PLANAR CUBIC CURVES

— FROM HESSE TO MUMFORD

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Abstract. What are the moduli space of elliptic curves and its compactification? In order to explain the issues involved, we discuss the case of planar cubic curves in detail. A basic idea for compactification is the GIT-stability of Mumford.

In arbitrary dimension, GIT-stability canonically compactifies the moduli space of abelian varieties over $\mathbf{Z}[\zeta_N, 1/N]$ for some large $N \geq 3$. In the smallest possible case, dimension one and $N = 3$, the problem is reduced to the study of planar cubic curves, more specifically Hesse cubic curves. Every GIT-stable cubic curve is isomorphic to a Hesse cubic curve.

1. Hesse cubic curves

Let $\mathbf{P}^2_k$ be the projective plane over an algebraically closed field k of characteristic different from 3. We could think of the base field k as the field $\mathbf{C}$ of complex numbers.

A Hesse cubic curve is by definition a cubic curve on the plane $\mathbf{P}^2_k$ defined by the following equation:

$$C(\mu) \ : \ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \tag{1}$$

for some $\mu \in k$, or $\mu = \infty$ (in which case we understand that $C(\infty)$ is the curve defined by $x_0x_1x_2 = 0$). Let ζ_3 is a primitive cube root of unity. For $\mu \neq \infty, 1, \zeta_3, \zeta_3^2$, the curve $C(\mu)$ is a nonsingular elliptic curve, while for $\mu = \infty, 1, \zeta_3, \zeta_3^2$ $C(\mu)$ consists of three nonsingular rational curves, pairwise intersecting at a distinct point so that the three irreducible components of $C(\mu)$ form a cycle.

When $k = \mathbf{C}$, if $C(\mu)$ is nonsingular, it is topologically a real 2-torus. Otherwise, it is a cycle of 3 rational curves, (or equivalently a union of three lines in $\mathbf{P}^2_k$ in general position), topologically a cycle of three real 2-spheres, which looks like a rosary of three beads. When μ approaches ∞ or ζ_3^k, then a real 2-torus is pinched locally at three distinct meridians into a cycle of three 2-spheres.

This class of curves was studied by Hesse in the middle of 19th century. His paper published in 1849 (see [Hesse]) is summarized as follows.

Date: December 15, 2003.

Theorem 1.1. (1) *Any nonsingular cubic curve can be converted into one of the Hesse cubic curves (1) under the action of* $\mathrm{SL}(3,k)$, *namely it is isomorphic to one of* $C(\mu)$ *for* $\mu \neq \infty, \mu^3 \neq 1$.

(2) *Every Hesse cubic curve* $C(\mu)$ *has nine inflection points , independent of* μ: $[1:-\beta:0],[0:1:-\beta],[-\beta:0:1]$ *where* $\beta^3=1$.

(3) $C(\mu)$ *is transformed isomorphically onto* $C(\mu')$ *under* $\mathrm{SL}(3,k)$ *with each of nine inflection points fixed if and only if* $\mu=\mu'$.

The first and third assertions of the theorem show that any isomorphism class of nonsingular cubic curves is represented by $\mu \in k$ with $\mu^3 \neq 1$. In other words,

$$k \setminus \{1, \zeta_3, \zeta_3^2\} = \text{the moduli space of nonsingular cubic curves with ordered nine inflection points}.$$

We mean by a moduli space the space naturally representing the isomorphism classes of some geometric objects as above. In this case the noncompact moduli space

$$k \setminus \{1, \zeta_3, \zeta_3^2\} = \operatorname{Spec} k[\mu, \frac{1}{\mu^3-1}]$$

can be compactified into $\mathbf{P}^1_k$ as the theorem of Hesse shows, where the exceptional values $\mu = 1, \zeta_3, \zeta_3^2$ and $\mu=\infty$ correspond to a singular Hesse cubic curve, that is a union of three lines in general position. It is remarkable that the compactified moduli space is also the moduli space of (isomorphism classes of) certain geometric objects, in this case Hesse cubic curves possibly singular.

In what follows we mean by "compactification of a moduli space" roughly what we saw above.

In this article, via diverse modern interpretations of the theorem of Hesse, we will present an analogy to it and construct for each large symplectic finite abelian group K a compactification $SQ_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$ of the moduli space of abelian varieties where ζ_N is a primitive N-th root of unity for N suitably chosen.

It is important to recognize that the problem of compactifying a moduli space is not the problem of finding all limits of some geometric objects. For example, the compactification of the moduli space of nonsingular Hesse cubic curves is $\mathbf{P}^1_k$. As this outcome suggests, one could say that the problem of compactification is to single out an important class or a relatively narrow class of limits only so that the class of limits may form a complete (or compact) algebraic variety set-theoretically.

This problem of compactifying a moduli space is algebro-geometrically quite interesting, whatever the objects to consider may be chosen. It was natural to ask whether one could construct compactifications of the moduli spaces at least for curves, abelian varieties and K3 surfaces because they are basic objects in algebraic geometry. As is well known, we have the Deligne-Mumford compactification for curves [DM69], whereas there are quite a lot of compactifications of the moduli of abelian varieties ([AMRT75], [FC90]).

Nevertheless in the present article we will construct *one and only one* new compactification $SQ_{g,K}$ of the moduli of abelian varieties. This compactification is natural enough because, as we will see below, there are three natural approaches to the compactification problem including GIT-stability and representation theory of Heisenberg groups, each of which leads us to the same compactification $SQ_{g,K}$.

Let us explain what is a modern interpretation of the theorem of Hesse. Let us look first at the curve

$$C(\infty)\ :\ x_0x_1x_2 = 0. \tag{2}$$

Let $C(\infty)^0 := C(\infty)\setminus\{\text{singular points}\}$. Then $C(\infty)^0$ has a group scheme structure, and as group schemes

$$C(\infty)^0 = \mathbf{G}_m \times (\mathbf{Z}/3\mathbf{Z}) \tag{3}$$

where $\mathbf{G}_m$ is the multiplicative group, so $\mathbf{G}_m = \mathbf{C}^*$ if $k = \mathbf{C}$. The family of Hesse cubic curves (1) is therefore an analogue of the so-called Tate curve over a complete discrete valuation ring (or the unit disc). Moreover theta functions on cubic curves, namely nonsingular elliptic curves or one-dimensional abelian varieties over $\mathbf{C}$, are Fourier series, and similarly functions on the group variety $\mathbf{G}_m$ has also Fourier series expansions. In this sense $\mathbf{G}_m$ is one of the nice limits of cubic curves. This viewpoint of degeneration of elliptic curves into nice group varieties was also very fruitful in higher dimension as is shown by the work of Grothendieck, Raynaud, Mumford, Faltings and Chai. This is the first modern interpretation of the theorem of Hesse, though this should now be considered classical.

From a different point of view, a finite group $G(3)$ of order 27 called the Heisenberg group acts on the Hesse cubic curves linearly. Let x_0, x_1, x_2 be the homogeneous coordinates of the plane $\mathbf{P}^2_k$ and V the vector space spanned by x_i $(i = 0, 1, 2)$. The Heisenberg group $G(3)$ is a subgroup of $\mathrm{GL}(V)$ generated by the following two linear transformations σ and τ of V:

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i = 0, 1, 2 \mod 3) \tag{4}$$

which are subject to the relation

$$\sigma^3 = \tau^3 = \mathrm{id}_V, \quad \sigma\tau = (\zeta_3 \cdot \mathrm{id}_V)\tau\sigma. \tag{5}$$

Let us regard $\mathbf{P}^2_k$ as the space of row 3-vectors. Then σ and τ induce automorphisms of $\mathbf{P}^2_k$ by

$$\begin{aligned} \bar\sigma : [x_0, x_1, x_2] &\mapsto [\sigma(x_0), \sigma(x_1), \sigma(x_2)] = [x_0, \zeta_3 x_1, \zeta_3^2 x_2],\\ \bar\tau : [x_0, x_1, x_2] &\mapsto [\tau(x_0), \tau(x_1), \tau(x_2)] = [x_1, x_2, x_0]. \end{aligned} \tag{6}$$

These restrict to automorphisms of Hesse cubic curves, which are the translations of the elliptic curves by their 3-torsion points. The vector space V is by (4) a three-dimensional representation of $G(3)$. This is often called the Schrödinger representation of $G(3)$, which is easily shown to be irreducible. Therefore by the famous lemma of Schur about irreducible representations, the basis x_0, x_1 and x_2, which are transformed by σ and τ as above, are uniquely determined up to constant multiples. This property enables us to identify x_0, x_1 and x_2 with a natural basis of theta functions over

C. In this sense theta functions usually defined over **C** have natural counterparts in positive characteristic. This is the second modern interpretation of the work of Hesse.

The third important interpretation of it is based on the GIT-stability of Mumford, which we will discuss later. We will see that the above three interpretations are essentially the same and that Hesse cubic curves and their equations are derived naturally from any of the interpretations. Thus each of the three interpretations is a route to one and the same compactification $\mathbf{P}^1_k$. Each interpretation provides us with a natural approach to the problem of compactifying the moduli of abelian varieties in higher dimension. We will explain them in more detail in the rest of the article.

2. Satake compactification and toroidal compactification

A compactification as a complex analytic space of the moduli space A_g of principally polarized abelian varieties was constructed by Satake in the 1950's, now known as the Satake compactification of A_g. Later as an application of the theory of torus embeddings, quite a lot of compactifications of complex spaces similar to A_g were constructed by Mumford et al. [AMRT75], which we call toroidal compactifications. Thereafter, toroidal compactifications were algebrized by Faltings and Chai [FC90] into compactifications of A_g or its analogues as a scheme over **Z**. A particular case of toroidal compactification, referred to as the Voronoi compactification of A_g, was discussed in [Namikawa76] in connection with proper degeneration of abelian varieties. This is very relevant to the subject of the present article.

Toroidal compactification is sufficiently general, and it seems that there is no other class of algebro-geometrically natural compactifications. However there is a feature missing in this compactification. As we explained for Hesse cubic curves in §1, we would like to demand a given compactification of the moduli space to be again the moduli space of *compact geometric objects of the same dimension*. All of the compactifications mentioned above do not meet the demand, though they are moduli spaces of noncompact objects by [FC90]. On the other hand it is still an open problem whether the Voronoi compactification in [Namikawa76] is a moduli space.

Therefore the first problem in this direction would be the existence of a compactification of A_g or its analogues which meets the above demand. Since the uniqueness of the compactification is not true for toroidal compactifications, there might exist many less important compactifications which meet the above demand. Thus a more important problem would be to single out a natural significant compactification and study its structure in detail.

The major purpose of this article is to report that there exists actually an algebro-geometrically natural compactification $SQ_{g,K}$ defined almost over **Z** (Theorem 10.5). It is immediate from its construction that it is projective. It is very likely that in the very near future it will be extended over **Z** in a way analogous to the Drinfeld compactification of modular curves [KM85]. However we do not know the precise relationship of $SQ_{g,K}$ with the Voronoi compactification yet. Compare also [Alexeev99].

