# Heterogeneity on production theory: a discrete geometric approach

Simona Settepanella[◇], Giovanni Dosi[†], Marco Grazzi[‡], Luigi Marengo[†1], and Federico Ponchio[†2]

[◇]Department of Mathematics, Hokkaido University, Japan
[†]Institute of Economics, Scuola Superiore Sant'Anna, Pisa, Italy
[‡]Department of Economics, University of Bologna, Italy
[†1]L.U.I.S.S., Rome, Italy
[†2]C.N.R., Pisa, Italy

May 26, 2015

**Abstract**

This paper presents the mathematical and computational details which provide the basis of the new methodology proposed in Dosi et al. 2013 to assess the level and the evolution of intra-industry heterogeneity and to measure industry and firm level productivity change. In particular in this work we show how geometry can be an effective tool to tackle some relevant issues in economics and how, with new computational methods, it is possible to switch from continuous models to discrete ones, the latter requiring a much smaller set of assumptions.

**JEL codes:** D24; D61; C67; C81; O30

**Keywords:** Production theory, Heterogeneous firms, Activity Analysis, Technical change, Zonotopes, Production functions

# 1 Introduction

This paper presents the mathematical and computational details that are the basis of the methodology proposed in Dosi et al. 2013 in which authors introduce a new framework based on concepts from mathematical geometry. This framework allows to account for the rich intra-industry heterogeneity and to measure firm and industry level productivity variation over time when firms are allowed to differ.

Standard economic theory has for long assumed - and largely is still assuming - that all firms within a single narrowly-defined industry are much similar, solely allowing for some noise that might (slightly) differentiate one firm from the other. However, the empirical firm-level literature that emerged in the last twenty years or so has shown a much richer picture (among the many Baily et al.; 1992; Baldwin and Rafiquzzaman; 1995; Bartelsman and Doms; 2000; Dosi; 2007; Syverson; 2011); one in which firms display a great deal of variation under many of their characteristics. So that, for instance, firms in the same industrial sector, defined at the 3 or 4 digit of the NACE classification, display differences in the levels of labor productivity of 5 or 6 times, looking at the top-to-bottom decile ratio (Dosi et al.; 2012).

The use of different and alternative techniques has also grown over time in response to increasing dissatisfaction with standard assumptions of production theory. Among such alternatives, the Data Envelopment Analysis (DEA) (see, among the many others Farrell; 1957; Charnes et al.; 1978; Daraio and Simar; 2007; Simar and Zelenyuk; 2011) has become quite popular also thanks to the availability of statistical software to perform empirical analysis. Other alternative approaches include the tropical geometry models (see, for instance, Baldwin and Klemperer 2015 and, in this Volume, Shiozawa 2015) and models based on algebraic geometry (see, for instance, Schmedders and Renner 2015 and, in this Volume, Tran 2015).

All these approaches share a common factor which is the use of a different mathematical approach that requires a smaller set of assumptions than the continuous case. As such, they all represent a step towards a higher degree of realism in economic analysis (Dosi; 2004).

Another approach that shares the same intent is that proposed in Dosi et al. 2013, which builds upon the analysis of revealed short-run production functions originlly put forth in Hildenbrand 1981. In Dosi et al. 2013 the authors propose a model based on Zonotopes in order to assess heterogeneity across firms. This is a purely geometrical and discrete model that, by means of computational algorithms on discrete models, enables to study economically meaningful quantities which are defined on a continuous model. In this short survey, we will give a brief description of geometric tools on which this model is based and then we will show how to retrieve and compute elasticity of substitution, a feature that is classically defined for differentiable functions. This is just one out of many possible examples that show how it is possible to compute standard economic measures relaxing some of the strong assumptions usually required in order to employ differentiable functions.

Another feature of the methodology proposed in Dosi et al. 2013 that we review here is the possibility to compute a measure of productivity change within an industry, without imposing a common production function on all firms making up the industry, or without any attempt to recover an efficient frontier, as in the DEA analysis. The proposed framework can also easily account for n-inputs and m-outputs and, crucially, the measures of heterogeneity and technical change do not require many of the standard assumptions typical of production theory.

This paper is organized as follow. In Section 2 the geometry of Zonotopes is explained and the idea behind the model in Dosi et al. 2013 is given. The mathematics that allows to assess elasticity of substitution is also explained. In Section 3 the algorithms to compute heterogeneity and elasticity are given and in Section 4 conclusions and farther possible developments are given.

