GLOBAL MCKAY CORRESPONDENCE FOR QUOTIENT SURFACE SINGULARITIES (THE FIRST DRAFT)

AKIRA ISHII AND IKU NAKAMURA, April 18, 2016

ABSTRACT. Let G be a finite subgroup of $\mathrm{GL}(2, \mathbf{C})$ acting on $\mathbf{C}^2 \setminus \{0\}$ freely. Then we prove a global isomorphism between certain coherent sheaves over G-$\mathrm{Hilb}(\mathbf{C}^2)$ which generalizes the McKay correspondence for a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$. This somewhat sharpens the correspondence in [15] and [14].

1. Introduction.

This is the first draft of a paper we are preparing, which is intended to be a continuation of [15], [14] and [20]. For an algebraically closed field k, let G be any finite small subgroup of $\mathrm{GL}(2, k)$, that is, a finite subgroup of $\mathrm{GL}(2, k)$ acting on $\mathbf{A}_k^2$ with the origin the unique fixed point of it. Throughout this article we assume that the characteristic of k is prime to the order $|G|$ of G. The G-orbit Hilbert scheme G-$\mathrm{Hilb}(\mathbf{A}_k^2)$ is the scheme parameterizing all the G-invariant zero-dimensional subschemes of $\mathbf{A}_k^2$ of length $|G|$, each with structure sheaf isomorphic to the group algebra $k[G]$ of G as G-module. It is by [14] the minimal resolution of the quotient $\mathbf{A}_k^2/G$.

Any point y of G-$\mathrm{Hilb}(\mathbf{A}_k^2)$ represents a G-invariant cluster Z_y of $\mathbf{A}_k^2$, that is, a G-invariant zero-dimensional subscheme of $\mathbf{A}_k^2$, which is defined by a G-invariant ideal I_y of the polynomial ring of two variables. For any y in the exceptional set of the resolution, let $\mathrm{Gen}(I_y)$ be the minimal G-invariant (sub)module generating I_y. Roughly speaking, [15] and [14] determined $\mathrm{Gen}(I_y)$ for every individual y, and [20] determined in part the ρ-part of the union of all such $\mathrm{Gen}(I_y)$ over the exceptional set for every nontrivial irreducible representation ρ of G when G is a subgroup of $\mathrm{SL}(2, k)$.

The purpose of this article is to *completely* determine the ρ-part of the union of all $\mathrm{Gen}(I_y)$ over the exceptional set *for every* ρ. If G is a finite subgroup of $\mathrm{SL}(2, k)$, this recovers the extended Dynkin diagram of the singularity, while [20] recovers only the Dynkin diagram. See Section 6.

Theorem 1.1. *Let G be a finite small subgroup of* $\mathrm{GL}(2, k)$*, and E the exceptional set of the minimal resolution G-*$\mathrm{Hilb}(\mathbf{A}_k^2)$ *of $\mathbf{A}_k^2/G$. Then the*

Date: April 18, 2016.

Research was supported in part by the Grants-in-aid (15K04819, S-23224001) for Scientific Research, JSPS.

2000 *Mathematics Subject Classification*. Primary 14J10, 14J17;Secondary 14D20.

Key words and phrases. Moduli, Hilbert scheme, McKay correspondence.

union of all $\mathrm{Gen}(I_y)$ *over* E *is an* O_F*-module* G*-isomorphic to*

$$(\bigoplus_{\rho\neq\rho_0} O_{E(\rho)}(-1)\otimes\rho)\bigoplus O_F(-F)\otimes\rho_0$$

where ρ *ranges over the set of all non-trivial irreducible representations of* G*, special in the sense of Definition 3.4.3, and* $E(\rho)$ *is an irreducible component of* E *associated to* ρ *and* F *is the fundamental divisor of* E*.*

See Theorem 5.5 and Corollary 5.6. Our main ingredient for the proof is the derived category method adopted by [3] and [14]. See also [16].

Here is an outline of the article. In Section 2, we recall reflexive modules and full modules. In Section 3, we review the local McKay correspondence for two-dimensional quotient singularities in [14]. In Section 4, we discuss the connection between tensor with the natural representation, extensions of the structure sheaf O_{Z_y}, and cup products of Ext groups. In Section 5 we formulate and prove the global McKay correspondence (the above theorem). In Section 6, we explain the case of D_5 in detail. In this section we discuss the connection of the coinvariant algebra of G with the extended Dynkin diagram when G is a subgroup of SL(2).

We are very grateful to Professor John McKay for his incessant encouragement and interest in the subject.

2. Reflexive modules and torsion free pullbacks

2.1. Derived category.

Definition 2.1.1. Let X be a scheme over a fixed field k. Let $A := Coh(X)$ be the category of coherent O_X-modules and $K(A)$ the category consisting of the bounded complexes of objects and morphisms in A.

We define a morphism $f : P^\bullet \to Q^\bullet$ in $K(A)$ to be *a quasi-isomorphism* if f induces an isomorphism on cohomology.

The derived category $D(A)$ is the localization of $K(A)$ at Qis (the monoid of quasi-isomorphisms.), that is, it is the category $K(A)$ modulo equivalence defined by Qis. See [11, Def., pp. 49–50]. Similarly we define $A^c := Coh^c(X)$ to be the category of coherent O_X-modules with complete supports, $K(A^c)$ the category consisting of the bounded complexes of objects and morphisms in A^c, and $D^c(X)$ the derived category of $K(A^c)$.

2.2. Reflexive O_X-modules and full O_Y-modules.

This subsection is taken from [1], [7], and [22]. Let Z be a scheme of finite type over k, F a coherent O_Z-module on Z. Then $\mathcal{F}$ is defined to be a *reflexive* O_Z-module iff $\mathcal{F}^{\vee\vee}\simeq F$, where $\mathcal{F}^\vee = Hom_{O_Z}(\mathcal{F}, O_Z)$.

Lemma 2.2.1. *Let* Z *be an irreducible surface,* $U = Z\setminus \mathrm{Sing}\,(Z)$ *and* $i : U \hookrightarrow Z$ *the inclusion. The following is true.*

1. *any torsion free module over a discrete valuation ring is free, hence, any torsion free sheaf on a nonsingular curve is locally free.*
2. *if* Z *is a nonsingular surface, any reflexive* O_Z*-module is locally free,*

3. *if Z is a normal surface, and if $\mathcal{F}$ is a reflexive O_Z-module, then $i_*(i^*\mathcal{F}) = \mathcal{F}$, and it is uniquely determined by its restriction to U,*
4. *if $i_*(i^*\mathcal{F}) = \mathcal{F}$, and $\mathcal{F}$ is locally free on U, then $\mathcal{F}$ is reflexive,*
5. *$\mathcal{G}^\vee$ is reflexive for any finite O_Z-module $\mathcal{G}$ if Z is normal quasi-projective.*

Proof. See [13, Cor. 1.4] for the parts (1) and (2). See [13, Prop. 1.6] for the part (3). See [13, Cor. 1.2] for the part (5). The part (4) is proved as follows. By the assumption, $\mathcal{F} \simeq i_*(i^*\mathcal{F}) \simeq i_*(i^*(\mathcal{F}^{\vee\vee}))$. In particular, $\mathcal{F}$ is torsion free, so $\mathcal{F}$ is a subsheaf of $\mathcal{F}^{\vee\vee}$. Since $\mathcal{F}^{\vee\vee}$ is a subsheaf of $i_*(i^*(\mathcal{F}^{\vee\vee}))$ which contains $\mathcal{F}$, we have $\mathcal{F} = \mathcal{F}^{\vee\vee}$, hence $\mathcal{F}$ is reflexive. □

Lemma 2.2.2. *Let X be an affine normal surface with rational singularities and $f : Y \to X$ the minimal resolution of X. Let M be a reflexive O_X-module. Let $\mathcal{M}$ be the torsion free pullback of M to Y, that is, $\mathcal{M} = f^*M/O_Y$-torsions. Then $f_*(\mathcal{M}) = M$, and*

(i) *$\mathcal{M}$ is locally free,*
(ii) *$\mathcal{M}$ is generated by global sections,*
(iii) *$R^1f_*(\mathcal{M}^\vee \otimes \omega_Y) = 0$.*

Conversely if $\mathcal{M}$ satisfies (i) and (iii), then $M := f_*(\mathcal{M})$ is reflexive.

Proof. (i) and (ii) are due to [18]. (iii) is due to [6]. We prove the converse.

Assume (i), (iii). Let $A = \mathcal{M} \otimes_{O_Y} \omega_Y^\vee$, $B = O_{nF}$, and $C = R^2f_*(A^\vee(-nF))$. Since f is proper and $A^\vee(-nF)$ is coherent, so is C. Since any fiber of f is at most 1-dimensional, by [8, Th. 4.1.5, p. 125], we have $C = 0$.