Unfortunately there is no *a priori* notion of a "natural compactification". Here we call our compactification $SQ_{g,K}$ algebro-geometrically *natural* mainly because it arises from the GIT-stability of Mumford and GIT-stability is natural enough as is widely accepted. But we hope that in the future $SQ_{g,K}$ will be proved to be natural also from the viewpoint of automorphic forms.

3. The space of closed orbits

3.1. **Example.** Now let us consider what the principle for singling out a nice compactification ought to be. One of the principles is suggested by GIT of Mumford, the geometric invariant theory [MFK94]. Let us look at the following example. Let $\mathbf{C}^2$ be the complex plane, (x, y) its coordinates. Let us consider the action of $\mathbf{C}^*$ on $\mathbf{C}^2$:

$$(\alpha, x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbf{C}^*) \tag{7}$$

What is (or ought to be) the quotient space of $\mathbf{C}^2$ by the action of $\mathbf{C}^*$? Let us decompose $\mathbf{C}^2$ into orbits first. Since the function xy is constant on any orbit, we see that there are four kinds of orbits:

$$\begin{gathered} O(a,1) = \{(x,y) \in \mathbf{C}^2; xy = a\} \ \ (a \neq 0), \\ O(0,1) = \{(0,y) \in \mathbf{C}^2; y \neq 0\}, \\ O(1,0) = \{(x,0) \in \mathbf{C}^2; x \neq 0\}, \\ O(0,0) = \{(0,0)\} \end{gathered} \tag{8}$$

where there are the closure relations of orbits

$$\overline{O(1,0)} \supset O(0,0), \ \overline{O(0,1)} \supset O(0,0).$$

It is tempting to define the quotient to be the orbit space, namely the set of all orbits, but its natural topology is not Hausdorff. In fact, if it is Hausdorff, then we see

$$O(1,0) = \lim_{x\to 0} O(1,x) = \lim_{x\to 0} O(x,1) = O(0,1) \tag{9}$$

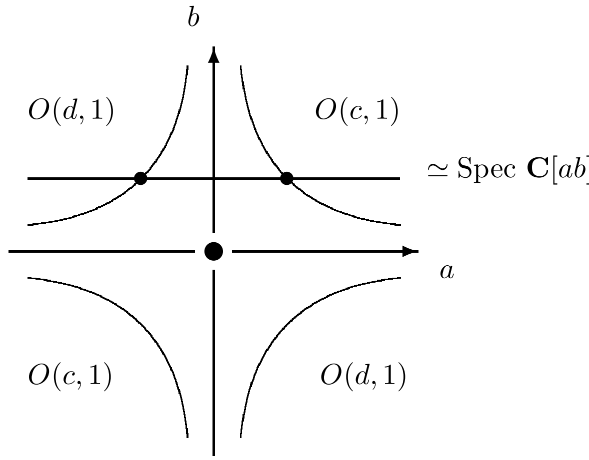
because $O(a,1) = O(1,a)$ $(a \neq 0)$. Hence the natural topology of the orbit space is not Hausdorff. In order to avoid this, we need to turn to the ring of invariants. By (7) the ring of invariants by the $\mathbf{C}^*$-action is a polynomial ring generated by xy. Thus we define the desirable quotient space to be

$$\mathbf{C}^2//\mathbf{C}^* = \{t; t \in \mathbf{C}\} \simeq \operatorname{Spec} \mathbf{C}[t]. \tag{10}$$

where $t = xy$. Since $t = xy$ is a function on $\mathbf{C}^2$, there is a natural morphism from $\mathbf{C}^2$ onto $\mathbf{C}^2//\mathbf{C}^*$. The three orbits $O(1,0)$, $O(0,1)$ and $O(0,0)$ are projected to the origin $t = 0$ of Spec $\mathbf{C}[t]$. Among the three orbits, $O(0,0)$ is the unique closed orbit, while $O(0,1)$ and $O(1,0)$ are not closed. Hence one could think that the origin $t = 0$ is represented by the unique closed orbit $O(0,0)$. This is a very common phenomenon for orbits spaces. Now we make an important remark to summarize the above:

Theorem 3.2. *The quotient space $\mathbf{C}^2//\mathbf{C}^*$ is set-theoretically the space of closed orbits.*

To be more precise, the theorem asserts the following: For any $a \in \mathbf{C}$ in the right hand side of (10), there is a unique closed orbit $O(a,1)$ or $O(0,0)$ respectively if $a \neq 0$ or $a = 0$.



The same is true in general. There is a notion of a semistable point, which we will define soon after stating the following theorems.

Theorem 3.3. (Seshadri-Mumford) *Let X be a projective scheme over a closed field k, G a reductive algebraic k-group acting on X. Then there exists an open subscheme X_{ss} of X consisting of all semistable points in X, and a quotient of X_{ss} by G in a certain reasonable sense.*

To be more precise, there exist a projective k-scheme Y and a G-invariant morphism π from X_{ss} onto Y such that

(1) *For any k-scheme Z on which G acts, and for any G-equivariant morphism $\phi : Z \to X$ there exists a unique morphism $\bar{\phi} : Z \to Y$ such that $\bar{\phi} = \pi\phi$,*
(2) *For given points a and b of X_{ss}*
$$\pi(a) = \pi(b) \text{ if and only if } \overline{O(a)} \cap \overline{O(b)} \neq \emptyset$$
where the closure is taken in X_{ss},
(3) *$Y(k)$ is regarded as the set of G-orbits closed in X_{ss}.*

We denote the (categorical) quotient Y by $X_{ss}//G$.

A reductive group in Theorem 3.3 is by definition an algebraic group whose maximal solvable normal subgroup is an algebraic torus; for example $\mathrm{SL}(n)$ and $\mathbf{G}_m$ are reductive.

We restate Theorem 3.3 in a much simpler form, though the statement of it is not precise.

Theorem 3.4. *Let X be a projective variety, G a reductive group acting on X. Then $X_{ss}//G$ is projective and it is identified with the set of G-orbits closed in X_{ss}.*

Let R be the graded ring of all G-invariant homogeneous polynomials on X. Then the (categorical) quotient Y of X_{ss} by G is defined to be

$$Y = \mathrm{Proj}\,(R).$$

The most important point to emphasize is the fact that

$$Y = \text{the space of orbits closed in } X_{ss}.$$

Now we give the definition of the term "semistable" in Theorem 3.3.

Definition 3.5. We keep the same notation as in Theorem 3.3. Let $p \in X$.

(1) the point p is said to be *semistable* if there exists a G-invariant homogeneous polynomial F on X such that $F(p) \neq 0$,
(2) the point p is said to be *Kempf-stable* if the orbit $O(p)$ is closed in X_{ss},
(3) the point p is said to be *properly-stable* if p is Kempf-stable and the stabilizer subgroup of p in G is finite.

We denote by X_{ps} or X_{ss} the set of all properly-stable points or the set of all semistable points respectively. Very often in the recent literatures "properly-stable" is only referred to as "stable", however in the present article we will use "properly-stable" for it in order to strictly distinguish it from "Kempf-stable".

Thus Theorem 3.3 tells us what the quotient space Y ought to be. What is the subset of X lying over Y? It is X_{ss}, namely the subset of X consisting of all points where at least a G-invariant homogeneous polynomial does not vanish. Any homogeneous polynomial now is not a function on X, instead the quotient of a pair of G-invariant homogeneous polynomials of equal degree is a function on X. Therefore X_{ss} is the subset of X where G-invariant functions are defined possibly by choosing a suitable denominator. Therefore

$$\begin{aligned} X \setminus X_{ss} &= \text{the common zero locus of all } G\text{-invariant} \\ &\qquad \text{homogeneous polynomials on } X \\ &= \text{the subset of } X \text{ where no } G\text{-invariant} \\ &\qquad \text{functions are defined } (0/0\ !). \end{aligned}$$

However since it is a very difficult task to determine the ring of all G-invariant homogeneous polynomials on X, so is it to determine X_{ss}. The geometric invariant theory is the theory in which Mumford intended to determine semistability or the subset X_{ss} without knowing explicitly the ring of all G-invariant homogeneous polynomials, instead by studying the geometric structure of X and the G-action on it. In lucky situations this is really the case, for instance, semistable vector bundles on a variety (Takemoto, Maruyama and Mumford), stable curves (Gieseker and Mumford) and in addition abelian varieties (Kempf) and their natural limits PSQASes as we will see in Theorem 10.3.

3.6. **Comparison of notions.** In order to apply Theorem 3.3 to moduli problems we compare various notions in GIT and moduli theories as follows:

$$\begin{aligned}
X &= \text{the set of geometric objects},\\
G &= \text{the group of isomorphisms of objects in } X,\\
//G &= \text{mod } G \text{ plus some extra relation in Theorem } 3.3\\
&\quad\ \text{called the orbit closure relation},\\
X_{ps} &= \text{the set of properly-stable geometric objects}\\
&= \text{the set of generic geometric objects},\\
X_{ss} &= \text{the set of semistable geometric objects}\\
&= X_{ps} \cup \text{moderately degenerating limits of objects in } X_{ps},\\
X_{ps}//G &= \text{the moduli space of generic geometric objects},\\
X_{ss}//G &= \text{compactification of the moduli space}.
\end{aligned}$$

We note that if $a, b \in X_{ps}$,

$$\begin{aligned}
\pi(a) = \pi(b) &\Longleftrightarrow \overline{O(a)} \cap \overline{O(b)} \neq \emptyset\\
&\Longleftrightarrow O(a) \cap O(b) \neq \emptyset\\
&\Longleftrightarrow O(a) = O(b)\\
&\Longleftrightarrow \ a \text{ and } b \text{ are isomorphic}.
\end{aligned}$$

Each point of X_{ps} gives a closed orbit and the moduli space $X_{ps}//G$ is an ordinary orbit space X_{ps}/G. Moreover it is compactified by $X_{ss}//G$. This is currently one of the most powerful principles for compactifying moduli spaces.

The first approximation to our moduli is $Y^0 := X_{ps}/G$. Therefore the first candidate for a compactification of our moduli could be $X_{ss}//G$. However in many cases X_{ss} is too big to determine explicitly. There are too many orbits in $X_{ss}//G$ which are unnecessary in understanding the space $X_{ss}//G$ itself. In this sense it is more practical to restrict our attention to Kempf-stable points, though the set of Kempf-stable points in X_{ss} is not even an algebraic subscheme of X_{ss} in general.