# 2 The geometric idea behind the Zonotope approach

In this section we illustrate the considerations that motivate the mathematical model in Dosi et al. 2013. We assume that it is possible to represent the *actual technique* of a production unit by means of a *production activity* represented by a vector (Koopmans; 1977; Hildenbrand; 1981)

$$a = (\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_{l+m}) \in \mathbb{R}_+^{l+m}.$$

A production unit, which is described by the vector $a$, produces during the current period $(\alpha_{l+1}, \ldots, \alpha_{l+m}) \in \mathbb{R}^m$ units of output by means of $(\alpha_1, \ldots, \alpha_l)$ units of input. In this framework it is possible to refer to the *size* of the firm as to the length of vector $a$, which can be regarded as a multi-dimensional extension of the usual measure of firm size, often proxied either by the number of employees, sales or value added. In fact, the length of the vector allows to employ both measures of input and output in the definition of firm size (see Dosi et al. 2013) .

## 2.1 Zonotopes

Zonotopes are the multi dimensional generalization of a solid known as Zonohedron. A Zonohedron is a convex polyhedron in which every face is a polygon with point symmetry or, equivalently, symmetry under rotations through an angle of 180 degrees. That is the Zonohedron is a very symmetric solid that generalize the cube and the parallelepiped. Indeed both those solids can be realized as a Zonohedron generated by three perpendicular vectors. Any Zonohedron may equivalently be described as the Minkowski sum of a set of line segments in three-dimensional space. That is, if $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^3$ is a finite family of vectors in the space, the *Zonohedron* generated by the family $\{a_i\}_{1 \leq i \leq N}$ is the solid

$$Y = \sum_{i=1}^{N} [0, a_i]$$

where

$$[0, a_i] = \{x_i a_i \mid x_i \in \mathbb{R}, 0 \leq x_i \leq 1\}$$

is the line segment associated to the vector $a_n$. In order to better understand the Minkowski sum of segments we can reduce to a 2-dimensional case, considering the Minkowski sum of vectors in the plane $\mathbb{R}^2$.

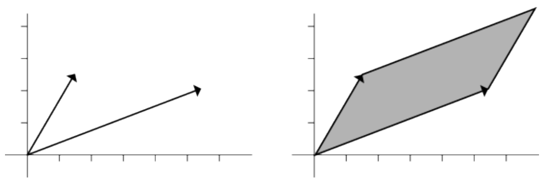

Figure 1: Minkowski sum of two planar vectors.

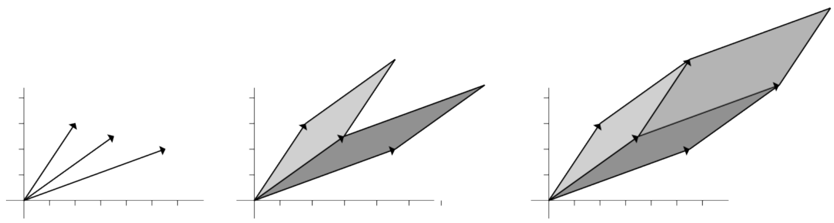

Figure 2: Minkowski sum of three planar vectors.

For example if we consider two vectors $a_1, a_2 \in \mathbb{R}^2$ their Minkowski sum is

$$Y = \{x_1a_1 + x_2a_2 \mid x_i \in \mathbb{R}, 0 \leq x_i \leq 1, i = 1, 2\}$$

that is the parallelogram illustrated in figure 1.

It is not difficult to see that, if we do the sum of three planar vectors $a_1, a_2, a_3 \in \mathbb{R}^2$ their sum is that represented in figure 2.

Analogously, a three dimensional Zonohedron is a solid such that any face is a parallelogram and if we do the sum of 10 vectors $\{a_n\}_{1\leq n\leq 10}$ as displayed in figure 3, we get the Zonohedron in figure 4.