Consider the exact sequence

$$0 \to A^\vee(-nF) \to A^\vee \to A^\vee \otimes_{O_Y} B \to 0.$$

Since $H^2(Y, A^\vee(-nF)) = H^0(X, C) = 0$ and $H^1(A^\vee) = 0$ by (iii), we have $H^1(A^\vee \otimes_{O_Y} B) = 0$ for every $n \geq 1$. Hence $\mathrm{Ext}^1_{O_Y}(A, B) \simeq H^1(A^\vee \otimes_{O_Y} B) = 0$. Since B is supported by F_{red}, by Serre duality on Y (by choosing a smooth compactification of Y),

$$\mathrm{Ext}^1_{O_Y}(O_{nF}, \mathcal{M}) = \mathrm{Ext}^1_{O_Y}(B, A \otimes \omega_Y) \simeq \mathrm{Ext}^1_{O_Y}(A, B)^\vee = 0.$$

By [9, Th. 2.8], $H^1_F(Y, \mathcal{M}) \simeq \mathrm{ind\,lim}\,\mathrm{Ext}^1_{O_Y}(O_{nF}, \mathcal{M}) = 0$. Let $U = Y \setminus F$. From the exact sequence

$$0 \to H^0_F(Y, \mathcal{M}) = 0 \to H^0(Y, \mathcal{M}) \to H^0(U, \mathcal{M}) \to H^1_F(Y, \mathcal{M}),$$

it follows $H^0(X, M) \simeq H^0(Y, \mathcal{M}) \simeq H^0(U, \mathcal{M}_U) \simeq H^0(U, M_U)$. So M is reflexive by Lemma 2.2.1 (4). □

Definition 2.2.3. We call an O_Y-module $\mathcal{M}$ a *full* O_Y-module (or simply a full sheaf following [6]) if the conditions (i)–(iii) are satisfied.

Corollary 2.2.4. *Under the same notation as in Lemma 2.2.2, there is a bijective correspondence between the following sets*

(i) *the set of indecomposable reflexive O_X-modules M,*
(ii) *the set of indecomposable full O_Y-modules $\mathcal{M}$.*

Corollary 2.2.5. *Under the same notation as in Lemma 2.2.2, every full O_Y-module $\mathcal{M}$ is determined by its restriction to $Y \setminus f^{-1}(\mathrm{Sing}\,(X))$.*

Proof. Clear from Lemma 2.2.1 (3) and Corollary 2.2.4. □

2.3. **The minimal resolution Y of the quotient $X = U/G$.** Let k be any algebraically closed field of any characteristic, and G a finite small subgroup of $\mathrm{GL}(2,k)$ so that we have a natural action of G on $\mathbf{A}^2_k$. We assume throughout this article that the order $|G|$ of G and the characteristic of k are coprime. Let $U = \mathbf{A}^2_k$, $X = U/G$ and let $\pi : U \to X$ be the natural morphism. The surface X has a unique singular point 0, which is a rational singularity [21]. Let $f : Y \to X$ be the minimal resolution of $X = U/G$. Thus we have a commutative diagram:

(2.3.1)
$$\begin{array}{ccc} Y \times_k U & \xrightarrow{\pi_U} & U \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X = U/G \end{array}$$

Let F be *the fundamental divisor* of the singularity $(X, 0)$. That is, the minimum of effective divisors D of Y such that $D \neq 0$, $\mathrm{Supp}\,(D) \subset f^{-1}(0)$, and $DE' \leq 0$ for any irreducible component E' of $f^{-1}(0)$.

The following is due to [22].

Theorem 2.4. *Let G be a finite small subgroup of $\mathrm{GL}(2,k)$ and $X = U/G$. Under the same notation as in Subsec. 2.3,*

1. *there is a bijective correspondence between the following sets*
   (i) *the set of irreducible components E_i of $E := f^{-1}(0)$,*
   (ii) *the set of indecomposable full O_Y-modules $\mathcal{M}_i$, special in the sense that $H^1(Y, \mathcal{M}_i^\vee) = 0$,*
   The correspondence $\mathcal{M}_i \mapsto E_i$ is given by
$$c_1(\mathcal{M}_i)E_j = \delta_{ij},$$
2. *the rank of $\mathcal{M}_i$ is equal to $c_1(\mathcal{M}_i)F$, the multiplicity of E_i in F.*

3. The local McKay correspondence

The following is due to [14].

Theorem 3.1. *Let $U = \mathbf{A}^2_k$, G a finite subgroup of $\mathrm{GL}(2,k)$ such that the order $|G|$ of G is prime to the characteristic of k. Then Y is connected and it is a minimal resolution of $X := U/G$.*

Definition 3.2. Let $\mathcal{Z}$ be the universal cluster over $Y = G\text{-Hilb}(U)$. Consider the commutative diagram:

(3.2.1)
$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & U \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X = U/G \end{array}$$

where $q := (\pi_U)_{\mathcal{Z}}$ and $p := (\pi_Y)_{\mathcal{Z}}$. Note that

$$\mathcal{Z} \times_Y (Y \setminus F) \simeq (U \times_X (X \setminus \{O\}).$$

Definition 3.3. We define two functors

$$\Psi : D_c^G(U) \to D_c(Y), \quad \Phi : D_c(Y) \to D_c^G(U)$$

by

$$\Psi(A) = [p_* \mathbf{L} q^*(A)]^G = \mathbf{R}(\pi_Y)_*(O_{\mathcal{Z}} \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} \pi_U^*(A)]^G,$$

$$\Phi(B) = \mathbf{R}(\pi_U)_*(O_{\mathcal{Z}}^{\vee} \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} \pi_Y^*(B \otimes_k \rho_0) \overset{\mathbf{L}}{\otimes}_{O_{Y \times_k U}} \pi_U^* K_U)[2].$$

where $O_{\mathcal{Z}}^{\vee}$ is the derived dual of $O_{\mathcal{Z}}$, that is, $\mathbf{R}\mathcal{H}om_{O_{Y \times_k U}}(O_{\mathcal{Z}}, O_{Y \times_k U})$.

Theorem 3.4. *Φ is fully faithful and Ψ is a left adjoint of Φ.*

See [14, Sec. 6].

Lemma 3.4.1. *If ρ is an irreducible representation of G, then*

$$\mathcal{M}_\rho := [p_* q^*(O_U \otimes_k \rho^*)]^G = [p_*(O_{\mathcal{Z}} \otimes_k \rho^*)]^G$$

is a full O_Y-module, whose corresponding reflexive O_X-module is given by

$$M_\rho := f_*(\mathcal{M}_\rho) = [\pi_*(O_U) \otimes_k \rho^*]^G.$$

Proof. The first half of the lemma is given by [14, Cor. 3.2]. The second half of the lemma follows from Corollaries 2.2.4 and 2.2.5. □

Remark 3.4.2. Let $\mathcal{F} = \pi_*(O_U)$. Let $W = X \setminus \{0\}$ and $i : W \hookrightarrow X$. Then

$$\Gamma(X, i_*\mathcal{F}) = \Gamma(W, \mathcal{F}_W) = \Gamma(U \setminus \{0\}, O_U) = \Gamma(U, O_U) = \Gamma(X, \mathcal{F}).$$

Since X is affine, this shows $i_*(\mathcal{F}) = \mathcal{F}$. By Lemma 2.2.1 (4), $\pi_*(O_U) = \mathcal{F}$ is reflexive. Moreover

$$p_*(O_{\mathcal{Z}}) = \bigoplus_\rho \mathcal{M}_\rho \otimes \rho, \quad \pi_*(O_U) = \bigoplus_\rho M_\rho \otimes_k \rho.$$

By Lemma 3.4.1. $p_*(O_{\mathcal{Z}})$ is the torsion free pullback of $\pi_*(O_U)$ by f.

Definition 3.4.3. An irreducible representation ρ of G is said to be *special* if its corresponding full O_Y-module $\mathcal{M} := \mathcal{M}(\rho) := [p_* q^*(O_U \otimes_k \rho^*)]^G$ is special, that is, $H^1(Y, \mathcal{M}^{\vee}) = 0$. We denote $\mathcal{M}(\rho_i)$ by $\mathcal{M}_i$. We number all irreducible representations ρ in the following manner: ρ_0 is trivial, ρ_i $(1 \le i \le m)$ is nontrivial special and ρ_j $(j \ge m+1)$ is nontrivial nonspecial.

Lemma 3.4.4. *Let ρ_i be an irreducible special representation of G, $\mathcal{M}_i = \mathcal{M}(\rho_i)$, and E_i the irreducible component of F with $c_1(\mathcal{M}_i)E_j = \delta_{ij}$ as in Theorem 2.4. Then*

$$\Psi(O_0 \otimes_k \rho_i^*) = \begin{cases} O_{E_i}(-1)[1] & \text{if } \rho_i \text{ is nontrivial special} \\ O_F & \text{if } i = 0, \text{ that is, } \rho_i \text{ is trivial,} \\ 0 & \text{if } \rho_i \text{ is nontrivial, nonspecial.} \end{cases}$$

See [14, Sec. 5] for the above lemma.

Lemma 3.5. *Under the same notation as above,*

$$G\text{-}\operatorname{Ext}^k_{O_U}(O_{Z_y},(O_0\otimes_k\rho_i)) := \operatorname{Hom}^k_{D^G_c(O_U)}(O_{Z_y},(O_0\otimes_k\rho_i))$$
$$=\begin{cases}\operatorname{Ext}^k_{O_Y}(O_F,O_y) & (i=0)\\ \operatorname{Ext}^{k-1}_{O_Y}(O_{E_i}(-1),O_y) & (1\le i\le m)\\ 0 & (i\ge m+1)\end{cases}$$
$$=\begin{cases}O_y & (i=0,k=0,1,y\in F),\\ O_y & (1\le i\le m,k=1,2,y\in E_i),\\ 0 & (\textit{otherwise}).\end{cases}$$

See [14, Sec. 7] for the above lemma. See also Def. 3.4.3.

Theorem 3.6. *(The local McKay correspondence) Let $\mathfrak{m}$ be the maximal ideal of O_U at the origin, $y\in Y$ and Z_y the G-invariant cluster of U corresponding to y and I_{Z_y} the ideal of O_U defining Z_y. Then the G-module $\operatorname{Gen}(I_y):=I_{Z_y}/\mathfrak{m}I_{Z_y}$ is given by*

$$\begin{cases}\rho_i\oplus\rho_0 & \textit{if } y\in E_i\setminus\cup_{j\ne i}E_j\\ \rho_i\oplus\rho_j\oplus\rho_0 & \textit{if } y\in E_i\cap E_j,\ i\ne j.\end{cases}$$

Proof. For later use, we recall how the proof in [14, Th. 7.1] proceeds in the following steps.