4. GIT-stability and stable critical points

4.1. **What is implied by GIT-stability?** The Morse theory is well known as a method of studying the topology of a differentiable manifold by a Morse function. In studying the topology it is important to know the critical exponents of the Morse function at critical points. When the critical exponent at a critical point is maximal, namely the Hessian of the Morse function is positive definite, the critical point is called a stable critical point. This is the case where the Morse function takes a local minimum at the point. If one considers the function as a sort of energy function in physics, the critical point corresponds to a stable point (or a stable physical state) where the energy attains its local minimum. To our knowledge, the term *stable* is used in this sense in most cases. However it may seem that GIT-stability has nothing to do with it, at least from the definition. Nevertheless as the following theorem of Kempf and Ness shows, GIT-stability does have to do with a stable critical point.

Let V a finite-dimensional complex vector space, G a reductive algebraic group acting on V. Let K be a maximal compact subgroup of G and $\| \ \|$ be a K-invariant Hermitian norm on V. For instance if $G = \mathrm{SL}(2, \mathbf{C})$, then $K = \mathrm{SU}(2)$. If one takes the example in 3.1, then $V = \mathbf{C}^2$, $G = \mathbf{C}^*$, $K = S^1 = \{w \in \mathbf{C}^*; \|w\| = 1\}$ and the K-invariant Hermitian norm on V is given by

$$\|(a, b)\| = |a|^2 + |b|^2,$$
$$\|g_\lambda \cdot (a, b)\| = |\lambda a|^2 + |\lambda^{-1} b|^2$$

where $g_\lambda = \mathrm{diag}(\lambda, \lambda^{-1})$.

Definition 4.2. Let $v \in V$, $v \neq 0$.

(1) the vector v is said to be *semistable* if there exists a G-invariant homegeneous polynomial F on V such that $F(v) \neq 0$,
(2) the vector v is said to be *Kempf-stable* if the orbit $O(v)$ is closed in V,
(3) the vector v is said to be *properly-stable* if p is Kempf-stable and the stabilizer subgroup of v in G is finite.

If v is Kempf-stable, then v is semistable. Let $\pi : V \setminus \{0\} \to \mathbf{P}(V)$ be the natural surjection. Then v is semistable (resp. Kempf-stable, properly-stable) if and only if $\pi(v)$ is semistable (resp. Kempf-stable, properly-stable).

For $v \neq 0$, we define $p_v(g) := \|g \cdot v\|$ on G. Then p_v is a function on an orbit $O(v)$, which is invariant by the action of K from the left. Then the following theorem is known.

Theorem 4.3. (Kempf-Ness)

(1) *p_v obtains its minimum on $O(v)$ at any critical point of p_v.*
(2) *The second order derivation of p_v at the minimum is "positive".*
(3) *v is Kempf-stable if and only if p_v obtains a minimum on $O(v)$.*

Thus summarizing the above, we see

$$v \text{ is Kempf-stable} \iff p_v \text{ has a stable critical point on } O(v).$$

In this sense, we can justify the term "stable" or "stability" in GIT.

In this connection we would like to add a few words about the history of coining the term "stability". The first edition of GIT was published in 1965, and then was followed by Deligne-Mumford's paper on stable curves in 1969 and the paper of Kempf and Ness in 1979. In the first edition of GIT the following theorem has been proved:

Theorem. Any nonsingular hypersurface of $\mathbf{P}^n$ is properly-stable.

This suggests that Mumford had an idea of compactifying the moduli space of nonsingular curves by stability at latest in 1965, though the notion of stable curves has maybe not been established yet because the article of Deligne and Mumford appeared in 1969. The term stability might come from the stable reduction theorem of abelian varieties (due to Grothendieck) and the stable curves that were probably being born at that time. On the other hand the theorem of Kempf and Ness is not so difficult to prove, though its discovery would not be so easy. Taking all of these into consideration, we suspect that Mumford was unaware of the connection of GIT-stability with Morse-stability like Theorem 4.3 when he first used the term stability in GIT. It might have been a mere accident. But it was a very excellent coining that described its essence very well as the subsequent history shows.

5. Stable curves of Deligne and Mumford

5.1. **The moduli space M_g of stable curves.** In the paper [DM69] Deligne and Mumford compactified the moduli space of nonsingular curves by adding stable curves — a class of curves with mild singularities. Roughly speaking we have

$$\begin{aligned} &\text{the moduli of smooth curves} \\ &\quad = \text{the set of all isomorphism classes of smooth curves} \\ &\quad \subset \text{the set of all isomorphism classes of stable curves} \\ &\quad = \text{the Deligne-Mumford compactification.} \end{aligned}$$

The Deligne-Mumford compactification M_g frequently appears in diverse branches such as the quantum gravity in physics and Gromov-Witten invariants in connection with mirror symmetry. Also widely known are Kontsevich's solution of Witten conjecture, and his cellular decomposition of M_g by ribbon graphs.

Let us recall the definition of stable curves:

Definition 5.2. Let C be a possibly reducible, connected projective curve of genus at least two. A curve C is called *moduli-stable* if the following conditions are satisfied:

(1) it is locally a curve on a nonsingular algebraic surface (or a two-dimensional complex manifold) defined by an equation $x = 0$ or $xy = 0$ in terms of local coordinates x, y,
(2) if a nonsingular rational curve C' is an irreducible component of C, then C' intersects the other irreducible components of C at least at three points.

It is easy to see that the automorphism group of any moduli-stable curve is finite. Since diverse stabilities appear in the context, we call a Deligne-Mumford stable curve a moduli-stable curve in what follows to distinguish the terminology strictly.

5.3. **Another stability.** Any moduli-stable curve is stable in the following sense [DM69]:

(i) Any given one parameter family of curves, after a suitable process of surgeries, namely after pulling back, taking normalizations and by contracting excessive irreducible rational components, can be modified into a one parameter family of moduli-stable curves,
(ii) if any fibre of the above family is moduli-stable, then fibres of the family are unchanged by the above surgeries.

What is the relationship between moduli-stability and GIT-stability?

Theorem 5.4. *For a connected curve C of genus greater than one, the following are equivalent:*

1. *C is moduli-stable,*
2. *Any Hilbert point of C of large degree is Kempf-stable,*
3. *Any Chow point of C of large degree is Kempf-stable.*

We note that a Hilbert point (or a Chow point) of a curve C is Kempf-stable if and only if it is properly-stable because the automorphism group of C of genus greater than one is finite. The equivalence of (1) and (2) is due to [Gieseker82], while the equivalence of (1) and (3) is due to [Mumford77]. A Hilbert point and a Chow point are the points which completely describe the embedding of C into the projective space, each being a point of a projective space of very big dimension. These are a kind of Plücker coordinates of the Grassman variety in the sense that the Plücker coordinates describe completely a subspace embedded in a fixed vector space.

5.5. **What is our problem?** A simple and clear theorem like Theorem 5.4 which intrinsically characterizes moduli-stable curves is our goal in our compactification problem of moduli spaces. In general the phenomenon of degeneration of algebraic varieties is so complicated that it may be far beyond classifying completely. Compactifying a moduli space may be achieved by minimizing the scale of degeneration of the algebraic varieties. Therefore one needs to collect only significant degeneration, and one needs to ignore less important ones. From this standpoint, it would be our central problem to understand the meaning of the most significant degeneration or the fundamental principle behind it.

So far in this section we reviewed the known results about moduli-stable curves. From the next section on we consider the same problems about abelian varieties. One of our goals is to complete the following diagram:

$$\begin{aligned}
\text{the}\,&\text{moduli of smooth AVs (= abelian varieties)}\\
&= \{\text{smooth polarized AVs} + \text{extra structure}\}/\ \text{isom.}\\
&\subset \{\text{smooth polarized AVs or}\\
&\qquad \text{singular polarized degenerate AVs} + \text{extra structure}\}/\ \text{isom.}\\
&= \ \text{the new compactification of the moduli of AVs}
\end{aligned}$$

It is also another important goal to characterize those degenerate varieties (schemes) which appear as natural limits of abelian varieties (Theorem 10.3).

6. Hesse cubic curves revisited

6.1. **Theta functions.** Let us start with an elliptic curve $E(\tau)$ over $\mathbf{C}$ having periods 1 and τ where τ is a point of the upper half plane $\mathbf{H}$, namely τ is a complex number with $\mathrm{Im}(\tau)$ positive. By definition two points z_1 and z_2 of $\mathbf{C}$ are identified in the elliptic curve $E(\tau)$ if and only if $z_1 - z_2$ is in the lattice $\mathbf{Z} + \mathbf{Z}\tau$.

For $k = 0, 1, 2$ we define theta functions by

$$\begin{aligned}\theta_k(q, w) &= \sum_{m\in\mathbf{Z}} e^{2\pi i(3m+k)^2\tau/6} e^{2\pi i(3m+k)z} \\ &= \sum_{m\in\mathbf{Z}} a(3m+k) w^{3m+k}\end{aligned}$$

where $a(x) = q^{x^2}$ $(x \in X)$, $q = e^{2\pi i\tau/6}$, $w = e^{2\pi iz}$, $X = \mathbf{Z}$ and $Y = 3\mathbf{Z}$.

It is easy to see

$$\begin{aligned}\theta_k(\tau, z+1) &= \theta_k(\tau, z), \\ \theta_k(\tau, z+\tau) &= q^{-9} w^{-3} \theta_k(\tau, z).\end{aligned} \tag{11}$$

Now we define a mapping $\Theta : E(\tau) \to \mathbf{P}^2_{\mathbf{C}}$ by

$$\Theta \ : \ z \ \mapsto \ [\theta_0(\tau, z), \theta_1(\tau, z), \theta_2(\tau, z)]. \tag{12}$$

Then Θ is well-defined and embeds $E(\tau)$ into $\mathbf{P}^2_{\mathbf{C}}$. It follows from Riemann-Roch theorem that the image of $E(\tau)$ is a cubic curve in $\mathbf{P}^2_{\mathbf{C}}$. Moreover the following transformation formulae are true:

$$\begin{aligned}\theta_k(\tau, z+\frac{1}{3}) &= \zeta_3^k \theta_k(\tau, z), \\ \theta_k(\tau, z+\frac{\tau}{3}) &= q^{-1} w^{-1} \theta_{k+1}(\tau, z).\end{aligned} \tag{13}$$

Therefore the transformation $z \mapsto z + (\tau/3)$ of $E(\tau)$ induces a linear transformation τ of $\mathbf{P}^2_{\mathbf{C}}$ defined in (4)

$$\begin{aligned}[\theta_0(\tau, z), \theta_1(\tau, z), \theta_2(\tau, z)] \ &\mapsto \ [\theta_0(\tau, z+\frac{\tau}{3}), \theta_0(\tau, z+\frac{\tau}{3}), \theta_0(\tau, z+\frac{\tau}{3})] \\ &= [\theta_1(\tau, z), \theta_2(\tau, z), \theta_0(\tau, z)].\end{aligned} \tag{14}$$

Thus we see that the image $\Theta(E(\tau))$ is invariant under $G(3)$. Therefore the cubic curve $\Theta(E(\tau))$ is the zero locus of a $G(3)$-invariant or a relative $G(3)$-invariant cubic polynomial.