When we consider vectors in a higher dimensional space $\mathbb{R}^n$ with $n > 3$, the Minkoski sum of $N$ vectors $\{a_i\}_{1\leq i\leq N} \in \mathbb{R}^n$ given by

$$Y = \{y \in \mathbb{R}^n_+ \mid y = \sum_{i=1}^{N} \phi_i a_i, 0 \leq \phi_i \leq 1\}$$

is called Zonotope. The vectors $\{a_i\}_{1\leq i\leq N} \in \mathbb{R}^n$ from which the Zonotope is formed are called its *generators*.[1]

Associated to a planar figure there is a well known numeric quantity: the area of the surface. It is a very easy remark that the area of the planar figure 2 is the sum of the 3 parallelograms generated by $a_1, a_2$, by $a_1, a_3$ and by $a_2, a_3$.
The same technique applies to the 3 dimensional case. In this case instead of the area we have the volume, but, analogously to the area for the 2-dimensional case, the volume of the Zonohedron generated by vectors $\{a_n\}_{1\leq n\leq 10}$ in figure 4 is the sum of the volumes

[1]The interested reader can refer to Ziegler (1995) for a survey on Zonotopes.

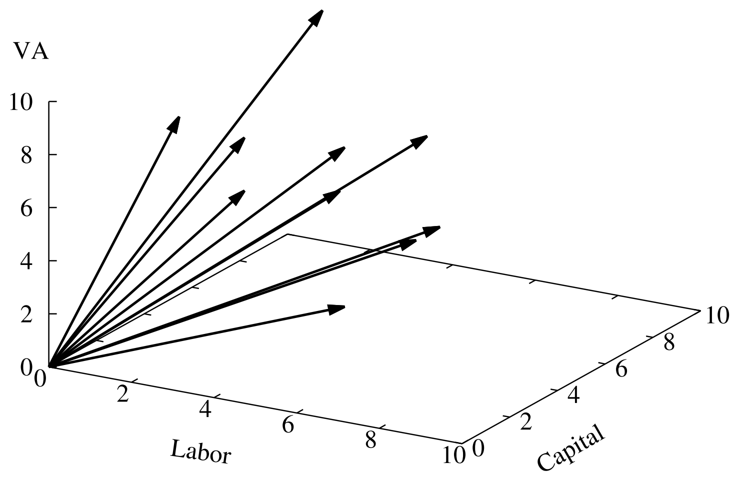


Figure 3: Vectors representing firms of the toy example, source Dosi et al. (2013).

of the $\binom{10}{3} = 120$ parallelepipeds $\{a_{i_1}, a_{i_2}, a_{i_3}\}_{1 \leq i_1 < i_2 < i_3 \leq 10}$ generated by any 3 of the ten vectors $\{a_n\}_{1 \leq n \leq 10}$ . Then in order to compute the volume of a Zonohedron we only need to compute the volume of all parallelepipeds from which it is formed.
It is well known that the volume Vol($P$) of a parallelepiped $P$ generated by vectors $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, $i = 1, 2, 3$, is the absolute value of the determinant of the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}.$$

Then, in general, if $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^3$ are vectors in $\mathbb{R}^3$, $A_{i_1,i_2,i_3}$ is the matrix whose rows are vectors $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ and $\Delta_{i_1,i_2,i_3}$ its determinant, the volume of the zonohedron $Y$ generated by $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^3$ is given by:

$$Vol(Y) = \sum_{1 \leq i_1 < i_2 < i_3 \leq N} \mid \Delta_{i_1,i_2,i_3} \mid$$

where $\mid \Delta_{i_1,i_2,i_3} \mid$ is the module of the determinant $\Delta_{i_1,i_2,i_3}$.
Now it is an easy remark that, in this framework, whatever holds in dimension 3 holds also in a generic dimension $n > 2$, that is if $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^n$ are vectors and $Y$ the Zonotope generated, then

$$Vol(Y) = \sum_{1 \leq i_1 < \ldots < i_n \leq N} \mid \Delta_{i_1,\ldots,i_n} \mid$$