1. Since $\operatorname{Gen}(I_{Z_y})$ is a nontrivial finite $O_0[G]$-module, it is a sum of irreducible $O_0[G]$-submodules,
2. if $1\le i\le m$ and $y\in E_i$, then by Lemma 3.5, there exists a nonzero element $\theta\in G\text{-}\operatorname{Ext}^1_{D_c(U)}(O_{Z_y},O_0\otimes_k\rho_i)$ and a nontrivial extension of O_U-modules with G-action corresponding to θ:

$$0\to O_0\otimes\rho_i\to\mathcal{F}\to O_{Z_y}\to 0, \tag{3.6.1}$$

   which is uniquely determined independently of the choice of θ,
3. there exists an O_U-submodule J of I_y such that (3.6.1) is the same as

$$0\to J/\mathfrak{m}I_y\to O_U/J\to O_{Z_y}\to 0. \tag{3.6.2}$$

   Since $J=\operatorname{Ann}_{O_U}(\mathcal{F})$, the ideal J is uniquely determined by y and ρ_i,
4. in particular, there are no two copies of the same ρ_i in $\operatorname{Gen}(I_{Z_y})$,
5. if $O_0\otimes_k\rho_i\subset\operatorname{Gen}(I_{Z_y})$ for some $i\ne 0$ and $y\notin E_i$, then Lemma 3.5 gives a G-invariant O_U-splitting ϕ of (3.6.2). Since $O_{Z_y}\simeq O_y[G]$ and $i\ne 0$, we have $\phi(1)=1$, from which we derive a contradiction,
6. we can argue for ρ_0 in the same manner.

□

4. Extensions of the socle and the cup products

4.1. The basic construction of extensions.

Definition 4.1.1. Let R be a commutative ring. For two extensions of R-modules

$$(s_1) \quad : \quad 0 \longrightarrow A_1 \longrightarrow C_1 \xrightarrow{\phi_1} F \longrightarrow 0,$$

$$(s_2) \quad : \quad 0 \longrightarrow A_2 \longrightarrow C_2 \xrightarrow{\phi_2} F \longrightarrow 0,$$

we define the Bare sum of (s_1) and (s_2) to be an exact sequence

$$0 \longrightarrow A_1 \oplus A_2 \longrightarrow C \xrightarrow{\phi} F \longrightarrow 0.$$

where $C := C_1 \times_F C_2 = \{(c_1, c_2) \in C_1 \times C_2; \phi_1(c_1) = \phi_2(c_2)\}$ and $\phi(c_1, c_2) = \phi_1(c_1) = \phi_2(c_2)$. See [5, p. 290].

Definition 4.1.2. Let

$$(s) \quad : \quad 0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\eta} F \longrightarrow 0.$$

be an exact sequence of R-modules. For any R-homomorphism $\psi : A \to B$, we define *the pushfoward* $\psi_*(s)$ of (s) by ψ to be an exact sequence:

$$\psi_*(s) \quad : \quad 0 \longrightarrow B \xrightarrow{(\mathrm{id}_B, 0)} C_\psi \xrightarrow{\eta \cdot p_2} F \longrightarrow 0.$$

where $C_\psi := B \times C/(\psi, \phi)(A)$.

Definition 4.1.3. For any R-homomorphism $\gamma : E \to F$, we define *the pullback* $\gamma^*(s)$ of (s) by γ to be an exact sequence:

$$\gamma^*(s) \quad : \quad 0 \longrightarrow A \xrightarrow{(0, \phi)} C^\gamma \xrightarrow{\eta \cdot p_2} E \longrightarrow 0$$

where $C^\gamma := E \times_F C := \{(e, c) \in E \times C; \gamma(e) = \eta(c)\}$.

Definition 4.2. Let G be a finite group, and R a commutative ring with G-action ρ. Let M be an R-module with G-action, that is,

$$g \cdot (rm) = \rho(g)(r)(g \cdot m) \quad (\forall g \in G, r \in R, m \in M).$$

We define the socle $\mathrm{Soc}(M)$ of M to be the sum of all G-invariant R-submodules of M which are irreducible G-modules.

For a cluster Z_y with $y \in F$, $O_{Z_y} = O_U/I_y$ is an O_U-module supported by the origin of U. Let $\mathfrak{m}$ be the maximal ideal of O_U defining O. Then

$$\mathrm{Soc}(O_{Z_y}) = [I_y : \mathfrak{m}]/I_y.$$

Theorem 4.3. [14] For $y \in F \subset Y = G\text{-Hilb}(U)$, $\mathrm{Soc}(O_{Z_y})$ is given by

$$[I_y : \mathfrak{m}]/I_y = \begin{cases} \rho_i \otimes \det(\rho_{\mathrm{nat}}) & y \in E_i, y \notin E_j (\forall j \neq i) \\ (\rho_i + \rho_j) \otimes \det(\rho_{\mathrm{nat}}) & y \in E_i \cap E_j \end{cases}$$

Proof. By Serre duality

$$G\text{-}\mathrm{Ext}^k_{O_U}(O_{Z_y}, O_0 \otimes_k \rho_i)^\vee \simeq G\text{-}\mathrm{Ext}^{2-k}_{O_U}(O_0 \otimes_k \rho_i \otimes_k \det \rho_{\mathrm{nat}}, O_{Z_y}),$$

where $\Omega^2_U = O_U \otimes \det \rho^\vee_{\mathrm{nat}}$. Then Theorem follows from Lemma 3.5. □

4.4. **The structure of the extension $O_U/\mathfrak{m}I_y$.** The purpose of this subsection is to study the exact sequence in Theorem 3.6 for $y \in F$, to be more precise, its structure under multiplication by linear polynomials in x and y:

$$\begin{array}{cccccccccc} (t_1) & : & 0 \longrightarrow & I_y/\mathfrak{m}I_y & \longrightarrow & O_U/\mathfrak{m}I_y & \xrightarrow{\phi} & O_{Z_y} & \longrightarrow & 0 \\ & & & \| & & \cup & & \cup & & \\ (t_2) & : & 0 \longrightarrow & I_y/\mathfrak{m}I_y & \to & [I_y : \mathfrak{m}]/\mathfrak{m}I_y & \to & [I_y : \mathfrak{m}]/I_y & \longrightarrow & 0. \end{array}$$

Let $\mathrm{Gen}(I_y) = I_y/\mathfrak{m}I_y = \bigoplus_{\rho\in\Lambda_y} W(\rho)$ for some G-invariant G-irreducible O_U-submodule $W(\rho) \neq 0$ by Theorem 3.6.

By Lemma 3.5 and by the proof of Theorem 3.6, for every $\rho \in \Lambda_y$, there exist a unique ideal J_ρ of O_U with $\mathfrak{m}I_y \subset J_\rho \subset I_y$ and a nontrivial extension

$$(\mathrm{ext}_\rho) \quad : \quad 0 \longrightarrow W(\rho) \to O_U/J_\rho \to O_{Z_y} \longrightarrow 0$$

where $I_y/J_\rho \simeq W(\rho)$ and $\mathfrak{m}I_y = \cap_{\rho\in\Lambda_y} J_\rho$ because $\mathrm{Gen}(I_y) = \bigoplus_{\rho\in\Lambda_y} W(\rho)$.

We choose and fix $0 \neq V(\xi) \subset \mathrm{Soc}(O_{Z_y})$. Since $V(\xi)$ is an O_U-submodule of O_{Z_y}, there exists an O_U-submodule $B(\xi)$ of $[I_y : \mathfrak{m}]$ such that $B(\xi)/I_y = V(\xi)$. Then we have a commutative diagram of exact sequences

$$\begin{array}{cccccccccc} (u_1) & : & 0 \longrightarrow & \bigoplus_{\rho\in\Lambda_y} W(\rho) & \longrightarrow & O_U/\mathfrak{m}I_y & \xrightarrow{\phi} & O_{Z_y} & \longrightarrow & 0, \\ & & & \| & & \cup & & \cup & & \\ (u_2) & : & 0 \longrightarrow & \bigoplus_{\rho\in\Lambda_y} W(\rho) & \to & B(\xi)/\mathfrak{m}I_y & \to & B(\xi)/I_y & \longrightarrow & 0. \\ & & & \downarrow & & \downarrow & & \| & & \\ (u_3) & : & 0 \longrightarrow & W(\rho) & \longrightarrow & B(\xi)/J_\rho & \to & B(\xi)/I_y & \longrightarrow & 0. \end{array}$$

Here we note the following:

(α) the exact sequence (u_1) is the same as (t_1),

(β) the exact sequence (u_1) is the Bare sum of all (ext_ρ),

(γ) the exact sequence (u_2) is the pullback of (u_1) by the inclusion $V(\xi) = B(\xi)/I_y \overset{j}{\hookrightarrow} O_{Z_y}$,

(δ) each (ext_ρ) is the pushfoward of (u_1) by the projection from $\mathrm{Gen}(I_y)$ to $W(\rho)$ where $O_U/J_\rho \simeq (W(\rho) \oplus (O_U/\mathfrak{m}I_y)) \,/\, \mathrm{Gen}(I_y)$,

(ε) the exact sequence (u_3) is the pullback of (ext_ρ) by $B(\xi)/I_y \overset{j}{\hookrightarrow} O_{Z_y}$,

(ζ) the exact sequence (u_3) is also the pushforward of (u_2) by the projection from $\mathrm{Gen}(I_y)$ to $W(\rho)$.