Now we recall that $G(3)$ has nine inequivalent one-dimensional representations 1_j $(0 \leq j \leq 8)$, among which 1_0 is the trivial representation. The group $G(3)$ acts on the space $W = H^0(\mathbf{P}^2_{\mathbf{C}}, O_{\mathbf{P}^2_{\mathbf{C}}}(3))$ of cubic polynomials via §1 (4). The space W is 10-dimensional, which is split as a $G(3)$-module as follows:

$$W = 1_0^{\oplus 2} \oplus 1_1 \oplus 1_2 \oplus \cdots \oplus 1_8.$$

For instance, $1_0^{\oplus 2}$ is generated by $G(3)$-invariant polynomials $x_0^3 + x_1^3 + x_2^3$ and $x_0x_1x_2$, while $x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3$ or $x_0^2x_1 + x_1^2x_2 + x_2^2x_0$ is a relative $G(3)$-invariant polynomial generating one of 1_j.

On the other hand we have a holomorphic family of elliptic curves $E(\tau)$ over $\mathbf{H}$, which are embedded into $\mathbf{P}^2_{\mathbf{C}}$ simultaneously. It follows that the family of cubic curves $\Theta(E(\tau))$ is defined by a holomorphic family of cubic polynomials. If $\Theta(E(\tau))$ for τ generic is the zero locus of a relative invariant polynomial in 1_j for some $j \geq 1$, then the entire family is the zero locus of the unique cubic polynomial because W has a unique copy of 1_j. This contradicts that the family is nontrivial, or algebro-geometrically nonconstant. It follows that $\Theta(E(\tau))$ is the zero locus of a $G(3)$-invariant polynomial.

Hence $\Theta(E(\tau))$ is defined by

$$C(\mu(\tau)) : x_0^3 + x_1^3 + x_2^3 - 3\mu(\tau)x_0x_1x_2 = 0 \tag{15}$$

for some $\mu(\tau)$ determined uniquely by τ. Thus we see that x_j is an algebraic counterpart of θ_j. This is the reason why we called x_j a natural basis of theta functions in §1.

6.2. **3-Torsion points.** Any Hesse cubic $C(\mu)$ has 9 inflection points independent of μ which are 3-torsions of the elliptic curve. To be explicit, they are $[1 : -\beta : 0], [0 : 1 : -\beta], [-\beta : 0 : 1]$ where $\beta^3 = 1$. They form an abelian group isomorphic to $\mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$. We choose the origin $e_0 = [1 : -1 : 0]$, and a basis of the group of 3-torsions

$$e_1 = [1 : -\zeta_3 : 0],\ e_2 = [-1 : 0 : 1].$$

The linear transformations σ and τ are the translations of $C(\mu)$ by e_1 and e_2 respectively. The group of 3-torsions carries a nondegenerate alternating bimultiplicative form $e_{C(\mu)}$ with values in the group of cube roots of unity, called the Weil pairing. For instance

$$e_{C(\mu)}(e_1, e_2) \overset{\text{def}}{=} \sigma\tau\sigma^{-1}\tau^{-1} = \zeta_3.$$

In fact, one checks $\sigma\tau(e_i) = \zeta_3\tau\sigma(e_i)$ for any i. Any elliptic curve E also have the Weil pairing e_E on the group of its 3-torsions, which is also nondegenerate alternating bimultiplicative. The elliptic curve $E(\tau)$ has the origin $z = 0$, and $z = 1/3$ and $z = \tau/3$ as a basis of the group of its 3-torsions, in which case we see easily $e_{E(\tau)}(1/3, \tau/3) = \zeta_3$.

Definition 6.3. The quadruple $(C(\mu), e_0, e_1, e_2)$ is called a level 3-structure of $C(\mu)$. Similarly we call the quadruple (E, e_0, e_1, e_2) a level 3-structure if e_0 is the origin of an elliptic curve E, and if e_1 and e_2 is a basis of the group of 3-torsions of E with $e_E(e_1, e_2) = \zeta_3$.

6.4. **Embedding and Proj.** Since the curve $E(\tau)$ is embedded into $\mathbf{P}^2_{\mathbf{C}}$ by Θ, we see

$$\begin{aligned} E(\tau) &\simeq \mathbf{C}^*/\{w \mapsto q^{2y}w; y \in 3\mathbf{Z}\} \\ &\simeq (\text{Proj } \mathbf{C}[a(x)w^x\vartheta, x \in X])/Y \\ &\simeq^* (\text{Proj } \mathbf{C}[a(x)w^x\vartheta, x \in X])^{Y-\text{inv}} \\ &\simeq \text{Proj } \mathbf{C}[\theta_k\vartheta, k = 0, 1, 2] \end{aligned} \tag{16}$$

where $a(x) = q^{x^2}$ for $x \in X$, $X = \mathbf{Z}$, $Y = 3\mathbf{Z}$ and ϑ is an indeterminate of degree one. The action of Y is defined via the ring homomorphism

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta. \tag{17}$$

The third rhs (= right hand side) of (16) is inserted in order to emphasize its formal similarity to the second and the fourth rhs, though the third rhs is mathematically incorrect because Y-invariant elements are infinite sums. In order to be mathematically correct, we need only to remove the third rhs.

Although (16) asserts only $E(\tau) \simeq \Theta(E(\tau))$, the expression (16) of $E(\tau)$ is quite suggestive for the compactification problem.

6.5. **One parameter family and Proj.** Let R be a complete discrete valuation ring, q a parameter of R, for instance $R = \mathbf{Q}_p$ or $k[[q]]$. Let $k(0)$ be the residue field of R and $k(\eta)$ the fraction field of R. If $R = \mathbf{Q}_p$ or $k[[q]]$, then $k(0) = \mathbf{F}_p$ or k respectively. There is no substantial difference even if we let $R = \mathbf{C}[[q]]$ in what follows. Since this subsection is a little too technical, the readers can skip it.

Now we define an analogue of (16) as follows: First let $X = \mathbf{Z}$, $Y = 3\mathbf{Z} \subset X$ and $a(x) = q^{x^2}$ $(x \in X)$. Then we define $\widetilde{R}$ by

$$\widetilde{R} = R[a(x)w^x\vartheta, x \in X]$$

and an action of Y on $\widetilde{R}$ by (17). Since $\widetilde{R}$ is not finitely generated over R, $\mathrm{Proj}\,(\widetilde{R})$ is not of finite type over R, but it is locally of finite type over R. In other words, there is a covering by infinitely many affine open subsets. Thus $\mathrm{Proj}\,(\widetilde{R})$ is a scheme locally of finite type over R.

Let $\mathcal{X}$ be the formal completion of $\mathrm{Proj}\,(\widetilde{R})$ along the fiber over the unique closed point 0 of Spec R. This is as a topological space a chain of infinitely many rational curves. Hence we have a topological quotient of it by Y, hence we have a quotient $\mathcal{X}/Y$ of $\mathcal{X}$ by Y as a formal scheme, namely the topological quotient of the chain of rational curves by Y equipped with formal ring structure. Since the action of Y on $\mathcal{X}$ keeps the invertible sheaf $O_{\mathcal{X}}(1)$ invariant, the sheaf $O_{\mathcal{X}}(1)$ descends to an ample sheaf $O_{\mathcal{X}}(1)/Y$ on the formal quotient $\mathcal{X}/Y$. Then a theorem of Grothendieck (Existence theorem [EGA, III, 5.4.5]) guarantees the existence of a projective scheme $\mathcal{Z}$ over R with a natural ample sheaf $O_{\mathcal{Z}}(1)$ such that the formal completion of $(\mathcal{Z}, O_{\mathcal{Z}}(1))$ along the closed fibre is isomorphic to the formal scheme $(\mathcal{X}/Y, O_{\mathcal{X}}(1)/Y)$. For short we call $\mathcal{Z}$ the algebrization of $\mathcal{X}/Y$ or a simplest Mumford family (associated with the degeneration data $a(x)$).

Thus we have the algebrization $\mathcal{Z}$ of $\mathcal{X}/Y$:

$$\begin{aligned} \mathcal{Z} &= (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X]/Y)^{\mathrm{alg}}, \\ \mathcal{Z}_0 &= (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X] \otimes (R/qR))/Y. \end{aligned} \tag{18}$$

where $\mathcal{Z}_0$ is the closed fibre of $\mathcal{Z}$. Note that $\mathcal{Z}_0 = \mathcal{X}_0/Y$.

6.6. **The special fibre $\mathcal{Z}_0$.** The explanation in the last subsection might be too technical. If one wishes to understand $\mathcal{Z}_0$ only one can proceed as follows. Since $\mathcal{Z}_0 = \mathcal{X}_0/Y$, we need to know $\mathcal{X}_0$ only. The scheme $\mathcal{X}$ locally of finite type is covered with affine opens V_n $(n \in \mathbf{Z})$

$$\begin{aligned} V_n &= \mathrm{Spec}\ R[a(x)w^x\vartheta/a(n)w^n\vartheta, x \in X] \\ &= \mathrm{Spec}\ R[a(n+1)w^{n+1}/a(n)w^n, a(n-1)w^{n-1}/a(n)w^n] \\ &= \mathrm{Spec}\ R[q^{2n+1}w, q^{-2n+1}w^{-1}] \\ &\cong \mathrm{Spec}\ R[x_n, y_n]/(x_ny_n - q^2) \end{aligned}$$

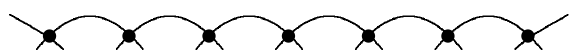
where $x_n = q^{2n+1}w$ and $y_n = q^{-2n+1}w^{-1}$. Hence

$$x_n = x_{n-1}^2 y_{n-1},\ y_n = x_{n-1}^{-1}.$$

Moreover we see

$$\begin{aligned}V_n \cap \{q=0\} &\cong \mathrm{Spec}\ k(0)[x_n, y_n]/(x_n y_n)\\ &= \{(x_n, y_n) \in k(0)^2; x_n y_n = 0\}.\end{aligned}$$

Thus $\mathcal{X}_0$ is a chain of infinitely many $\mathbf{P}^1_{k(0)}$. The action of Y on $\mathcal{X}_0$ sends $V_n \overset{S_{-3}}{\to} V_{n+3} \overset{S_{-3}}{\to} V_{n+6} \to \cdots$, $(x_n, y_n) \overset{S_{-3}}{\mapsto} (x_{n+3}, y_{n+3}) = (x_n, y_n)$ so that we have a cycle of 3 rational curves as the quotient $\mathcal{X}_0/Y$.