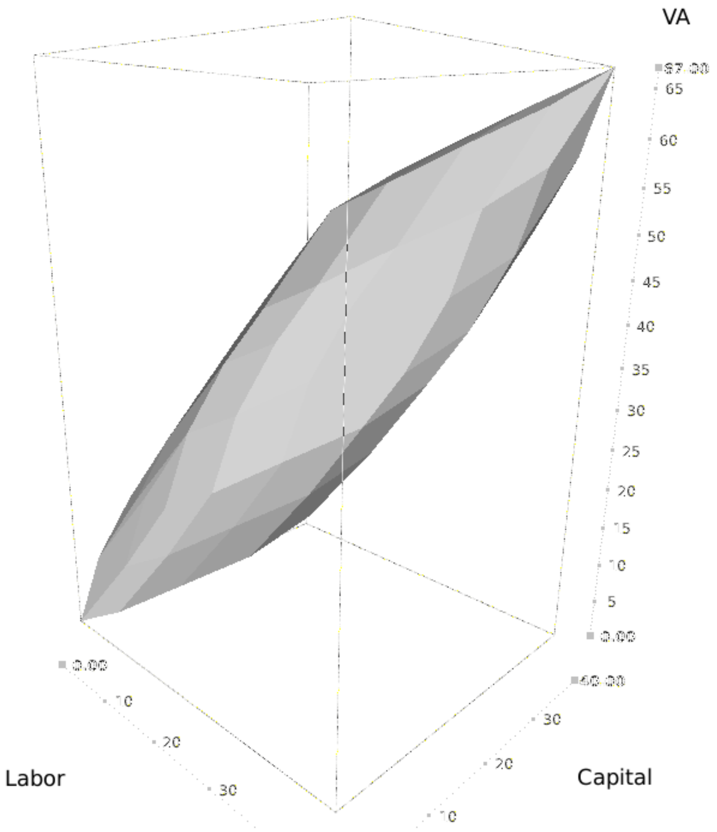


Figure 4: 3D representation of the zonohedron generated by vectors in Fig. 3, source Dosi et al. (2013).

where $| \Delta_{i_1,\dots,i_n} |$ is the absolute value of the determinant $\Delta_{i_1,\dots,i_n}$ of the matrices $A_{i_1,\dots,i_n}$ whose rows are vectors $\{a_{i_1}, \dots, a_{i_n}\}$.
In Dosi et al. 2013 vectors $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^n$ represent the production activities of firms in an industry with N enterprises.

## 2.2 Heterogeneity by means of Zonotopes

The Basic idea behind the use of the Zonotopal approach, brought from Hildenbrand (1981) is the following. Let us consider the basic case of an industry with only two firms, $a_1 = (i_1, o_1)$ and $a_2 = (i_2, o_2)$ where $i_1, i_2$ are the units of input and $o_1, o_2$ are the units of output of the two firms. Let us assume that the two firms use exactly the same techniques to produce their outputs. This is equivalent to the fact that the two vectors $a_1, a_2 \in \mathbb{R}^2$ are proportional. That is if the firm $a_1$ utilizes input $i_1$ to get output $o_1$ then the firm $a_2$ will enter input $i_2 = K \cdot i_1$ to get output $o_2 = K \cdot o_1$ as they use exactly the same technique. Indeed if this was not so, then it would simply means that $a_2$ is doing something different, i.e. it is using different technique[2]. Being proportional for the two vectors means, from a geometric point of view, that they lie in the same line in the plane $R^2$. On the contrary let's assume that the two firms are the most different possible, that is, as a limit case (even if not feasible), one is producing without inputs, i.e. $a_1 = (0, o_1)$ and the other one is not producing even with not zero input, i.e. $a_2 = (i_2, 0)$. In this case the resulting vectors are perpendicular.

From this example it is clear that the difference in the vectors representing the firms represents the difference between firms and that somehow the angle or the space between those vectors is a measure of their differences in productivity. For example if the vectors in Figure 1 represent two firms, the one closer to the output axis (the y axis) clearly consumes less input for producing a similar quantity of output compared to the second firm.

But how can we asses this difference, i.e. this heterogeneity? Clearly the proportional and perpendicular cases above are the two extreme cases hence we would expect that the first one **minimizes** the measure of heterogeneity, that is it gives null heterogeneity, while the second one **maximizes** the heterogeneity.

The first question that arises is: *which is the measure that is 0 in the first case and maximum in the second one?*

The natural answer would be to check the angle between the two vectors. But then let's consider the case of three vectors as the one in figure 2. This represents a different industry, indeed it is the one in figure 1 with one more vector. Then a second question arises: *are the industries represented in figure 1 and 2 different? Is their heterogeneity different?*

In Dosi et al. 2013 authors think that those two industries are different with different heterogeneity. Then it is a simple remark that something else but the angle has to be chosen to assess heterogeneity.