Lemma 4.4.1. *Let $y \in F$. Let $V^\natural(\xi)$ be any $k[G]$-submodule (not necessarily an O_U-submodule) of $B(\xi)/\mathfrak{m}I_y$ lifting $V(\xi)$. Then*

1. *the k-module $V^\natural(\xi) + \mathrm{Gen}(I_y)$ is uniquely determined by $V(\xi)$,*
2. *$S_1 \cdot V^\natural(\xi)$ is a k-submodule of $I_y/\mathfrak{m}I_y$ determined uniquely by $V(\xi)$, which we denote by $S_1 \cdot V(\xi)$,*

where $S_1 := \mathfrak{m}/\mathfrak{m}^2$ is identified with the space of linear polynomials on $\mathbf{A}^2_k$.

Proof. We note first that since $V(\xi) \subset [I_y : \mathfrak{m}]/I_y = \mathrm{Soc}(I_y)$, we can choose $V^\natural(\xi) \subset [I_y : \mathfrak{m}]/\mathfrak{m}I_y$. Let $V_1^\natural(\xi)$ be another lifting of $V(\xi)$. Then $V_1^\natural(\xi) + I_y/\mathfrak{m}I_y = V^\natural(\xi) + I_y/\mathfrak{m}I_y$. The first assertion is thus clear.

We prove the second. Since $V^\natural(\xi) \subset [I_y : \mathfrak{m}]/\mathfrak{m}I_y$, we have $\mathfrak{m}V^\natural(\xi) \subset I_y/\mathfrak{m}I_y$, hence $\mathfrak{m}^2 V^\natural(\xi) = 0$. It follows

$$S_1 \cdot V_1^\natural(\xi) \subset S_1 \cdot (V^\natural(\xi) + I_y/\mathfrak{m}I_y) = S_1 \cdot V^\natural(\xi) \subset I_y/\mathfrak{m}I_y.$$

Therefore $S_1 \cdot V_1^\natural(\xi) = S_1 \cdot V^\natural(\xi)$. □

Lemma 4.4.2. *Let $y \in F$. Let $V_\rho^\natural(\xi)$ be any $k[G]$-submodule (not necessarily an O_U-submodule) of $B(\xi)/J_\rho$ lifting $V(\xi)$. Then*

1. *the k-module $V_\rho^\natural(\xi) + I_y/J_\rho$ is uniquely determined by $V(\xi)$,*
2. *$S_1 \cdot V_\rho^\natural(\xi)$ is a k-submodule of I_y/J_ρ, uniquely determined by $V(\xi)$,*
3. *$S_1 \cdot V_\rho^\natural(\xi) = S_1 \cdot V^\natural(\xi) + J_\rho/J_\rho$,*

where $I_y/J_\rho \simeq W(\rho)$.

Proof. The first two assertions are proved in the same manner as before. We prove the third. By our choice of $V^\natural(\xi)$ in Lemma 4.4.1, ϕ maps $V^\natural(\xi)$ isomorphically onto $V(\xi)$, hence $V^\natural(\xi)+J_\rho/J_\rho \simeq V(\xi)$. Hence we can choose $V^\natural(\xi) + J_\rho/J_\rho$ as $V_\rho^\natural(\xi)$. By the second assertion we have

$$S_1 \cdot V_\rho^\natural(\xi) = S_1 \cdot V^\natural(\xi) + J_\rho/J_\rho.$$

This proves Lemma. □

Corollary 4.4.3. *Under the same notation, the following are equivalent:*

1. *the exact sequence (u_3), which is by (ε) the pullback of (ext_ρ) via the inclusion $V(\xi) \hookrightarrow O_{Z_y}$, is a trivial extension,*
2. *there exists a G-invariant O_U-module $V_\rho^\natural(\xi)$ of $B(\xi)/J_\rho$ lifting $V(\xi)$,*
3. *every lifting $V_\rho^\natural(\xi)$ of $V(\xi)$ to $B(\xi)/J_\rho$ is a G-invariant O_U-module,*
4. *$S_1 \cdot V_\rho^\natural(\xi) = \{0\}$,*
5. *$S_1 \cdot V(\xi) \cap W(\rho) = \{0\}$.*

Corollary 4.4.4. *Under the same notation, the following are equivalent:*

1. *the exact sequence (u_3) does not split,*
2. *no lifting $V_\rho^\natural(\xi)$ of $V(\xi)$ is a G-invariant O_U-module,*
3. *$S_1 \cdot V_\rho^\natural(\xi) = W(\rho)$,*
4. *$S_1 \cdot V(\xi) \supset W(\rho)$.*

4.5. **Translation of the splitting.** We choose and fix irreducible representations ρ and χ of G such that $W(\rho) \subset \mathrm{Gen}(I_y)$ and $V(\xi) \subset \mathrm{Soc}(O_{Z_y})$.

The extension (ext_ρ) is uniquely determined by ρ in view of the proof (Step 3) of Theorem 3.6. Since every irreducible submodule $V(\xi)$ of O_{Z_y} is unique, the extension (4.5.1) is, in view of (ε), isomorphic to the pullback (u_3) of (ext_ρ) via the inclusion $j : V(\xi) \hookrightarrow O_{Z_y}$:

$$(4.5.1) \qquad 0 \longrightarrow W(\rho) \to B(\xi)/J_\rho \to V(\xi) = B(\xi)/I_y \longrightarrow 0.$$

Associated to (4.5.1), we have a natural cup product

$$\begin{aligned}\text{Hom}^0_{D^G_c(U)}(O_0\otimes_k\xi, O_{Z_y})\times\text{Hom}^1_{D^G_c(U)}(O_{Z_y}, O_0\otimes_k\rho)&\\ \to\text{Hom}^1_{D^G_c(U)}(O_0\otimes_k\xi, O_0\otimes_k\rho).&\end{aligned}\tag{4.5.2}$$

By Corollary 4.4.3, the following are equivalent:

(a) the cup product (4.5.2) is zero,
(b) the extension (4.5.1) is trivial,
(c) $S_1\cdot V(\xi)\cap W(\rho)=\{0\}$.

Similarly by Corollary 4.4.4, the following are equivalent:

(i) the cup product (4.5.2) is nonzero,
(ii) the extension (4.5.1) is nontrivial,
(iii) $S_1\cdot V(\xi)\supset W(\rho)$.

4.6. **Translation of the cup product.** Let $\rho,\xi\in\text{Irr}(G)$, $\Omega^2_U=O_U\otimes_k\det\rho_{\text{nat}}^\vee$ and $\alpha=\xi\otimes\det\rho^\vee_{\text{nat}}$. Suppose α is nontrivial and special. We recall Serre duality for a nonsingular surface Z:

$$\text{Ext}^p_{D_c(Z)}(B^\bullet, A^\bullet\otimes_{O_Z}\omega_Z)\simeq\text{Ext}^{2-p}_{D_c(Z)}(A^\bullet, B^\bullet)^\vee.\tag{4.6.1}$$

We apply it to $Z=U$ and Y to obtain the following for $\alpha\neq\rho_0$,

$$\begin{aligned}\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\xi, O_{Z_y})&\simeq\text{Hom}^{2-k}_{D^G_c(U)}(O_{Z_y}, O_0\otimes_k\xi\otimes_k\Omega^2_U)^\vee\quad\text{by (4.6.1)}\\&=\text{Hom}^{2-k}_{D^G_c(U)}(O_{Z_y}, O_0\otimes_k\alpha)^\vee\\&\simeq\text{Ext}^{1-k}_{O_Y}(O_{E(\alpha)}(-1), O_y)^\vee\quad\text{(Lemma 3.5)}\\&\simeq\text{Ext}^{1-k}_{O_Y}(O_{E(\alpha)}(-1), O_y\otimes_{O_Y}\omega_Y)^\vee\\&\simeq\text{Ext}^{1+k}_{O_Y}(O_y, O_{E(\alpha)}(-1))\quad\text{by (4.6.1)}.\end{aligned}$$

If ρ is nontrivial and special, then by Lemma 3.5

$$\text{Hom}^k_{D^G_c(U)}(O_{Z_y}, O_0\otimes_k\rho)\simeq\text{Ext}^{k-1}_{O_Y}(O_{E(\rho)}(-1), O_y).$$

We also see

$$\begin{aligned}\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\rho_0\otimes_k\det(\rho_{\text{nat}}), O_{Z_y})\simeq\text{Ext}^{2-k}_{O_Y}(O_F, O_y)^\vee\simeq\text{Ext}^k_{O_Y}(O_y, O_F),\\ \text{Hom}^k_{D^G_c(U)}(O_{Z_y}, O_0\otimes_k\rho_0)\simeq\text{Ext}^k_{O_Y}(O_F, O_y).\end{aligned}$$

We see by $O_0=\omega_0=O_0^\vee\otimes_{O_Y}\omega_U[2]$,

$$\begin{aligned}\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\xi, O_0\otimes_k\rho)&\simeq\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\xi,(O_0^\vee\otimes_k\rho)\otimes_{O_Y}\omega_U[2])\\&\simeq\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\rho^*,(O_0\otimes_k\xi)^\vee\otimes_{O_Y}\omega_U[2])\\&\simeq\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\rho^*, O_0^\vee\otimes_k\xi^*\otimes_{O_Y}\omega_U[2])\\&\simeq\text{Hom}^k_{D^G_c(U)}(O_0\otimes_k\rho^*, O_0\otimes_k\xi^*),\end{aligned}$$

which is further translated in Subsec. 4.7.