The generic fibre $\mathcal{X}_\eta$ is an elliptic curve over $k(\eta)$, isomorphic to a nonsingular Hesse cubic curve over $k(\eta)$, as is shown by the representation theory of $G(3)$. In the complex case $k(0) = \mathbf{C}$, $\mathcal{X}$ near $q = 0$ is isomorphic to a family of Hesse cubic curves with $C(\infty)$ central fibre. In the notation of [Kodaira63, p. 565] it is a singular fibre of type ${}_1\mathrm{I}_3$. Therefore $\mathcal{X}$ is isomorphic (after its completion along the closed fibre) to the family constructed by Kodaira with the parameter of the disc chosen suitably. The expression (18) is however a modified form of [Mumford72]. This way of understanding or defining the limit $C(\infty)$ as $\mathcal{Z}_0$ is very useful in the sense that one can easily generalize it in the higher-dimensional cases. In fact one can define new limits in arbitrary dimension by choosing $a(x)$, X and Y suitably. Any Kempf-stable degenerate abelian variety PSQAS in Theorem 10.3 is obtained in this way.

7. Moduli theory of cubic curves

7.1. **Stability of cubic curves.** Now let us recall the compact moduli theories of cubic curves. Let k be an algebraically closed field of characteristic $\neq 3$. There are two compact moduli theories of cubic curves from the viewpoint of GIT. First we recall that $\mathrm{SL}(3,k)$ acts on $H^0(\mathbf{P}^2, O(1))$, hence on $V := H^0(\mathbf{P}^2, O(3))$=the space of ternary cubic forms.

Definition 7.2. Le $f \in V$. Then

(1) f is *semistable* if there is an $\mathrm{SL}(3,k)$ invariant homogeneous polynomial H on V such that $H(f) \neq 0$.
(2) f is *Kempf-stable* if the $\mathrm{SL}(3,k)$-orbit of f in V is closed.

We recall that Kempf-stable implies semistable, whereas properly-stable is Kempf-stable with finite stabilizer. The stabilizer group of a Kempf-stable point can be infinite. Let us say that a cubic curve is Kempf-stable/semistable if the equation defining the curve is Kempf-stable/semistable. See Table 1.

Table 1. Stability of cubic curves

curves (sing.)	stability	stab. gr.
smooth elliptic	Kempf-stable	finite
3 lines, no triple point	Kempf-stable	2 dim
a line+a conic, not tangent	semistable, not Kempf-stable	1 dim
irreducible, a node	semistable, not Kempf-stable	$\mathbf{Z}/2\mathbf{Z}$
3 lines, a triple point	not semistable	1 dim
a line+a conic, tangent	not semistable	1 dim
irreducible, a cusp	not semistable	1 dim

7.3. $(\mathbf{G}_m)^2$**-stability.** The following example illustrates Kempf-stability and semistability of cubic curves well enough. Let

$$C_{a,b,c} \ : \ ax_0^3 + bx_1^3 + cx_2^3 - x_0x_1x_2 = 0.$$

The diagonal subgroup $G \simeq (\mathbf{G}_m)^2$ of SL(3) on the parameter space Spec $k[a,b,c]$ acts by

$$(a,b,c) \mapsto (sa, tb, uc)$$

where $stu = 1$, and $s,t,u \in \mathbf{G}_m$. This is very similar to the example in §3. We easily see

(i) $(\mathbf{G}_m)^2$-Kempf-stable points are $abc \neq 0$ or $(a,b,c) = (0,0,0)$,
(ii) $(\mathbf{G}_m)^2$-semistable points which are not $(\mathbf{G}_m)^2$-Kempf-stable are $abc = 0$ except $(0,0,0)$.

We see that $C_{a,b,c}$ is a smooth elliptic curve if $abc \neq 0$, and $C_{a,b,c}$ is a 3-gon if $(a,b,c) = (0,0,0)$. We also see that the curves $C_{a,b,c}$ with $abc = 0$, $(a,b,c) \neq (0,0,0)$ are either an irreducible curve with a node or a union of a line and a conic.

We note also that for any of the above curves,

$$\begin{aligned} (\mathbf{G}_m)^2\text{-Kempf-stable} &\iff \mathrm{SL}(3)\text{-Kempf-stable},\\ (\mathbf{G}_m)^2\text{-semistable} &\iff \mathrm{SL}(3)\text{-semistable}. \end{aligned}$$

Thus we see that for a cubic curve C,

$$\begin{aligned} C \ \text{Kempf-stable} &\iff C \ \text{smooth or a 3-gon}\\ &\iff C \ \text{isomorphic to a Hesse cubic}\\ &\iff C \ \text{invariant under (a GL(3)-conjugate of) } G(3),\\ C \ \text{Kempf-stable} &\ \text{and singular}\\ &\iff C \ \text{a 3-gon}\\ &\iff C \ \text{a special fibre of "the Tate curve"}. \end{aligned}$$

In other words, C is Kempf-stable and singular if and only if

$$C \simeq (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X]/Y) \otimes R/qR, \tag{19}$$

where $X = \mathbf{Z}$, $Y = 3\mathbf{Z}$, $R = k[[q]]$, $a(x) = q^{x^2}$ for $x \in X$.

We wish to emphasize that (19) is very similar to (16).

7.4. **Moduli spaces of cubic curves.** The first compactification of the moduli of nonsingular cubic curves is the quotient

$$SQ_{1,1} := V(\text{semistable})/\!/\,\mathrm{SL}(3) \simeq \mathbf{P}^1.$$

This is the theory of the j-invariant of elliptic curves.

The second compactification is given in Theorem 7.5. A pair of cubic curves with level 3-structure is defined to be isomorphic if there is an linear isomorphism of the cubic curves mapping e_i to e_i. By the theorem of Hesse, the cubic curves $C(\mu)$ and $C(\mu')$ with level 3-structure are isomorphic iff $\mu = \mu'$ and the isomorphism is the identity morphism of $C(\mu)$. Thus we see that

$$\begin{aligned}
&\{\text{smooth cubics } + \text{level 3 structure}\}/\,\text{isom.}\\
&\qquad = \{\text{smooth Hesse cubics } + \text{ level 3 structure}\}/\,\text{isom.}\\
&\qquad = \{\text{smooth Hesse cubics } + \text{ level 3 structure}\}.
\end{aligned}$$

The following is a prototype for all the rest.

Theorem 7.5. *Let $G(3)$ be the Heisenberg group of level 3. Then*

$$\begin{aligned}
SQ_{1,3} : &= \left\{\begin{array}{l}\textit{Kempf-stable cubic curves}\\ \textit{with level 3-structure}\end{array}\right\}/\textit{isom.}\\
&= \left\{\begin{array}{l}\textit{cubic curves invariant under } G(3)\\ \textit{with level 3-structure}\end{array}\right\}\\
&= \left\{\begin{array}{l}\textit{Hesse cubics}\\ \textit{with level 3-structure}\end{array}\right\},
\end{aligned}$$

which compactifies

$$\begin{aligned}
A_{1,3} : &= \left\{\begin{array}{l}\textit{smooth cubic curves}\\ \textit{with level 3-structure}\end{array}\right\}/\textit{isom.}\\
&= \left\{\begin{array}{l}\textit{smooth Hesse cubics}\\ \textit{with level 3-structure}\end{array}\right\}.
\end{aligned}$$

The compactification $SQ_{1,3}$ ($\simeq \mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$) is projective over $\mathbf{Z}[\zeta_3, 1/3]$, and it is the fine moduli scheme for families of Kempf-stable cubic curves with level 3 structure over reduced base schemes.

We note that $A_{1,3}$ is $\mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$ with four points ∞, ζ_3^k deleted.

7.6. **Relationship between the compactifications.** Now we discuss the relationship between two compactifications $SQ_{1,1}$ and $SQ_{1,3}$. The natural map from the second compactification to the first compactification

$$\bar{j} : SQ_{1,3} \to SQ_{1,1},\ \bar{j}(\mu) := j(C(\mu))$$

is given by forgetting level 3-structures of cubic curves. This is written explicitly as

$$\bar{j}(\mu) = 27\frac{\mu^3(\mu^3+8)^3}{(\mu^3-1)^3}.$$

The function $\bar{j}$ is invariant under the transformations of μ:

$$\gamma_1 : \mu \mapsto \frac{\mu+2}{\mu-1}, \quad \gamma_2 : \mu \mapsto \frac{\zeta_3\mu+2}{\mu-\zeta_3^2}$$

which generate the Galois group of the morphism $\bar{j}$. These are induced from the linear transformations of $\mathbf{P}^2_{\mathbf{C}}$

$$g_1 : (x_0, x_1, x_2) \mapsto (y_0, y_1, y_2),$$
$$g_2 : (x_0, x_1, x_2) \mapsto (z_0, z_1, z_2)$$

where

$$y_k = \zeta_3^{2k}x_0 + \zeta_3^{2+k}x_1 + \zeta_3 x_2,$$
$$z_k = x_0 + x_1 + x_2 + (\zeta_3^2 - 1)x_k.$$

These have been observed already in [Hesse, p. 83 (1844)].

Moreover we see

$$\begin{cases} g_1(e_0) = e_0, \\ g_1(e_1) = -e_2, \\ g_1(e_2) = e_1, \end{cases} \qquad \begin{cases} g_2(e_0) = e_0, \\ g_2(e_1) = e_1 + e_2, \\ g_2(e_2) = e_2. \end{cases}$$

We know that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ generate $\mathrm{SL}(2, \mathbf{F}_3)$, a group of order 12, which is isomorphic to the Galois group of the morphism $\bar{j}$.

8. PSQASes

Let R be a complete discrete valuation ring, q a uniformizing parameter of R, $k(\eta)$ the fraction field of R and $X = \mathbf{Z}^g$.

Definition 8.1. A collection $a(x)$ $(x \in X)$ is called a degeneration data (of Faltings-Chai) if the following conditions are satisfied:

1. $a(0) = 1$, $a(x) \in k(\eta)^\times := k(\eta) \setminus \{0\}$ $(\forall x \in X)$,
2. $b(x, y) := a(x+y)a(x)^{-1}a(y)^{-1}$ $(x, y \in X)$ is a bilmultiplicative form on X,
3. $B(x, y) := \mathrm{val}_q(a(x+y)a(x)^{-1}a(y)^{-1})$ $(x, y \in X)$ is a positive definite bilinear form on X.

It is an algebraic analogue of periods of an abelian variety over $k(\eta)$. For a positive bilinear form $P(x, y)$ on X let $a(x) = q^{P(x,x)}$. Then it is a degeneration data of Faltings-Chai. For instance $a(x) = q^{x^2}$, $B(x, y) = 2xy$ $(x, y \in \mathbf{Z})$ in §1.