Geometry gives a natural answer. Indeed if we consider any two vectors $a_1, a_2$ in the plane $\mathbb{R}^2$, the *area* of the parallelogram $P$ generated by them is area$(P) = \mid a_1 \mid\mid a_2 \mid \sin\theta$ where $\mid a \mid$ is the norm of the vector, that is its length, and $\theta$ is the angle between the two vectors. Since $\sin\theta = 0$ for $\theta = 0$ and $\sin\theta = 1$ for $\theta = \frac{\pi}{2}$, then the area of $P$ is zero when the vectors are proportional and it is maximum when they are perpendicular. In particular this area coincides with the area of their Minkowski sum and this quantity is clearly different when we consider the Minkowski sum of three vectors $a_1, a_2, a_3 \in \mathbb{R}^2$ instead of 2.

Then it seems that the area and, more in general, the volume of the Minkowski sum $Y$ of a family $\{a_i\}_{1\leq i\leq N}$ of vectors is definitively a good candidate to assess heterogeneity

[2]Let us remark that, right now, we don't distinguish change in output due to changes in efficiency from that due to technical change as we consider both of them reasons for *heterogeneity* of firms. The problem to distinguish them will be addressed in subsequent papers (see Conclusions)

between firms. This shouldn't be a surprise since measures based on Zonotopes are already used in social sciences to construct inequality indices (see, for instance, Koshevoy (1995, 1998) and, for a survey, Savaglio (2002)).
Unfortunately the volume of the Minkowski sum of vectors grow when the number of vectors grow as it is a sum of the volumes generated by any three vectors. Then changes are needed in order to obtain a pure measure of heterogeneity that is independent from the number of vectors and from the used units.
Again geometry helps. Indeed the most natural answer to this problem is to consider not simply the volume of the Minkowski sum $Y$ but the percentage of space $Y$ occupies inside the biggest possible volume that *contains* it when the diagonal $\sum_{i=1}^{N} a_i$ is fixed. This is equivalent, in the example in figure 1 to consider the area of the Minkowski sum $P$ of the two vectors $a_1$ and $a_2$ quotiented by the area of the rectangle $R$ that has the same diagonal of $P$. That is we consider the quotient:

$$\frac{\text{Area}(P)}{\text{Area}(R)} = \frac{\mid a_1 \mid\mid a_2 \mid \sin\theta}{\mid a_1 \mid\mid a_2 \mid} = \sin\theta \quad . \tag{1}$$

Clearly the rectangle $R$ contains the parallelogram $P$ and the larger is the angle between the two vectors $a_1, a_2$, the higher is the area of $P$ and the smaller is the quotient in equation (1).
This is the idea that motivate authors in Dosi et al. 2013 to define heterogeneity as the *ratio* of the volume $\text{Vol}(P)$ of the Zonotope $Y$ generated by the production activities $\{a_n\}_{1\leq n\leq N}$ over the *total* volume $\text{Vol}(P_Y) = \prod_{i=1}^{N} \mid a_i \mid$ of an industry $P_Y$ with production activity $d_Y = \sum_{n=1}^{N} a_n$ .[3]

**Remark 2.1** *Even if angles are not a good candidates to assess heterogeneity it is evident that, among the two firms represented by vectors in Figure 1, the one represented by the vector closer to the input axis (i.e. that form a smaller angle with inputs axis) is lesser productive than the second firm. That is angles of those vectors with input axis are a good indicator of productivity of a firm and hence, more in general, in an n-dimensional space* $\mathbb{R}^n$ *the angle of a vector* $a_i$ *with the space of inputs is informative of the productivity of the firm represented by* $a_i$*. This is the motivating example for Dosi et al. 2013 to use those angles as a index of productivity and to use theirs variation to assess change in productivity.*

## 2.3 Elasticity of substitution

In Dosi et al. 2013 authors compute the elasticity of substitution associated to production activities of an industry $\{a_i\}_{1\leq i\leq N} \in \mathbb{R}^n$. Here we illustrate the mathematical aspect of their computations while in the subsequent Section the algorithm used will be described.

[3]We refer to the original paper Dosi et al. 2013 for examples on numerical and real data.