4.7. **The case where $G \subset \mathrm{SL}(2)$.** In this subsection, we assume $G \subset \mathrm{SL}(2)$. Since $G \subset \mathrm{SL}(2)$, both Ψ and Φ are equivalences of the categories such that $\Psi\Phi = \mathrm{id}_{D_c(Y)}$ and $\Phi\Psi = \mathrm{id}_{D_c^G(U)}$. This is proved by the same argument as in [3] and [14, Th. 6.2].

In what follows we consider a pair ξ and ρ such that $\xi \subset \mathrm{Soc}(O_{Z_y})$ and $\rho \subset \mathrm{Gen}(I_y)$. Since $G \subset \mathrm{SL}(2)$, $\det(\rho_{\mathrm{nat}}) = \rho_0$ and $\xi = \alpha$.

First we assume $\alpha \neq \rho_0$. If neither α nor ρ are trivial, then

$$\begin{aligned}\mathrm{Hom}^k_{D_c^G(U)}(O_0 \otimes_k \alpha, O_0 \otimes_k \rho) &\simeq \mathrm{Hom}^k_{D_c^G(U)}(O_0 \otimes_k \rho^*, O_0 \otimes_k \alpha^*)\\ &\simeq \mathrm{Hom}^k_{D_c(Y)}(\Psi(O_0 \otimes_k \rho^*), \Psi(O_0 \otimes_k \alpha^*))\\ &= \mathrm{Ext}^k_{O_Y}(O_{E(\rho)}(-1), O_{E(\alpha)}(-1)).\end{aligned}$$

If $\rho = \rho_0$ but $\alpha \neq \rho_0$, then

$$\begin{aligned}\mathrm{Hom}^k_{D_c^G(U)}(O_0 \otimes_k \alpha, O_0 \otimes_k \rho) &\simeq \mathrm{Hom}^k_{D_c^G(U)}(O_0 \otimes_k \rho^*, O_0 \otimes_k \alpha^*)\\ &\simeq \mathrm{Hom}^k_{D_c^G(U)}(\Psi(O_0 \otimes_k \rho_0^*), \Psi(O_0 \otimes_k \alpha^*))\\ &= \mathrm{Ext}^k_{O_Y}(O_F, O_{E(\alpha)}(-1)[1])\\ &= \mathrm{Ext}^{k+1}_{O_Y}(O_F, O_{E(\alpha)}(-1))\\ &= \mathrm{Ext}^{1-k}_{O_Y}(O_{E(\alpha)}(-1), O_F)^\vee.\end{aligned}$$

Hence the cup product in $D_c^G(U)$

$$\begin{aligned}&\mathrm{Hom}^0_{D_c^G(U)}(O_0 \otimes_k \alpha, O_{Z_y}) \times \mathrm{Hom}^1_{D_c^G(U)}(O_{Z_y}, O_0 \otimes_k \rho)\\ &\qquad \to \mathrm{Hom}^1_{D_c^G(U)}(O_0 \otimes_k \alpha, O_0 \otimes_k \rho).\end{aligned} \tag{4.7.1}$$

is translated into the following cup product in $D_c(Y)$

$$\begin{aligned}&\mathrm{Ext}^1_{O_Y}(O_y, O_{E(\alpha)}(-1)) \times \mathrm{Hom}_{O_Y}(O_{E(\rho)}(-1), O_y)\\ &\qquad \to \mathrm{Ext}^1_{O_Y}(O_{E(\rho)}(-1), O_{E(\alpha)}(-1))\end{aligned} \tag{4.7.2}$$

if $\rho \neq \rho_0$, $\alpha \neq \rho_0$. If $\rho = \rho_0$ and $\alpha \neq \rho_0$, then (4.7.1) is translated into

$$\begin{aligned}&\mathrm{Ext}^1_{O_Y}(O_y, O_{E(\alpha)}(-1)) \times \mathrm{Ext}^1_{O_Y}(O_F, O_y)\\ &\qquad \to \mathrm{Ext}^2_{O_Y}(O_F, O_{E(\alpha)}(-1)).\end{aligned} \tag{4.7.3}$$

By Serre duality (4.6.1), this cup product is equivalent to

$$\begin{aligned}&\mathrm{Ext}^1_{O_Y}(O_y, O_{E(\alpha)}(-1)) \times \mathrm{Hom}_{O_Y}(O_{E(\alpha)}(-1), O_F)\\ &\qquad \to \mathrm{Ext}^1_{O_Y}(O_y, O_F).\end{aligned} \tag{4.7.4}$$

Lemma 4.8. *Suppose $G \subset \mathrm{GL}(2)$. Let F be the fundamental divisor, and C any irreducible component of F. Then we have*

$$\mathrm{Ext}^q_{O_Y}(O_y, O_C(-1)) \simeq \mathrm{Ext}^{2-q}_{O_Y}(O_C(-1), O_y)^\vee = \begin{cases} k & (q = 1, 2, y \in C) \\ 0 & (otherwise) \end{cases}$$

$$\mathrm{Ext}^q_{O_Y}(O_y, O_F) \simeq \mathrm{Ext}^{2-q}_{O_Y}(O_F, O_y)^\vee = \begin{cases} k & (q = 1, 2, y \in F) \\ 0 & (otherwise) \end{cases}$$

Lemma 4.9. *Suppose $G \subset \mathrm{GL}(2)$. Let F be the fundamental divisor, and E, C any irreducible component of F with $E \neq C$. Then we have*

$$\mathrm{Ext}^q_{O_Y}(O_C(-1), O_E(-1)) = \begin{cases} k & (q = 1, CE = 1) \\ 0 & (otherwise) \end{cases}$$

$$\mathrm{Ext}^q_{O_Y}(O_E(-1), O_E(-1)) = \begin{cases} k & (q = 0) \\ k^{\oplus(1+d)} & (q = 2, E^2 = -2 - d) \\ 0 & (otherwise) \end{cases}$$

$$\mathrm{Ext}^q_{O_Y}(O_F, O_E(-1)) = \begin{cases} k^{\oplus e} & (q = 2, FE = -e < 0) \\ 0 & (q = 2, FE = 0) \\ 0 & (otherwise) \end{cases}$$

where $d = 0$ and $e \leq 2$ if $G \subset \mathrm{SL}(2)$ [1].

The proof of the lemmas is easy, so we omit it.

Lemma 4.10. *Suppose $G \subset \mathrm{SL}(2)$. Then the following is true:*

1. *the cup product of (4.7.2) is* $\begin{cases} \neq 0 & \text{if } \{y\} = E(\alpha) \cap E(\rho) \\ 0 & (otherwise) \end{cases}$
2. *the cup product of (4.7.3) is* $\begin{cases} \neq 0 & \text{if } FE(\alpha) \neq 0, y \in E(\alpha) \\ 0 & (otherwise) \end{cases}$

Proof. First we prove (1). Let $E = E(\alpha)$, $C = E(\rho)$, $A = O_E(-1)$ and $B = O_C(-1)$. Assume $E \neq C$. It suffices to consider the case $\{y\} = E \cap C$ by Lemma 4.8. Hence $EC = 1$. Since $y \in E$, we have an exact sequence

$$(u) \quad : \quad 0 \to O_E(-1) \to O_E \to O_y \to 0$$

which is a nontrivial extension given by a nonzero element of $\mathrm{Ext}^1_{O_Y}(O_y, A)$. Let $\gamma \in \mathrm{Hom}_{O_Y}(B, O_y)$ be a nonzero class. The cup product (4.7.2) is the pullback $\gamma^*(u)$ of (u), which is given explicitly by

$$0 \to O_E(-1) \to O_{E+C}(L) \to O_C(-1) \to 0$$

where L is the unique line bundle of $E + C$ such that $L_E = O_E$ and $L_C = O_C(-1)$. This is a nontrivial extension because the following is nontrivial:

$$0 \to O_E(-1) \to O_{E+C} \to O_C \to 0.$$

[1] $e = 2$ in the A_1 case; otherwise $e \leq 1$. See Lemma 4.10, Step 3.

If $E = C$, then $\gamma^*(u)$ is trivial because $\mathrm{Ext}^1_{O_Y}(A, A) = 0$ by Lemma 4.9.

Next we prove (2) in 3 steps. Let $E = E(\alpha)$. We note that $FE \neq 0$ implies $FE = -2$ in the case A_1, while $FE = -1$ in the other cases.

Step 1. Suppose $FE = -1$. It suffices to consider the case $y \in E$. Let $H = F - E$. Let $\theta \in \mathrm{Hom}_{O_Y}(O_E(-1), O_F)$ be a nonzero element. From the exact sequence $0 \to O_E(-1) \to O_F \to O_H \to 0$, we obtain an injection ϕ

$$0 = \mathrm{Hom}(O_y, O_H) \to \mathrm{Ext}^1(O_y, O_E(-1)) \xrightarrow{\phi} \mathrm{Ext}^1(O_y, O_F).$$

Since $\mathrm{Hom}_{O_Y}(O_E(-1), O_F)$ is one-dimensional by Lemma 4.9, ϕ is the cup product (4.7.4). Thus (4.7.4) is nonzero.

Step 2. If $FE = 0$, then $\mathrm{Ext}^2_{O_Y}(O_F, O_E(-1)) = 0$ by Lemma 4.9, hence the cup product (4.7.3) is zero.