Definition 8.2. (Totally degenerate case) Let $X = \mathbf{Z}^g$, Y a sublattice of X of finite index, $K := (X/Y) \oplus$ dual of (X/Y) and $a(x)$ $(x \in X)$ a degeneration data of Faltings-Chai. Let $(\mathcal{Z}, \mathcal{L})$ be the simplest Mumford family associated with $a(x)$

$$(\mathcal{Z}, \mathcal{L}) := \{(\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X], O_{\mathrm{Proj}}(1))/Y\}^{alg}. \tag{20}$$

We note that the generic fibre $(\mathcal{Z}, \mathcal{L}) \otimes k(\eta)$ is an abelian variety over $k(\eta)$. Associated with a given polarized abelian variety $(G_\eta, \mathcal{L}_\eta)$ over $k(\eta)$, one can construct a degeneration data of Faltings-Chai or its analogue also in the partially degenerate case (which is very complicated in general). Starting with the degeneration data one can define the simplest Mumford family $(\mathcal{Z}, \mathcal{L})$ such that the generic fibre $(\mathcal{Z}_\eta, \mathcal{L}_\eta)$ is isomorphic to $(G_\eta, \mathcal{L}_\eta)$.

The polarized scheme (Z, L) is said to be a K-PSQAS over $k(0) := R/qR$ if (Z, L) is a special fibre $(\mathcal{Z}, \mathcal{L}) \otimes k(0)$ of the simplest Mumford family $(\mathcal{Z}, \mathcal{L})$. We note that a nonsingular PSQAS is by definition an abelian scheme. A PSQAS is an abbreviation of *a projectively stable quasi-abelian scheme*.

8.3. **Limits of theta functions — an example.** As we saw in §6, the elliptic curves $E(\tau)$ are embedded in $\mathbf{P}^2$ by theta functions θ_k. In this subsection we first calculate the limits of theta functions when $\mathrm{Im}(\tau)$ goes to infinity, and then determine the limit of $\Theta(E(\tau))$. The theta functions on $E(\tau)$ are defined as follows:

$$\theta_k(q, w) = \sum_{m \in \mathbf{Z}} q^{(3m+k)^2} w^{3m+k} \quad (k = 0, 1, 2)$$

where $q = e^{2\pi i \tau/6}$, $w = e^{2\pi i z}$. There is a relation between them:

$$C(\mu(q)) : \theta_0^3 + \theta_1^3 + \theta_2^3 = 3\mu(q)\theta_0\theta_1\theta_2.$$

Our aim here is to prove that $C(\infty)$ is the 3-gon $x_0x_1x_2 = 0$ by calculating the limits of θ_k explicitly when $\mathrm{Im}(\tau)$ goes to infinity, without proving $\lim_{\tau\to\infty} \mu(\tau) = \infty$. See also [Namikawa76].

In this situation, we see that the degeneration data of Faltings-Chai for the family $C(\mu(q))$ over Spec R is given by $a(x) = q^{x^2}$ $(x \in X)$. Thus in particular, we have a natural choice of the lattice $X = \mathbf{Z}$ and its sublattice $Y = 3\mathbf{Z}$. It is obvious that there is an expression of θ_k in terms of $a(x)$:

$$\theta_k(q, w) = \sum_{y \in Y} a(y + k) w^{y+k} \quad (k = 0, 1, 2).$$

As we will see below, our calculation of limits of theta functions explains how a polyhedral decomposition (called the Delaunay decomposition) of $X \otimes_{\mathbf{Z}} \mathbf{R}$ gets involved in describing the limit of $\Theta(E(\tau))$. In general the Delaunay decomposition, which we will define later, is a polyhedral decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R} = \mathbf{R}^g$ describing limits of theta functions.

Now we turn to the calculation of the limit. Let $R = \mathbf{C}[[q]]$, $I = qR$, $w = q^{-1}u$ for $u \in R \setminus I$ and $\overline{u} = u \mod I$. Then the power series θ_k converge I-adically:

$$\begin{aligned}
\theta_0(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{9m^2-3m} u^{3m} \\
&= 1 + q^6u^3 + q^{12}u^{-3} + q^{30}u^6 + \cdots , \\
\theta_1(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2-3m-1} u^{3m+1} \\
&= u + q^6u^{-2} + q^{12}u^4 + \cdots , \\
\theta_2(q, q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2-3m-2} u^{3m+2} \\
&= q^2u^2 + q^2u^{-1} + q^{20}u^5 + q^{20}u^{-4} + \cdots .
\end{aligned}$$

Hence in $\mathbf{P}^2$

$$\lim_{q\to 0}\,[\theta_k(q,q^{-1}u)] = [1,\overline{u},0]$$

Similarly in $\mathbf{P}^2$

$$\lim_{q\to 0}\,[\theta_k(q,q^{-3}u)] = [0,1,\overline{u}],$$
$$\lim_{q\to 0}\,[\theta_k(q,q^{-5}u)] = [\overline{u},0,1].$$

On the other hand let $w = q^{-2\lambda}u$ and $u \in R\setminus I$.

$$\lim_{q\to 0}\,[\theta_k(q,q^{-2\lambda}u)] = \begin{cases} [1,0,0] & (\text{if } -1/2<\lambda<1/2),\\ [0,1,0] & (\text{if } 1/2<\lambda<3/2),\\ [0,0,1] & (\text{if } 3/2<\lambda<5/2).\end{cases} \tag{21}$$

When λ ranges in $\mathbf{R}$, the same calculation shows that the same limits repeat mod $Y = 3\mathbf{Z}$. Thus we see that $\lim_{\tau\to\infty} C(\mu(\tau))$ is the 3-gon $x_0x_1x_2 = 0$.

Definition 8.4. For $\lambda \in X\otimes_{\mathbf{Z}}\mathbf{R}$ fixed, we define a function F_λ on the lattice $X = \mathbf{Z}$ by

$$F_\lambda(a) = a^2 - 2\lambda a.$$

We also define a Delaunay cell $D(\lambda)$ to be the convex closure of all $a \in X$ which attain the minimum of $F_\lambda(a)$.

By computations we see

$$D(j+\frac{1}{2}) = [j,j+1] := \{x\in\mathbf{R};\ j\le x\le j+1\},$$
$$D(\lambda) = \{j\}\quad (\text{if } j-\frac{1}{2}<\lambda<j+\frac{1}{2}),$$
$$[\bar\theta_k]_{k=0,1,2} := \lim_{q\to 0}[\theta_k(q,q^{-2\lambda}u))]_{k=0,1,2}$$
$$\bar\theta_k = \begin{cases}\bar u^j & (\text{if } j\in D(\lambda)\cap(k+3\mathbf{Z}))\\ 0 & (\text{if } D(\lambda)\cap(k+3\mathbf{Z})=\emptyset).\end{cases}$$

For instance $D(\frac{1}{2})\cap(0+3\mathbf{Z}) = \{0\}$, $D(\frac{1}{2})\cap(1+3\mathbf{Z}) = \{1\}$ and

$$\lim_{q\to 0}[\theta_k(q,q^{-1}u))] = [\bar\theta_0,\bar\theta_1,\bar\theta_2] = [\bar u^0,\bar u,0] = [1,\bar u,0].$$

Similarly for any $\lambda = j+(1/2)$, we have an algebraic torus as a limit

$$\{[\bar u^j,\bar u^{j+1}]\in\mathbf{P}^1;\bar u\in\mathbf{G}_m\}\simeq\mathbf{G}_m\ (=\mathbf{C}^*).$$

Next one needs to consider the cases $\frac{1}{2}<\lambda<\frac{3}{2}$, $\lambda=\frac{3}{2}$, $\frac{3}{2}<\lambda<\frac{5}{2}$ in order.

Let σ be a Delaunay cell, and $O(\sigma)$ the stratum of $C(\infty)$ consisting of limits of $(q,q^{-2\lambda}u)$ for $\lambda\in\sigma$. If σ is one-dimensional, then $O(\sigma) = \mathbf{C}^*$, while $O(\sigma)$ is one point if σ is zero-dimensional. Thus we see that $C(\mu(\infty))$ is a disjoint union of $O(\sigma)$, σ being Delaunay cells mod Y, in other words, it is stratified in terms of the Dalaunay decomposition mod Y.

Let $\sigma_j = [j, j+1]$ and $\tau_j = \{j\}$. Then the Dalaunay decomposition in this case and the stratification of $C(\infty)$ are given in Figure 1.

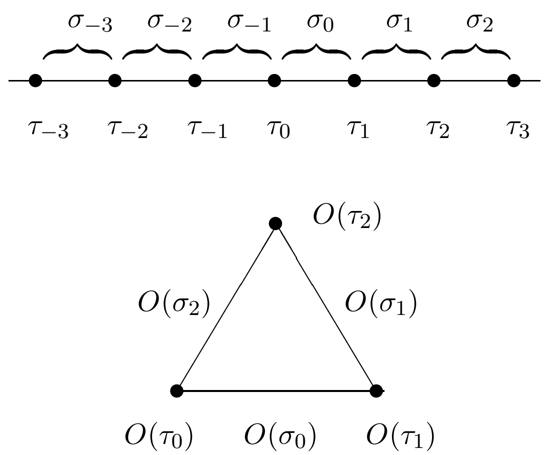


FIGURE 1

8.5. **The general case.** Now we consider the general case. Our PSQAS is defined as a special fibre of the simplest Mumford family $(\mathcal{Z}, \mathcal{L})$ given in §6. Let X be an integral lattice of rank g and B a positive symmetric integral bilinear form on X associated with the degeneration data for $(\mathcal{Z}, \mathcal{L})$.

For $\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}$ fixed, we define:

Definition 8.6. A Delaunay cell σ is a convex hull spanned by the integral vectors (which we call Delaunay vectors) attaining the minimum of the function

$$B(x, x) - 2B(\lambda, x) \quad (x \in X).$$

When λ ranges in $X \otimes_{\mathbf{Z}} \mathbf{R}$, we will have various Delaunay cells. All of them constitute a locally finite polyhedral decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, invariant under the translation by X. We call this the Delaunay decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, which we denote by Del_B.

There are two types of Delaunay decompositions inequivalent under the action of $\mathrm{SL}(2, \mathbf{Z})$. See Figure 2.

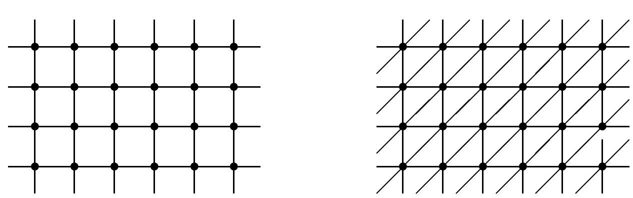


FIGURE 2

Definition 8.7. Let $\sigma \in \mathrm{Del}_B$ be a Delaunay cell. A Voronoi cell $V(\sigma)$ dual to σ is defined to be

$$V(\sigma) = \{\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}; \sigma \text{ is a Delaunay cell for } \lambda\}.$$

The Voronoi decomposition Vor_B is by definition the decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$ consisting of polyhedra $V(\sigma)$ $(\sigma \in \mathrm{Del}_B)$. This covers $X \otimes_{\mathbf{Z}} \mathbf{R}$.

