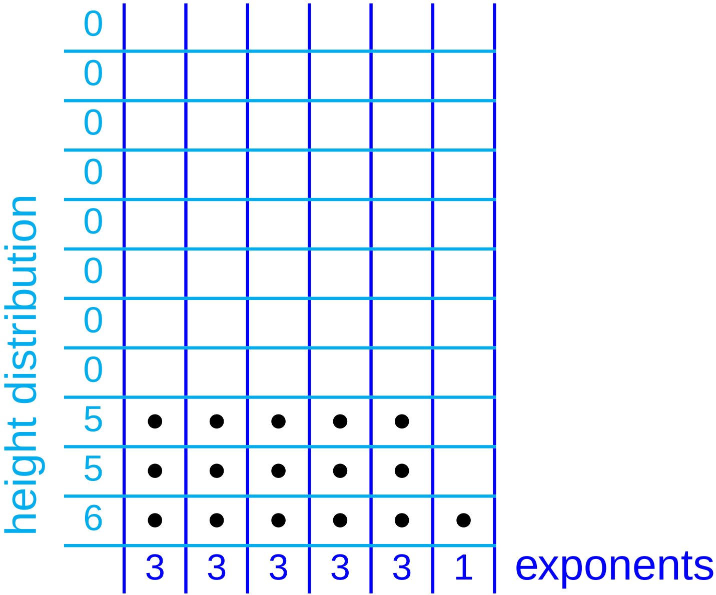
Inductive use of MAT (E_6) : $I = \Phi_1^+$



Inductive use of MAT (E_6) : $I = \Phi_2^+$



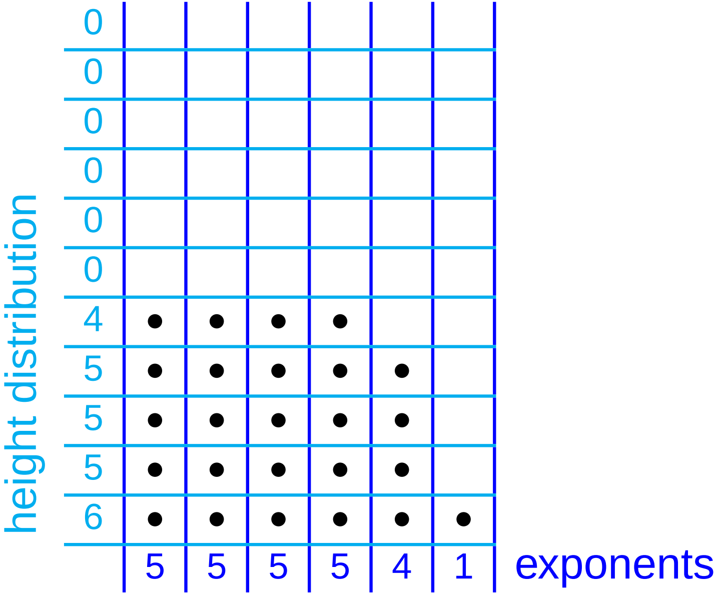
Inductive use of MAT (E_6) : $I = \Phi_3^+$



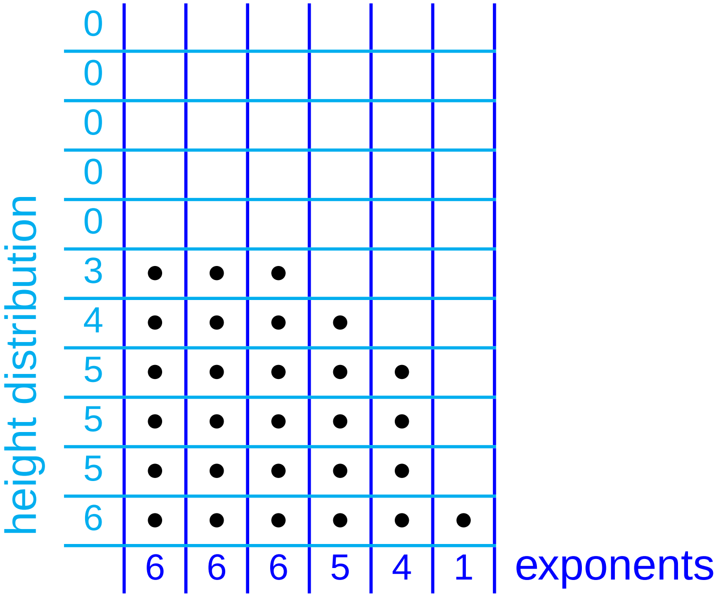
Inductive use of MAT (E_6) : $I = \Phi_4^+$