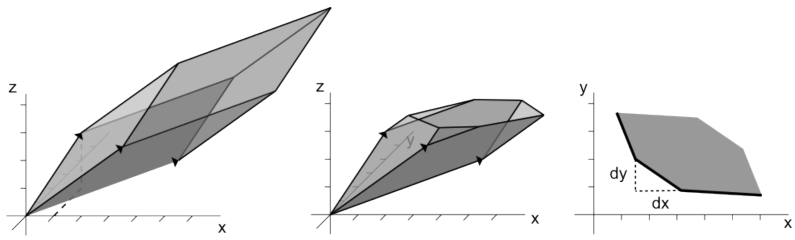


Figure 5: A Zonotope (on the left) is cut along the xy-plane (middle). The efficient part of the resulting polygon (right), marked with a black line is the isoquant. The Marginal Rate of Technical Substitution (MRTS) is the ratio $dy/dx$.

The concept of *isoquant* describes the hypotetical curve of equal production in a 2-dimensional capital-labor graph. In the Zonotope approach in Dosi et al. 2013 an isoquant is represented by a cut of the Zonotope along a plane of constant production. For example, in a 3-dimensional case, i.e. vectors $a_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, in which the output is represented by the third coordinate $z_i$, an isoquant is obtained cutting the Zonotope with a plane parallel to the $xy$-plane, i.e. a plane with points having constant output $z = c$. With such a cut we obtain a convex polygon, representing all the possible combination of existing firms generating a fixed level of production. In this case the isoquant is the optimal frontier of this polygon where either labor or cost must be incremented to increase production (see figure 5).

Given a production function $z = f(x, y)$, the slope

$$MRTS = \frac{d\,f/dy}{d\,f/dx}$$

of the isoquant $f(x, y) = c$, $c \in \mathbb{R}^+$ constant, is the Marginal Rate of Technical Substitution (MRTS) and represents locally the ratio at which labor can be exchanged with capital while keeping a constant output. In the Zonotope case the isoquant is a part of the boundary of a polygon. Hence it is a piecewise linear function and its MRTS is a piecewise constant function. The MRTS depends only on the direction of the gradient of $f$ which is perpendicular to the isoquant in that point.

The *elasticity of substitution* represents how much (in relative terms) the ratio of production factors changes with respect to the MRTS in a given isoquant. For example if, in a given isoquant, we change the labor/capital ratio by 10% and the MRTS changes by 1%, then the elasticity of substitution is approximately 10 along that isoquant.

The formula for computing the elasticity of substitution for a production function $z = f(x, y)$ is:

$$ES = \frac{d\,\ln(y/x)}{d\,\ln(MRTS)} = \frac{\frac{d(y/x)}{y/x}}{\frac{d(\frac{df}{dy}/\frac{df}{dx})}{\frac{df}{dy}/\frac{df}{dx}}} \quad .$$

In the discrete case of Zonotopes the isoquants are piecewise linear functions, hence if we simply compute the elasticity of substitution in the classical way we get that its value is either infinity on linear pieces, or not defined in the vertices of the polygon. This is a meaningless result and, in fact, we are dealing with a discrete setting not a continuous one hence we cannot use classical differential tools, but differential methods on discrete sets are needed. There are several ways to approach differential problems in discrete settings as, for example, the *smoothing*, i.e. a process to create an approximating function of a discrete data set that attempts to capture important patterns in the data, while leaving out noise; the *interpolation*, i.e. a method of constructing new data points within the range of a discrete set of known data points and *curve fitting*, i.e. the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constraints. The method used in Dosi et al. 2013 is essentially based on smoothing. They compute the elasticity of substitution using finite differences on the vertices of the isoquant and smoothing the result across neighbouring vertices to remove the error due to the discrete sampling of $f$ (see Section 3.2 for a detailed algorithm).

# 3 Computational aspects and algorithms

In this section we give all computational aspects associated to tools introduced in the previous Section.

## 3.1 The volume of a Zonotope

The implicit representation of a Zonotope given in the previous chapter is not very convenient for many computations, we prefer to generate an explicit representation as a set of parallelogram faces. We will need to compute the coordinates of each vertex and which four vertices belongs to a face. This will allow us to easily slice the Zonotope at a desired production level and compute marginal rates and elasticity.

We will exploit a few known properties of the 3-dimensional Zonotopes, as a brute force approach is too computationally expensive. The process for higher dimensions is analogous.