The Delaunay decomposition will describe PSQASes as follows:

Theorem 8.8. *Let Z be a PSQAS, X the integral lattice, Y the sublattice of X of finite index and B the positive integral bilinear form on X all of which were defined in §7. Let σ,τ be Delaunay cells in* Del_B.

1. *For each σ there exists a subscheme $O(\sigma)$ of Z, invariant under the torus-action, which is a torus of dimension $\dim \sigma$ over k such that Z is the disjoint union of $O(\sigma)$. Let $Z(\sigma)$ be the closure of $O(\sigma)$.*
2. *$\sigma \subset \tau$ iff $O(\sigma) \subset O(\tau)$.*
3. *$\sigma \subset \tau$ iff $Z(\sigma) \subset Z(\tau)$.*
4. *$Z = \bigcup_{\sigma \in \mathrm{Del} \bmod Y} Z(\sigma)$.*
5. *The local scheme structure of Z is completely described by the bilinear form B.*

8.9. **Why is Theorem 8.8 true?** This is because

(i) the linear system spanned by (the natural limits of) the following theta functions on a given PSQAS is very ample:

$$\theta_k = \sum_{y \in Y} a(k+y) w^{k+y}, \quad \theta_{k+z} = \theta_k \ (\forall z \in Y)$$

where

$$a(x) = q^{B(x,x)/2 + C(x) + D} \cdot (\text{a unit in } R)$$

and $C(x)$ is a linear function in $x \in X$ and D is a constant by Definition 8.2,

(ii) the distribution of minima of the function

$$B(x,x) - 2B(\lambda, x) + 2C(x) + 2D \quad (x \in X)$$

is the same up to translation as that of the function

$$B(x,x) - 2B(\lambda, x) \quad (x \in X).$$

Let $\lambda \in V(\sigma)$ for a Delaunay cell $\sigma \in \mathrm{Del}_B$. By the same computation as before, we see (by forgetting zero terms)

$$\begin{aligned} &\lim_{q \to 0} [\theta_k(q, q^{-B(\lambda, \cdot)} u^{\cdot})]_{k \in X/Y} \\ &\qquad = [u^a \cdot (\text{a unit in } R/I)]_{a \in \sigma} \in \mathbf{P}^{\sharp(\sigma \cap X) - 1}. \end{aligned}$$

Thus we have a stratum $O(\sigma)$ of Z in Theorem 8.8

$$O(\sigma) = \{[u^a \cdot (\text{a unit in } R/I)]_{a \in \sigma \cap X}; u \in \mathbf{G}_m^g]\} \simeq \mathbf{G}_m^{\dim \sigma}.$$

In general Z can be nonreduced. However even in that case the linear system spanned by the limit of θ_k is very ample so that we are able to describe Z set-theoretically as a disjoint union of algebraic tori. See also [Namikawa80].

9. Voronoi cells

The Voronoi cells are by definition polytopes dual to the Delaunay cells. To be more precise, let X be an integral lattice and B a positive integral bilinear form on X. We redefine maximal-dimensional Voronoi cells.

Definition 9.1. For $n \in X$ we define a Voronoi cell $V(n)$ by

$$V(n) = \{x \in X \otimes \mathbf{R}; \|x - n\| \leq \|x - a\| \text{ for any } a \in X\}$$

where $\|y\| = \sqrt{B(y, y)}$. All $V(n)$ $(n \in X)$ are translates of $V(0)$ and the union of all $V(n)$ cover $X \otimes \mathbf{R}$. The Voronoi decomposition Vor_B is the polyhedral decomposition consisting of $V(n)$ and their faces.

9.2. **Two-dimensional Voronoi cells.** Let B_1 and B_2 be integral symmetric matrices as follows:

$$B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then the Voronoi cell $V_i(0)$ associated with B_i is the following convex closure of the 4 or 6 points respectively:

$$V_1(0) = \langle \pm(1/2, 1/2),\ \pm(1/2, -1/2) \rangle,$$
$$V_2(0) = \langle \pm(1/3, 2/3),\ \pm(1/3, -1/3),\ \pm(2/3, 1/3) \rangle.$$

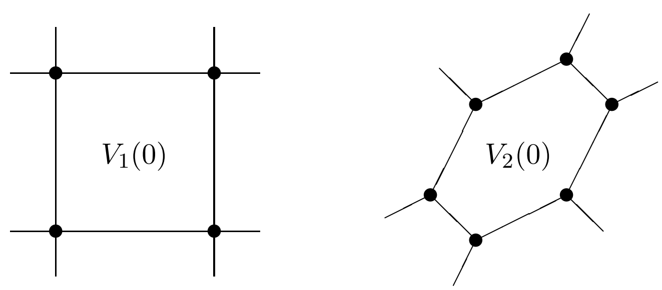


Figure 3

9.3. **A three-dimensional Voronoi cell - the most degenerate case.** Let B be an integral symmetric matrix:

$$B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

In order to compute the Voronoi cell $V(0)$, we need to know all the Delaunay cells in Del_B containing the origin where Del_B is the Delaunay decomposition associated with B. The Delaunay cells in Del_B containing the origin are listed as follows; let e_i be the standard i-th unit vector of $X \otimes \mathbf{R} = \mathbf{R}^3$;

then the Delaunay cells are the convex closures:

$$\langle 0, e_i\rangle,\ \langle 0, e_i+e_j\rangle, \langle 0, e_1+e_2+e_3\rangle,$$
$$\langle 0, e_i, e_i+e_j\rangle,\ \langle 0, e_i, e_1+e_2+e_3\rangle,\ \langle 0, e_i+e_j, e_1+e_2+e_3\rangle,$$
$$\langle 0, e_i, e_i+e_j, e_1+e_2+e_3\rangle$$

and their translates by X containing the origin. For instance, the Delaunay cell $\langle 0, e_i, e_i+e_j\rangle$ has exactly two translates $\langle 0, -e_i, e_j\rangle$, $\langle 0, -e_j, -e_i-e_j\rangle$ by X besides itself which contain the origin. Therefore there are exactly 1 zero-dimensional Delaunay cells, $14 = 6+6+2$ one-dimensional Delaunay cells, $36 = 18+9+9$ two-dimensional Delaunay cells and 24 three-dimensional Delaunay cells containing the origin. For instance $14 = 6+6+2$ one-dimensional Delaunay cells are listed as follows:

$$\pm\langle 0, e_i\rangle, \pm\langle 0, e_i+e_j\rangle, \pm\langle 0, e_1+e_2+e_3\rangle.$$

It follows that the unique three-dimensional Voronoi cell $V(0)$ has 14 two-dimensional faces, 36 edges (=one-dimensional faces) and 24 vertices. Since there are three kinds of two-dimensional Voronoi cells, we denote them respectively by $H_{\pm e_i}$, $S_{\pm(e_i+e_j)}$, $H_{\pm(e_1+e_2+e_3)}$, each of which is labelled by the vertex, different from the origin, of the one-dimensional Delaunay cell dual to it.

The Voronoi cells $H_{\pm e_i}$ and $H_{\pm(e_1+e_2+e_3)}$ are hexagons, while $S_{\pm(e_i+e_j)}$ are squares. Hence $V(0)$ has 8 hexagons and 6 squares as two-dimensional faces, 36 edges and 24 vertices. Any hexagon is adjacent to three hexagons and three squares, while any square is adjacent to four hexagons. Therefore no pair of squares meet. Two hexagons and a square meet at each vertex. Thus we see that $V(0)$ is a truncated octahedron, a polytope obtained from an octahedron by truncating all its 6 vertices. Note that in general one must leave part of the edge of the old polyhedron as an edge of the new polyhedron when one truncates (at a vertex).

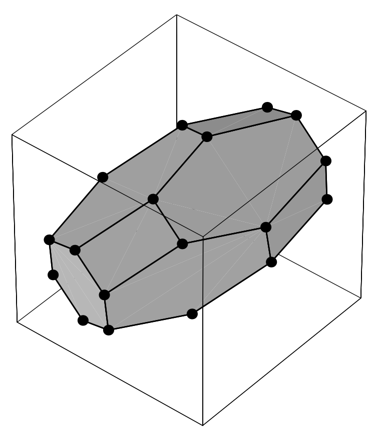


FIGURE 4. A truncated octahedron V_3

9.4. **The other cases.** In dimension three three are five Delaunay decompositions inequivalent under the action of $\mathrm{SL}(3, \mathbf{Z})$ including the case in 9.3. The first two types are a product of one-dimensional Delaunay decomposition with those in 8.5. Therefore there are two Voronoi decompositions corresponding to these decompositions, a cube mod X and a hexagonal pillar mod X.

The other types of Voronoi 3-cells are a truncated octahedron V_3 we saw in 9.3, a dodecahedron or a polytope V_5 which we will explain below.

A dodecahedron is a polytope V_4 with 14 vertices, 24 edges and 12 square faces as is shown in Figure 5. The polytope V_5 has 18 vertices, 28 edges and 12 faces. There are 4 hexagon faces between two umbrellas, each consisting of 4 square faces. Each hexagon is adjacent to two hexagons and 4 squares. Any square face is adjacent to two hexagons and two squares. This is V_5 shown in Figure 5.

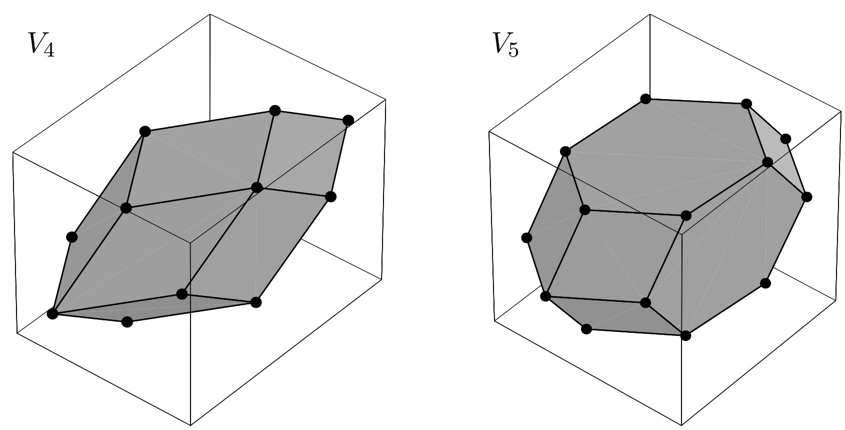


FIGURE 5

10. COMPACTIFICATION OF THE MODULI IN HIGHER DIMENSION

10.1. **Symplectic finite abelian groups.** Let K be a finite symplectic abelian group, namely a finite abelian group with e_K a nondegenerate alternating bimultiplicative form, which we call a symplectic form on K. Let $e_{\min}(K)$ (resp. $e_{\max}(K)$) be the minimum (resp. the maximum) of elementary divisors of K. To be more explicit, let $K = H \oplus H^\vee$, $H = (\mathbf{Z}/e_1\mathbf{Z}) \oplus \cdots \oplus (\mathbf{Z}/e_g\mathbf{Z})$ with $e_1|e_2|\cdots|e_g$ and $H^\vee = \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{G}_m)$. Then $e_{\min}(K) = e_1$ and $e_{\max}(K) = e_g$. Moreover we set

$$e_K(z + \alpha, w + \beta) = \beta(z)\alpha(w)^{-1}$$

for $z, w \in H$, $\alpha, \beta \in H^\vee$. Then e_K is a symplectic form on K. Let $N := e_{\min}(K)$ and $M := e_{\max}(K)$.