height distribution						
0						
0						
0						
0						
0						
0						
0						
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	4	4	4	4	4	1

Inductive use of MAT (E_6) : $I = \Phi_5^+$



Inductive use of MAT (E_6) : $I = \Phi_6^+$



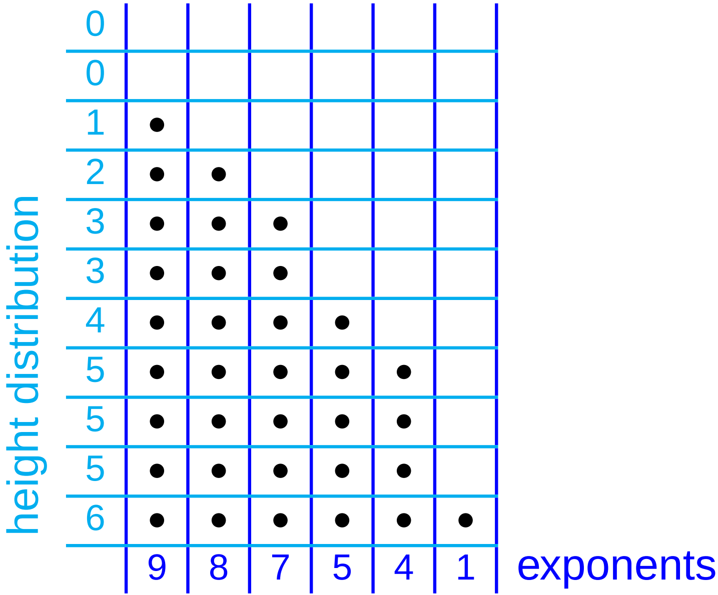
Inductive use of MAT (E_6) : $I = \Phi_7^+$

height distribution						
0						
0						
0						
0						
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	7	7	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_8^+$

height distribution						
0						
0						
0						
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	8	8	7	5	4	1

Inductive use of MAT (E_6) : $I = \Phi_9^+$



Inductive use of MAT (E_6) : $I = \Phi^+_{10}$

height distribution						
0						
1	•					
1	•					
2	•	•				
3	•	•	•			
3	•	•	•			
4	•	•	•	•		
5	•	•	•	•	•	
5	•	•	•	•	•	
5	•	•	•	•	•	
6	•	•	•	•	•	•
exponents	10	8	7	5	4	1

height distribution						
1	●					
1	●					
1	●					
2	●	●				
3	●	●	●			
3	●	●	●			
4	●	●	●	●		
5	●	●	●	●	●	
5	●	●	●	●	●	
5	●	●	●	●	●	
6	●	●	●	●	●	●
exponents	11	8	7	5	4	1

Summary

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1. The celebrated dual-partition formula for a root system by Shapiro-Steinberg-Kostant-Macdonald is generalized to the class of ideals.
2. The theory of free arrangements (the multiple addition theorem (MAT)) provides a base of our proof.
3. Our proof needs combinatorial and geometric properties concerning the height of positive roots.

A Natural Question

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- Is there any nice characterization of the set

$$\mathcal{F} := \{F \subseteq \Phi^+ \mid \mathcal{A}(F) \text{ is free}\},$$

  where $\mathcal{A}(F)$ is the corresponding subarrangement to F?

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Remark. At least we know

$$\mathcal{F} \supseteq \{wI \mid w \in W,\ I \text{ is an ideal}\}.$$

I stop here.

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Thanks for your attention!