Step 3. If $FE = -2$, it is the A_1 case, where $F = E$, $FE = E^2 = -2$ and $\mathrm{Ext}^2_{O_Y}(O_F, O_E(-1))$, hence $\mathrm{Hom}_{O_Y}(O_E(-1), O_F)$ is 2-dimensional by Lemma 4.9. There exists a nonzero $\theta \in \mathrm{Hom}_{O_Y}(O_E(-1), O_F)$ such that

$$0 \to O_E(-1) \xrightarrow{\theta} O_F\ (= O_E) \to O_{y'} \to 0$$

is exact for some $y' \neq y$, $y' \in E$. Then we have an injection ϕ

$$0 = \mathrm{Hom}(O_y, O_{y'}) \to \mathrm{Ext}^1(O_y, O_E(-1)) \xrightarrow{\phi} \mathrm{Ext}^1(O_y, O_F)$$

By Lemma 4.8, ϕ is an isomorphism and the cup product (4.7.4) is nonzero. This proves (2). □

Summarizing Subsections 4.4–4.7, we obtain the following.

Theorem 4.11. *Suppose $G \subset \mathrm{SL}(2)$. Under the above notation especially with $\xi = \alpha$, let $y \in F$, $V(\alpha) \subset \mathrm{Soc}(I_y)$* [2] *and $W(\rho) \subset \mathrm{Gen}(I_y)$. Then the following are equivalent:*

1. *$W(\rho) \subset S_1 \cdot V(\alpha)$ in $\mathrm{Gen}(I_y)$,*
2. *the extension (4.5.1) is nontrivial,*
3. *the cup product (4.5.2) is nonzero,*
4. *the cup product (4.7.1) is nonzero,*
5. *(4.7.2) is nonzero if $\rho \neq \rho_0$, while (4.7.3) is nonzero if $\rho = \rho_0$,*
6. *$E(\alpha)E(\rho) = 1$ if $\rho \neq \rho_0$, while $E(\alpha)(-F) \neq 0$* [3] *if $\rho = \rho_0$.*

5. The global McKay correspondence

5.1. **The ideals n_Y and I_Y.** Let $U = \mathbf{A}^2_k$, $X = U/G$, $Y = G$-Hilb(U) and $f : Y \to X$ the natural morphism, which is the minimal resolution of X. Let $\pi_Y : Y \times_k U \to Y$ and $\pi_U : Y \times_k U \to U$ be the first and the second projection. Since $Y \times_X X \simeq Y$, Y is a closed subscheme of $Y \times_k X$. Let I_Y be the ideal of $O_{Y \times_k X}$ defining Y, and $n_Y = I_Y O_{Y \times_k U}$.

[2]hence $\alpha \neq \rho_0$ by Theorem 3.6

[3]this is equivalent to $\alpha = \rho_{\mathrm{nat}}$ if it is not the A_n-case

Let $\mathcal{Z}$ be the universal subscheme of $Y \times_k U$, and $\mathcal{I}$ the ideal sheaf of $O_{Y\times_k U}$ defining $\mathcal{Z}$. We have a commutative diagram of structure sheaves:

$$\begin{array}{ccc} O_{\mathcal{Z}} & \stackrel{q^*}{\longleftarrow} & O_U \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ O_Y & \stackrel{f^*}{\longleftarrow} & O_X. \end{array}$$

where

$$O_Y \simeq O_Y \otimes_{O_X} O_X \simeq O_Y \otimes_k O_X/I_Y,$$

$$O_Y \otimes_k O_X \stackrel{\mathrm{id}_Y^*\otimes\pi^*}{\to} O_Y \otimes_k O_U \stackrel{p^*\otimes q^*}{\to} O_{\mathcal{Z}},$$

$$O_Y \otimes_k O_X/I_Y \simeq O_Y \stackrel{p^*}{\to} O_{\mathcal{Z}} = O_Y \otimes_k O_U/\mathcal{I}.$$

5.2. **The fundamental divisor F.** Let F be the fundamental divisor of the singularity $(X, 0)$ (see Subsec. 2.3). For a finite subgroup of $\mathrm{SL}(2, k)$ other than cyclic groups, that is, for every rational double singularity other than A_n singularities, there is a unique irreducible component E_{i_0} of $f^{-1}(0)$ such that $FE_{i_0} = -1$ and $FE_j = 0$ $(j \neq i_0)$. Let $C = F_{\mathrm{red}}$. Since

$$O_F(-F) \otimes O_{E_i} \simeq \begin{cases} O_{E_{i_0}}(1) & (i = i_0) \\ O_{E_j} & (i \neq i_0), \end{cases}$$

$O_E(-F)$ is a line bundle of degree one on E.

As is well known, the dual graph of E is identified with the Dynkin diagram Γ of the (corresponding) Lie algebra. The extended Dynkin diagram is obtained by adding to Γ as a vertex the highest root α_0 with $(\alpha_0)^2 = 2$. Then it is natural to identify $-F$ with α_0 in the extended Dynkin diagram.

Lemma 5.2.1. *Let $\mathfrak{m}$ be the (maximal) ideal of O_U defining the origin 0, and I_F the ideal of O_Y defining the fundamental divisor F. Then*

1. *$O_{Y\times_X U} \simeq O_Y \otimes_k O_U/\mathfrak{n}_Y$, that is, $\mathfrak{n}_Y$ is the ideal of $O_Y \otimes_k O_U$ defining the fiber product $Y \times_X U$.*
2. *$\mathcal{Z} \subset Y \times_X U$, that is, $\mathfrak{n}_Y \subset \mathcal{I}$,*
3. *$\pi_U^*\mathfrak{m} + \mathfrak{n}_Y = \pi_U^*\mathfrak{m} + I_F O_{Y\times_k U} = \pi_U^*\mathfrak{m} + \mathcal{I}$,*
4. *$\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y\mathcal{I} = \mathfrak{m}\mathcal{I} + \mathcal{I}^2 = \mathfrak{m}\mathcal{I} + I_F\mathcal{I}$.*

Proof. Let $\mathfrak{m}_X$ be the (maximal) ideal of O_X defining the singular point 0. The ideal I_Y of $O_Y \otimes_k O_X$ is generated by $(f^*a) \otimes 1 - 1 \otimes a$ for $a \in \mathfrak{m}_X$. Therefore $\mathfrak{n}_Y$ is generated by $(f^*a) \otimes 1 - 1 \otimes (\pi^*a)$ for $a \in \mathfrak{m}_X$. Hence $O_Y \otimes_k O_U/\mathfrak{n}_Y$ is the structure sheaf of the fiber product $Y \times_X U$. Since $\mathcal{Z} \subset Y \times_X U$, we have $I_Y \subset \mathcal{I}$, so that $\mathfrak{n}_Y \subset \mathcal{I}$. This proves part (2).

The inclusion $\pi_U^*\mathfrak{m} + \mathfrak{n}_Y \subset \pi_U^*\mathfrak{m} + \mathcal{I}$ is clear. We shall prove the converse. Since $\mathfrak{n}_Y$ is generated by $(f^*a)\otimes 1 - 1\otimes(\pi^*a)$ for $a \in \mathfrak{m}_X$, we have $(f^*a)\otimes 1 \in \pi_U^*\mathfrak{m} + \mathfrak{n}_Y$ because $\pi^*a \in \mathfrak{m}$. Thus $\pi_U^*\mathfrak{m} + \mathfrak{n}_Y = \pi_U^*\mathfrak{m} + I_F O_{Y\times U}$.

We see

$$\begin{aligned} O_{Y\times_k U}/(\pi_U^*\mathfrak{m}+I_F O_{Y\times_k U}) &\simeq (O_Y/I_F)\otimes_{O_{Y\times_k U}}(O_U/\mathfrak{m}) = O_F, \\ O_{Y\times_k U}/(\pi_U^*\mathfrak{m}+\mathcal{I}) &\simeq (O_{Y\times_k U}/\mathcal{I})\otimes_{O_{Y\times_k U}}(O_U/\mathfrak{m}) \\ &\simeq O_{\mathcal{Z}}\otimes_{O_{Y\times_k U}}(O_U/\mathfrak{m}) = O_F \end{aligned}$$

by Lemma 3.4.4 (2). It follows that

$$\pi_U^*\mathfrak{m}+I_F O_{Y\times U} = \pi_U^*\mathfrak{m}+\mathcal{I}.$$

This proves Lemma. □

Lemma 5.2.2. *Let $\mathfrak{n} := (\mathfrak{m}\cap k[x,y]^G)O_U$. Then the following is true:*

1. $I_{Z_y}\simeq \mathcal{I}/\mathfrak{m}_y\mathcal{I}\simeq (\mathcal{I}/\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U))$,
2. $\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U) = \mathfrak{m}_y\mathcal{I}$,
3. $\mathfrak{n}_Y+\mathfrak{m}_y\otimes_k O_U = O_Y\otimes_k\mathfrak{n}+\mathfrak{m}_y\otimes_k O_U$ *if* $y\in F$,
4. $(\mathfrak{n}_Y+\mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I}\simeq\mathfrak{n}$ *if* $y\in F$.

Proof. Since $O_{\mathcal{Z}}$ is O_Y-flat,

$$\begin{aligned} O_{Z_y} &\simeq O_{\mathcal{Z}}\otimes_{O_Y} O_y = (O_{Y\times_k U}/\mathcal{I})\otimes_{O_Y} O_y \\ &\simeq (O_Y\times_k O_U)/(\mathcal{I}+\mathfrak{m}_y\otimes_k O_U) \\ &\simeq (O_y\times_k O_U)/(\mathcal{I}+\mathfrak{m}_y\otimes_k O_U)/(\mathfrak{m}_y\otimes_k O_U) \\ &\simeq (O_y\times_k O_U)/\left(\mathcal{I}/\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U)\right), \end{aligned}$$

whence $I_{Z_y}\simeq\mathcal{I}/\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U)$. This proves the part (1).