Let $\mu_M = \{z \in \mathbf{G}_m; z^M = 1\}$ and $G(K)$ the Heisenberg group, that is a central extension of μ_M by K with its commutator form equal to e_K. If k is algebraically closed, $G(K)$ is unique up to isomorphism. If $g = 1$, $e_1 = 3$, then $G(K) = G(3)$ in the notation of §1.

Theorem 10.2. *Suppose $e_{\min}(K) \geq 3$. Let (Z, L) be a g-dimensional K-PSQAS over an algebraically closed field k of characteristic prime to $|K|$. Then (Z, L) is $G(K)$-equivariantly embedded into $\mathbf{P}(V(K) \otimes k)$ by the natural limits of theta functions. In particular, the image of (Z, L) is a $G(K)$-invariant closed subscheme of $\mathbf{P}(V(K) \otimes k)$.*

By Theorem 10.2, any K-PSQAS (Z, L) is a point of the Hilbert scheme

$$\mathrm{Hilb}_K := \mathrm{Hilb}^P_{\mathbf{P}(V(K))}$$

where $P(n) = \chi(Z, nL) = n^g\sqrt{|K|}$, $g = \dim Z$.

In what follows we say that $\operatorname{Spec} k$ is a geometric point of $\operatorname{Spec} \mathbf{Z}[\zeta_M, 1/M]$ if k is an algebraically closed field containing ζ_M and M is invertible in k. We denote by $\mathrm{SL}_{\pm}(V(K) \otimes k)$ the subgroup of $\mathrm{GL}(V(K) \otimes k)$ consisting of matrices with determinant ± 1.

Theorem 10.3. *Suppose $e_{\min}(K) \geq 3$. Let $M = e_{\max}(K)$. Let $\operatorname{Spec} k$ be a geometric point of $\operatorname{Spec} \mathbf{Z}[\zeta_M, 1/M]$ and $(Z, L) \in \mathrm{Hilb}_K(k)$. Suppose that (Z, L) is smoothable into an abelian variety whose Heisenberg group is isomorphic to $G(K)$. Then the following are equivalent:*

(1) *the n-th Hilbert points of (Z, L) are Kempf-stable for any large n,*
(2) *a subgroup of $\mathrm{SL}_{\pm}(V(K) \otimes k)$ conjugate to $G(K)$ stabilizes (Z, L),*
(3) *(Z, L) is a K-PSQAS over k.*

We note that any nonsingular PSQAS is an abelian scheme, which is known to be Kempf-stable by [Kempf78]. For comparison with Theorem 5.4, we restate Theorem 10.3 in a much simpler form:

Theorem 10.4. *Any of the following three objects is the same:*

(1) *a degenerate abelian variety whose Hilbert points are* Kempf-stable*,*
(2) *a degenerate abelian variety which is* stable under the action of $G(K)$*, the Heisenberg group,*
(3) *a K-PSQAS, namely a degenerate abelian variety which is* moduli-stable *(in the sense similar to stable curves).*

Theorem 10.5. *Assume $e_{\min}(K) \geq 3$. Let $M = e_{\max}(K)$. Then there is a projective $\mathbf{Z}[\zeta_M, 1/M]$-subscheme $SQ_{g,K}$ of Hilb_K such that for any geometric point $\operatorname{Spec} k$ of $\operatorname{Spec} \mathbf{Z}[\zeta_M, 1/M]$,*

$$\begin{aligned}
SQ_{g,K}(k) &= \left\{ (Z, L);\ \begin{array}{l} (Z, L) \text{ is a Kempf-stable} \\ \text{degenerate AV over } k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /isom. \\
&= \left\{ (Z, L);\ \begin{array}{l} (Z, L) \text{ is a degenerate AV over } k \\ \text{invariant under } G(K) \\ \text{with level } G(K)\text{-structure} \end{array} \right\} \\
&= \left\{ (Z, L);\ \begin{array}{l} (Z, L) \text{ is a } K\text{-PSQAS over } k \\ \text{invariant under } G(K) \\ \text{with level } G(K)\text{-structure} \end{array} \right\},
\end{aligned}$$

which compactifies

$$A_{g,K}(k) = \left\{(Z,L);\ \begin{array}{l}(Z,L) \textit{ is an abelian variety over } k\\ \textit{with level } G(K)\textit{-structure}\end{array}\right\}/\textit{isom.}$$
$$= \left\{(Z,L);\ \begin{array}{l}(Z,L) \textit{ is an abelian variety over } k\\ \textit{invariant under } G(K)\\ \textit{with level } G(K)\textit{-structure}\end{array}\right\}.$$

By definition $G(3) = G(K)$ and $SQ_{1,3} = SQ_{1,K}$ if $K = (\mathbf{Z}/3\mathbf{Z})^2$.

Theorem 10.6. *Suppose* $e_{\min}(K) \geq 3$. *Let* $M := e_{\max}(K)$. *The functor* $\mathcal{SQ}_{g,K}$ *of projectively stable quasi-abelian schemes over reduced base schemes with level* $G(K)$*-structure is representable by the projective* $\mathbf{Z}[\zeta_M, 1/M]$*-scheme* $SQ_{g,K}$.

Theorem 10.5 or Theorem 10.6 shows that $SQ_{g,K}$ is a nice compactification of the moduli space of abelian varieties. It is shown by Theorem 10.3 to be also natural from the viewpoint of GIT. If one chooses $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$, all of the above theorems are reduced to Theorem 7.5.

11. Recent topics

11.1. **The other compactifications.** By Theorem 10.6 there exists a fine moduli scheme of nonsingular abelian schemes and Kempf-stable degenerate abelian schemes possibly singular. Some of the Kempf-stable degenerate abelian schemes are however nonreduced as is contrary to the expectation of many specialists. How about the existence of a compactification of the moduli space by reduced degenerate abelian schemes? The first trial in this direction is due to [Alexeev99]. Unfortunately his compactification is of dimension $g(g+1)/2 + \sqrt{|K|} - 1$, larger than $g(g+1)/2$ if $\sqrt{|K|}$ is strictly larger than one where $\sqrt{|K|}$ is equal to the dimension of the space of global sections of a given polarization. Therefore the interior of his compactification is not the ordinary moduli of abelian varieties with level structure except for the principally polarized case.

Recently the author [Nakamura01] constructed a compactification of the moduli by reduced degenerate abelian schemes, which are almost the same as the reduced models of our Kempf-stable degenerate abelian schemes. This compactification is a complete algebraic space of dimension $g(g+1)/2$ as desired, which is however only proved to be a coarse moduli. It is still an open problem whether it is a fine moduli. This would be a proper substitute for $SQ_{g,K}$ if we stick to the idea of compactifying the moduli by compact reduced schemes. And the author conjectures that this would be one of toroidal compactifications.

11.2. **The one-dimensional case.** The classical level structure and our level structure (referred to as Heisenberg level structure in [NT01]) are the same over any closed field, while they could be different over a non-closed field. This has been thoroughly investigated by [NT01] in the one-dimensional case. For both level structures we have the fine-moduli scheme for elliptic curves with level n-structure for $n \geq 3$ and both of the fine-moduli schemes are isomorphic as schemes. However their universal curves over the fine moduli are the same if n is odd, or different if n is even. This means that the functors of elliptic curves with these level n-structure are the same for $n \geq 3$ odd, or different for $n \geq 4$ even.

This explains why the universal elliptic curve with our level 4-structure

$$\begin{aligned}\mu_0(x_0^2 + x_2^2) - \mu_1 x_1 x_3 &= 0\\ \mu_0(x_1^2 + x_3^2) - \mu_1 x_0 x_2 &= 0\end{aligned}$$

has no rational points over the base rational curve. See [Nakamura99, p. 712]. The universal elliptic curve for our level structure has a $G(K)$-linearized invertible sheaf, though it has no rational points over the base curve. On the other hand the universal elliptic curve for the classical level structure has no $G(K)$-linearized invertible sheaf, though it has rational points over the base curve.

The universal elliptic curve with our level 4-structure is converted into the universal elliptic curve with classical level 4-structure in an elementary way. See [NT01, p. 320].

11.3. **Voronoi cells and crystals.** If $g = 2$ (resp. $g = 3$) there are exactly two (resp. five) different combinatorial types of Voronoi cells $V(0)$. This fact is equivalent to the existence of two or five different (= inequivalent under unimodular linear transformations) types of Delaunay decompositions in respective dimensions. The two types of Voronoi cells $V(0)$ in two dimension were given in §9.

The most remarkable is that any of the five types of $V(0)$ in three dimension have the same combinatorial type as a crystal of a certain kind of mineral — salt, garnet, calcite ($6[CaCO_3]$), apophyllite ($KCa_4(Si_4O_{10})_2F{\cdot}8H_2O$) and sphalerite (zinc blende ZnS), though very often the same mineral could show two or more different combinatorial types of crystals. As is well known a crystal of salt is a cube. A crystal of calcite could be a hexagonal pillar. A crystal of sphalerite (zinc blend) is a truncated octahedron V_3 we saw in §9. A crystal of garnet could have the same combinatorial type as a dodecahedron, which is the same as V_4 in §9. The polytope V_5 in §9 has the same combinatorial type as a crystal of apophyllite. All of these polytopes are three-dimensional Voronoi cells for positive symmetric matrices B chosen suitably. We note that a hexagon is one of the Voronoi cells $V(0)$ for $g = 2$, which is also identified with a crystal of mica.

No theoretical explanation (for instance, from the viewpoint of statistical physics) seems to be known so far for this similarity between 3-dimensional Voronoi cells and some of crystals.

Acknowledgment. In the fall of 1972, Professor Ueno sent to Professor Namikawa an unpublished note about his calculation of limits of abelian surfaces and their theta functions similar to §8. The author, who was in Nagoya university at that time, was shown the unpublished note by Professor Namikawa. Being inspired by the note, the author started studying the problem of compactifying the moduli of abelian varieties. Thirty years have passed since then. [Nakamura99] and the present article might clarify to some extent a few of the issues the author did not understand well enough when he wrote [Nakamura75]. The author would like to thank both Professor Namikawa and Professor Ueno for having led him to the problem and for their numerous suggestions since that time. The author also thanks many mathematician colleagues for numerous advises during the preparation of the article [Nakamura99]. Last but not least he also thanks Professor S. Zucker for his suggestions for improving our English in the present article.

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