- The vertices of the Zonotope can be computed as the sum of a subset $\{a_i\}_{i \in I}$, $I \subset \{1, \dots, N\}$, of the vectors $\{a_i\}_{1 \leq i \leq N} \in \mathbb{R}^3$. Let us remark that not all subset of vectors define a vertex.

- Each vertex $v$ has an opposing vertex $\overline{v}$ which is the sum of all the other vectors, i.e. if $v = \sum_{i \in I} a_i$ and $J = \{1, \dots, N\} \setminus I$ then $\overline{v} = \sum_{i \in J} a_i$.

- Each pair of vectors $a_i, a_j \in \{a_i\}_{1\leq i\leq N}$ define a couple of opposing parallel faces whose normal $n_{i,j}$ is perpendicular to both $a_i$ and $a_j$.

- The vectors that generate the base vertex of a face share the same sign when computing the dot product with the normal of the face.

To computing the faces then we can iterate over all the pair of vectors $a_i, a_j$ the following process: compute the normal $n_{i,j}$ and the sign $s_{i,j,k} \in \{0, 1\}$ of the dot product of $n_{i,j}$ with each input vector $a_k$, zero when negative, one when positive.
This computation is $O(n^3)$ in the number of vectors. The first (base) vertex of the face is computed by $\sum_{k\neq i,j} s_{i,j,k}a_k$, the others by adding $a_i$, $a_j$ and $a_i + a_j$.

Once the zonohedron is in explicit form, we can easily compute the volume as sum of all the volumes of the pyramids with each face as base and the center of the zonohedron as apex.

To extend this algorithm in higher dimensions $\mathbb{R}^n$, i.e. $\{a_i\}_{1\leq i\leq N} \in \mathbb{R}^n$, we need to consider that a *face* has dimension $n - 1$ and is defined by $n - 1$ vectors $\{a_i\}_{i\in I}$, $I \subset \{1 \leq i \leq N\}$ of cardinality $n - 1$, and has an unique normal vector $n_I$ associated, which can be computed using the generalized cross product. It has $2^{n-1}$ vertices where the base vertex is computed as in the 3-dimensional case and all the others are computed adding all the possible combinations of the $n - 1$ vectors $\{a_i\}_{i\in I}$. Similarly the volume can be computed as a sum of $n$ dimensional pyramids.

## 3.2 The elasticity of substitution

As already specified in Section 2.3 the Zonotope approach is a discrete one. Then, instead of use partial derivatives of a global production function $f$ as in the continuous setting, we can compute the isoquant at a certain production level and use finite differences to compute the elasticity of substitution. The algorithm to compute elasticity of substitution in $\mathbb{R}^3$ is based on few steps.

1. Compute the intersection of the Zonotope with a plane $z = c$ of fixed production (as in Figure 5 (b)), that is find the intersection of each face with the plane $z = c$. This is equivalent to compute the intersection of four segments with a plane and sort the resulting oriented segments in a clockwise direction to form a closed polygon.

2. Remove the non efficient boundary of the polygon obtained at point 1. . This can be done by looking at the direction of the segments: in the efficient segments the $y$ increases and $x$ decreases going from the first to the second vertex. All the others are inefficient (see Figure 5 (c)).

The sequence of vertices $v_i = (x_i, y_i)$ left in the efficient part of the polygon essentially corresponds to our isoquant and we can compute the MRTS as

$$MRTS_i = -\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

and the unfiltered elasticity of substitution as

$$E_i = \frac{\ln(y_{i+1}/x_{i+1}) - \ln(y_i/x_i)}{\ln(MRTS_{i+1}) - \ln(MRTS_i)}.$$

The resulting values are then smoothed using a gaussian filter in logaritmic space to find the value of elasticity of substitution

$$ES_i = \sum w_{ij} E_j$$

where $w_{ij} = g(|\ln(y_i/x_i) - \ln(y_j/x_j)|)$ and $g = ae^{-(ax)^2/2}/\sqrt{2\pi}$ is the gaussian function. The filtering is necessary to reduce the sampling noise due to the piecewise linear approximation of the production function with a zonotope.