Since $O_{\mathcal{Z}}$ is O_Y-flat again, the following is an exact sequence:

$$0\to\mathcal{I}\otimes_{O_{Y\times_k U}} O_y\to O_y\otimes_k O_U\ (\simeq O_U)\to O_{Z_y}(\simeq O_{\mathcal{Z}}\otimes_{O_{Y\times_k U}} O_y)\to 0,$$

so that we have $I_{Z_y}=\mathcal{I}/\mathfrak{m}_y\mathcal{I}$. Hence we have

$$\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U)=\mathfrak{m}_y\mathcal{I},$$

which proves the part (2). Since $\mathfrak{n}_Y$ is generated by $f^*a\otimes 1-1\otimes\pi^*a$ for $a\in\mathfrak{m}_X$ and $\pi^*\mathfrak{m}_X=\mathfrak{n}$, we have $\mathfrak{n}_Y+\mathfrak{m}_y\otimes_k O_U = O_Y\otimes_k\mathfrak{n}+\mathfrak{m}_y\otimes_k O_U$ for $y\in F$, which is the part (3). It follows that

$$\begin{aligned} (\mathfrak{n}_Y+\mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I} &= (\mathfrak{n}_Y+\mathfrak{m}_y\mathcal{I})/\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U) \\ &\simeq \mathfrak{n}_Y/\mathfrak{n}_Y\cap\mathcal{I}\cap(\mathfrak{m}_y\otimes_k O_U) \\ &= \mathfrak{n}_Y/\mathfrak{n}_Y\cap(\mathfrak{m}_y\otimes_k O_U)\quad\text{by Lemma 5.2.1 (2)} \\ &\simeq \mathfrak{n}_Y+\mathfrak{m}_y\otimes_k O_U/(\mathfrak{m}_y\otimes_k O_U) \\ &\simeq (O_Y\otimes_k\mathfrak{n}+\mathfrak{m}_y\otimes_k O_U)/(\mathfrak{m}_y\otimes_k O_U)\simeq\mathfrak{n}. \end{aligned}$$

This completes the proof. □

Definition 5.2.3. We define

$$\begin{aligned} \mathcal{V} &:= \mathcal{I}/(\mathfrak{m}\mathcal{I}+\mathfrak{n}_Y), \\ \mathcal{V}^\dagger &:= \mathcal{I}/(\mathfrak{m}\mathcal{I}+I_F\mathcal{I})\simeq(\mathcal{I}/\mathfrak{m}\mathcal{I})\otimes_{O_Y} O_F. \end{aligned}$$

Lemma 5.2.4. *For $y \in F$,*

$$\mathcal{V} \otimes_{O_Y} O_y \simeq I_{Z_y}/(\mathfrak{m} I_{Z_y} + \mathfrak{n});\ \mathcal{V}^\dagger \otimes_{O_Y} O_y \simeq I_{Z_y}/\mathfrak{m} I_{Z_y}.$$

Proof. Let $y \in Y$ and $\mathfrak{m}_y$ the maximal ideal of $O_{Y,y}$. Hence $O_y = O_{Y,y}/\mathfrak{m}_y = O_y$. Since $I_{Z_y} = \mathcal{I}/\mathfrak{m}_y\mathcal{I}$, we have

$$\begin{aligned}
\mathcal{V} \otimes_{O_Y} O_y &\simeq (\mathcal{I}/(\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y)) \otimes_{O_Y} O_y \\
&= \mathcal{I}/(\mathfrak{m}\mathcal{I} + \mathfrak{n}_Y + \mathfrak{m}_y\mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y\mathcal{I})/\left(\mathfrak{m}(\mathcal{I}/\mathfrak{m}_y\mathcal{I}) + (\mathfrak{n}_Y + \mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I}\right) \\
&\simeq I_{Z_y}/(\mathfrak{m} I_{Z_y} + \mathfrak{n}), \\
\mathcal{V}^\dagger \otimes_{O_Y} O_y &= \mathcal{I}/(\mathfrak{m}\mathcal{I} + I_F\mathcal{I} + \mathfrak{m}_y\mathcal{I}) = \mathcal{I}/(\mathfrak{m}\mathcal{I} + \mathfrak{m}_y\mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y\mathcal{I})/(\mathfrak{m}\mathcal{I} + \mathfrak{m}_y\mathcal{I})/\mathfrak{m}_y\mathcal{I}) \\
&\simeq (\mathcal{I}/\mathfrak{m}_y\mathcal{I})/m(\mathcal{I}/\mathfrak{m}_y\mathcal{I}) \simeq I_{Z_y}/\mathfrak{m} I_{Z_y}.
\end{aligned}$$

□

Definition 5.3. For a coherent $O_{Y\times U}$-module J, we define a functor

$$\Psi_J : D_c^G(U) \to D_c(Y)$$

by

$$\Psi_J(A) = [p_*(\mathbf{L}\pi_U^*(A) \overset{\mathbf{L}}{\otimes}_{O_{Y\times_k U}} J)]^G = \mathbf{R}(\pi_Y)_*(\mathbf{L}\pi_U^*(A) \overset{\mathbf{L}}{\otimes}_{O_{Y\times_k U}} J)]^G$$

where $A \in D_c^G(U)$. Note that $\Psi = \Psi_{O_{\mathcal{Z}}}$.

Lemma 5.4. *The following is true:*

1. $\Psi(O_0 \otimes_k \rho_i^*) = \begin{cases} O_F & (i = 0), \\ O_{E_i}(-1)[1] & (\rho_i \textit{ : nontrivial special}), \end{cases}$
2. $\Psi_{O_{Y\times_k U}}(O_0 \otimes_k \rho_i^*) = \begin{cases} O_Y & (i = 0), \\ 0 & (i \neq 0), \end{cases}$
3. $\Psi_{\mathcal{I}}(O_0 \otimes_k \rho_i^*) = \begin{cases} O_Y(-F) & (i = 0), \\ O_{E_i}(-1) & (i \neq 0), \\ 0 & \textit{(otherwise).} \end{cases}$

Proof. The part (1) follows from Lemma 3.4.4. Since $O_{Y\times_k U}$ is $O_{Y\times_k U}$-flat, by definition,

$$\begin{aligned}
\Psi_{O_{Y\times_k U}}(O_0 \otimes_k \rho_i^*) &= [p_*(O_Y \otimes_k (O_0 \otimes_k \rho_i^*))]^G \\
&= [O_Y \otimes_k \rho_i^*]^G = \begin{cases} O_Y & (i = 0) \\ 0 & (i \neq 0) \end{cases}
\end{aligned}$$

The part (3) follows from the parts (1) and (2) and the exact sequence:

$$\longrightarrow \Psi^{-2}_{\mathcal{I}}(O_0 \otimes_k \rho_0^*) \to \Psi^{-2}_{O_{Y\times_k U}}(O_0 \otimes_k \rho_0^*) \to \Psi^{-2}_{O_{\mathcal{Z}}}(O_0 \otimes_k \rho_0^*)$$
$$\longrightarrow \Psi^{-1}_{\mathcal{I}}(O_0 \otimes_k \rho_0^*) \to \Psi^{-1}_{O_{Y\times_k U}}(O_0 \otimes_k \rho_0^*) \to \Psi^{-1}_{O_{\mathcal{Z}}}(O_0 \otimes_k \rho_0^*)$$
$$\longrightarrow \Psi^{0}_{\mathcal{I}}(O_0 \otimes_k \rho_0^*) \longrightarrow \Psi^{0}_{O_{Y\times_k U}}(O_0 \otimes_k \rho_0^*) \to \Psi^{0}_{O_{\mathcal{Z}}}(O_0 \otimes_k \rho_0^*) \longrightarrow 0.$$

This proves the part (3). □

The goal of this section is to prove the following

Theorem 5.5. *(The global McKay correspondence)*

$$\mathcal{V} \simeq \sum_{\rho_i \neq \rho_0} O_{E_i}(-1) \otimes_k \rho_i,$$
$$\mathcal{V}^\dagger \simeq \mathcal{V} \oplus O_F(-F) \otimes_k \rho_0,$$

where ρ_i ranges over all non-trivial special irreducible representations of G.

Corollary 5.6. *Let $\mathfrak{n} = (\mathfrak{m} \cap k[x,y]^G)O_U$. Then*

$$\mathcal{V} \otimes_{O_Y} O_y = \begin{cases} \rho_i & (y \in E_i \setminus \cup_{j\neq i} E_j) \\ \rho_i \oplus \rho_j & (y \in E_i \cap E_j,\ i \neq j), \end{cases}$$
$$\mathcal{V}^\dagger \otimes_{O_Y} O_y = \begin{cases} \rho_i \oplus \rho_0 & (y \in E_i \setminus \cup_{j\neq i} E_j) \\ \rho_i \oplus \rho_j \oplus \rho_0 & (y \in E_i \cap E_j,\ i \neq j). \end{cases}$$

We note $E_i = C(\rho_i)$ and $\mathrm{Gen}(I_{Z_y}) = I_{Z_y}/\mathfrak{m}I_{Z_y}$ in Theorem 1.1.