# 4 Conclusions

In this short paper we have shown how the combination of discrete mathematical models together with the recent availability of more powerful calculators are preparing the stage for new possibilities to model economic and social phenomena in which one is able to relax many of the strong standard assumptions. In this respect, the works of Dosi et al. (2013) and Tran (2015) already represent a case to the point. The result of this ongoing process - still very much at an early stage - is a potentially amazing new quantity of mathematical tools in discrete settings that allow to approximate continuous models without the requirement to impose many restrictive conditions on the data themselves. Indeed, geometric models in discrete setting require way less conditions to be defined and studied with respect to classical models based on continuity. This suggests that not only economics might be changing, but also the mathematics used therein and that new mathematical instruments, such as geometric and discrete ones, could be the adequate tools to foster such change. Indeed those new tools will allow to create models that will not require extra conditions to be enforced on the data in order to be implemented.

# References


Baily, M. N., Hulten, C. and Campbell, D. (1992). Productivity dynamics in manufacturing establishments, *Brookings Papers on Economic Activity: Microeconomics* **4**: 187–249.

Baldwin, E. and Klemperer, P. (2015). Understanding preferences: "demand types", and the existence of equilibrium with indivisibilities, *http://www.nuff.ox.ac.uk/users/klemperer/tropical.pdf* .

Baldwin, J. R. and Rafiquzzaman, M. (1995). Selection versus evolutionary adaptation: Learning and post-entry performance, *International Journal of Industrial Organization* **13**(4): 501–522.

Bartelsman, E. J. and Doms, M. (2000). Understanding productivity: Lessons from longitudinal microdata, *Journal of Economic Literature* **38**(2): 569–594.

Charnes, A., Cooper, W. W. and Rhodes, E. (1978). Measuring the efficiency of decision making units, *European Journal of Operations Research* **2**: 429–444.

Daraio, C. and Simar, L. (2007). Conditional nonparametric frontier models for convex and nonconvex technologies: a unifying approach, *Journal of Productivity Analysis* **28**(1): 13–32.

Dosi, G. (2004). A very reasonable objective still beyond our reach: Economics as an empirically disciplined social science, *in* M. Augier and J. G. March (eds), *Models of a Man, Essays in memory of Herbert A. Simon*, Cambridge, MA, MIT Press.

Dosi, G. (2007). Statistical regularities in the evolution of industries. a guide through some evidence and challenges for the theory, *in* F. Malerba and S. Brusoni (eds), *Perspectives on Innovation*, Cambridge University Press.

Dosi, G., Grazzi, M., Marengo, L. and Settepanella, S. (2013). Production theory: accounting for firm heterogeneity and technical change, *LEM working paper* (22).

Dosi, G., Grazzi, M., Tomasi, C. and Zeli, A. (2012). Turbulence underneath the big calm? The micro-evidence behind Italian productivity dynamics, *Small Business Economics* **39**(4): 1043–1067.

Farrell, M. J. (1957). The measurement of productive efficiency, *Journal of the Royal Statistical Society. Series A (General)* **120**(3): 253–290.

Hildenbrand, W. (1981). Short-run production functions based on microdata, *Econometrica* **49**(5): 1095–1125.

Koopmans, T. C. (1977). Examples of production relations based on microdata, *in* G. C. Harcour (ed.), *The Microeconomic foundations of macroeconomics*, London: Macmillan Press.

Koshevoy, G. (1995). Multivariate lorenz majorization, *Social Choice and Welfare* **12**: 93–102.

Koshevoy, G. (1998). The lorenz zonotope and multivariate majorizations, *Social Choice and Welfare* **15**: 1–14.

Savaglio, E. (2002). Multidimensional inequality: A survey, *U of Siena, Economics Working Paper* (362).

Schmedders, K. and Renner, P. J. (2015). A polynomial optimization approach to principal-agent problems, *Econometrica* **83**(2): 729–769.

Shiozawa, Y. (2015). Trade theory and exotic algebra, *Evolutionary and Institutional Economics Review* **12**.

Simar, L. and Zelenyuk, V. (2011). Stochastic FDH/DEA estimators for frontier analysis, *Journal of Productivity Analysis* **36**(1): 1–20.

Syverson, C. (2011). What determines productivity?, *Journal of Economic Literature* **49**(2): 326–65.

Tran, N. M. (2015). The perron-rank family, *Evolutionary and Institutional Economics Review* **12**.

Ziegler, G. M. (1995). *Lectures on Polytopes*, Graduate Texts in Mathematics, Springer-Verlag.