Proof of Theorem 5.5 and Corollary 5.6. By Lemma 5.4, we have

$$\mathcal{I}/\mathfrak{m}\mathcal{I} = p_*(\mathbf{L}\pi_U^*(O_0) \overset{\mathbf{L}}{\otimes}_{O_{Y\times_k U}} \mathcal{I}) \simeq \sum_{\rho:\text{irred.}} \Psi_{\mathcal{I}}(O_0 \otimes_k \rho^*) \otimes \rho$$
$$\simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_k \rho_i \bigoplus O_Y(-F) \otimes_k \rho_0,$$

which is an isomorphism in $D_c(Y)$. However since the rhs is concentrated to degree zero only, it is an isomorphism of O_Y-modules. It follows

$$\mathcal{V}^\dagger = \mathcal{I}/(\mathfrak{m}\mathcal{I} + I_F\mathcal{I}) \simeq (\mathcal{I}/\mathfrak{m}\mathcal{I}) \otimes_{O_Y} O_F$$
$$\simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_k \rho_i \bigoplus O_F(-F) \otimes_k \rho_0.$$

It remains to compute $\mathcal{V}$. By Lemma 5.2.4,

$$I_{Z_y}/\mathfrak{m}I_{Z_y} = \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_{O_Y} O_y \otimes_k \rho_i \bigoplus O_y(-F) \otimes_k \rho_0,$$
$$I_{Z_y}/(\mathfrak{m}I_{Z_y} + \mathfrak{n}) \simeq \bigoplus_{i=1}^n O_{E_i}(-1) \otimes_{O_Y} O_y \otimes_k \rho_i$$

where every generator of $O_F(-F) \otimes_k \rho_0 \subset \mathcal{V}^\dagger$ is the image of a G-invariant polynomial in $\mathcal{I}$, which reduces to zero in $\mathcal{V}$ because $\mathfrak{m} \cap k[x,y]^G \subset \mathfrak{n}$. Hence

$$\mathcal{V} = \bigoplus_{i=1}^{n} O_{E_i}(-1) \otimes_k \rho_i.$$

This proves Theorem 5.5 and Corollary 5.6. □

6. The case D_5

In this section we discuss the case D_5 in detail.

6.1. **The binary dihedral group $\mathbf{D}_3$.** Let $S = k[x,y]$ and $U = \mathbf{A}_k^2 = \operatorname{Spec} S$. The simple singularity D_5 is the quotient singularity of U by the binary dihedral group $G := \mathbf{D}_3$ of order 12, which is generated by σ and τ:

$$\sigma = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\epsilon := e^{2\pi i/6}$. We have $\sigma^6 = \tau^4 = 1$, $\sigma^3 = \tau^2$ and $\tau\sigma\tau^{-1} = \sigma^{-1}$. The group G acts on U from the right by $(x,y) \mapsto (x,y)g$ for $g \in G$, hence, τ acts on the ring S is defined to be $\tau(x) = -y$ and $\tau(y) = x$. The ring of G-invariants in S is generated by three elements

$$A_6 := x^6 + y^6,\ A_8 := xy(x^6 - y^6),\ A_4 := x^2y^2.$$

The quotient U/G is isomorphic to the hypersurface $4A_4^4 + A_8^2 - A_4A_6^2 = 0$. Let $\mathfrak{n}$ be the ideal of S generated by A_4, A_6 and A_8.

ρ	$\operatorname{Tr}\rho$	1	σ	τ
ρ_0	χ_0	1	1	1
ρ_1	χ_1	1	1	-1
ρ_2	χ_2	2	1	0
ρ_3	χ_3	2	-1	0
ρ_4	χ_4	1	-1	i
ρ_5	χ_5	1	-1	$-i$

Table 1. Character table of $\mathbf{D}_3$ (of type D_5)

6.2. **The coinvariant algebra of D_5.** We consider the case of D_5. Let the coinvariant algebra of D_5 be $\operatorname{Coinv}(D_5) := S/\mathfrak{n}$. The irreducible decompositions of S and $\operatorname{Coinv}(D_5)$ are given in Table 2.

The algebra $\operatorname{Coinv}(D_5)$ admits a quiver structure induced from multiplication of the symmetric algebra, which is given by Figure 1. This gives the extended Dynkin diagram $\widetilde{D}_5$. See Subsec. 6.4. It is clear that one has the Dynkin diagram (that is, the dual graph) of the exceptional set of the minimal resolution $Y := \operatorname{Hilb}^G(U)$ by removing the vertex at ρ_0.

k	S_k	$S_k(S/\mathfrak{n})$
0	ρ_0	ρ_0
1	ρ_2	ρ_2
2	$\rho_1+\rho_3$	$\rho_1+\rho_3$
3	$\rho_2+\rho_4+\rho_5$	$\rho_2+\rho_4+\rho_5$
4	$\rho_0+2\rho_3$	$2\rho_3$
5	$2\rho_2+\rho_4+\rho_5$	$\rho_2+\rho_4+\rho_5$
6	$\rho_0+2\rho_1+2\rho_3$	$\rho_1+\rho_3$
7	$3\rho_2+\rho_4+\rho_5$	ρ_2

TABLE 2. Irreducible decompositions of S and Coinv (D_5)

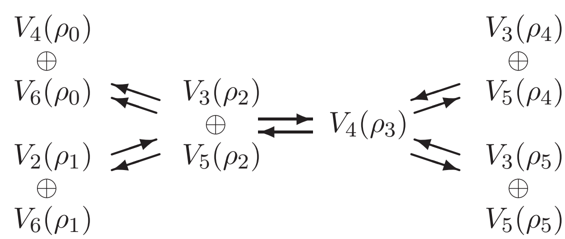

FIGURE 1. Quiver structure of D_5

Contrary to the ordinary simply-laced diagrams, we draw the extended Dynkin diagram as in Figure 1. See also [20]. In the rest of this section we explain this extended Dynkin diagram of D_5 by applying Subsection 4.4. In what follows, let $E(\rho_i)=E_i$, which is the same as $C(\rho_i)$ in [15].

6.3. **The curves $E(\rho_i)$ and the points $P(\rho_i,\rho_j)$.** Let

$$V_2(\rho_1)=\{xy\},\quad V_6(\rho_1)=\{x^6-y^6\},$$
$$V_3(\rho_2)=\{x^2y,-xy^2\},\quad V_5(\rho_2)=\{y^5,-x^5\},$$

where $\{a_1,a_2,\cdots\}$ denotes the k-vector space spanned by $a_1,a_2,\cdots$.

Following [15] we define for a nonzero G-submodule W of $V_2(\rho_1)+V_6(\rho_1)$,

$$I_1(W)=W+\sum_{k=1}^{5}S_kV_2(\rho_1)+\mathfrak{n}.$$

We see that

$$E(\rho_1)=\{I_1(W);\rho_1\simeq W\subset V_2(\rho_1)\oplus V_6(\rho_1)\},$$
$$P(\rho_1,\rho_2)=\{I_1(V_6(\rho_1))\}=\{V_6(\rho_1)S+V_3(\rho_2)S+\mathfrak{n}\}.$$

Thus the subset $E(\rho_1)$ is identified with $\mathbf{P}(V_2(\rho_1)\oplus V_6(\rho_1))$, a smooth rational curve consisting of all nontrivial G-submodules of $V_2(\rho_1)\oplus V_6(\rho_1)$.

Similarly we see

$$\begin{aligned} E(\rho_2) &= \{I_2(W); \rho_2\simeq W\subset V_3(\rho_2)\oplus V_5(\rho_2)\},\\ P(\rho_2,\rho_3) &= \{I_2(V_5(\rho_2))\} = \{V_5(\rho_2)S + S_1\cdot V_3(\rho_2)S + \mathfrak{n}\}, \end{aligned}$$

where

$$I_2(W) = W + \sum_{k=1}^{5} S_k V_3(\rho_2) + \sum_{k=1}^{5} S_k V_5(\rho_2) + \mathfrak{n}.$$

6.4. **The extended Dynkin diagram.** The list of generators of $\mathrm{Gen}(I_y)$ $(y\in E)$ is given by

$$\begin{aligned} V_4(\rho_0)\oplus V_6(\rho_0) &= \{x^2y^2\}\oplus\{x^6+y^6\},\\ V_2(\rho_1)\oplus V_6(\rho_1) &= \{xy\}\oplus\{x^6-y^6\},\\ V_3(\rho_2)\oplus V_5(\rho_2) &= \{x^2y, -xy^2\}\oplus\{y^5, -x^5\},\\ V_4(\rho_3) &= \{y^4, x^4\}\oplus\{x^3y, -xy^3\},\\ V_3(\rho_4)\oplus V_5(\rho_4) &= \{x^3+iy^3\}\oplus\{xy(x^3-iy^3)\},\\ V_3(\rho_5)\oplus V_5(\rho_5) &= \{x^3-iy^3\}\oplus\{xy(x^3+iy^3)\}, \end{aligned}$$

where we set $V_4'(\rho_3) = \{y^4, x^4\}$ and $V_4''(\rho_3) = \{x^3y, -xy^3\}$.

Let A and B be one of the G-modules in the above list. Then we draw an arrow from A to B if and only if $S_1\cdot A$ contains B in $\mathrm{Gen}(I_y)$ for some $y\in \mathrm{Sing}\,(F_{\mathrm{red}})$. Thus we have Figure 1 with double *directed* arrows which describes the quiver structure.

Now we explain Figure 1 in detail. For instance, two arrows starting from $V_3(\rho_2)$ in Figure 1 are induced from $S_1\cdot V_3(\rho_2) = V_4(\rho_0)\oplus V_4''(\rho_3)$. Geometrically, this happens at the point $y := P(\rho_2,\rho_3)$, where we see

$$\begin{aligned} I_y &= V_5(\rho_2)S + S_1\cdot V_3(\rho_2)S,\\ \mathcal{V}^\dagger\otimes_{O_Y} O_y &= \mathrm{Gen}(I_y) = V_5(\rho_2)\oplus V_4''(\rho_3)\oplus V_4(\rho_0). \end{aligned}$$

This gives the arrows from $V_3(\rho_2)$ to $V_4(\rho_0)$ and $V_4(\rho_3)$.

The ideal I_y is also given by

$$I_y = V_4''(\rho_3)S + S_1\cdot V_4'(\rho_3)S + V_4(\rho_0)S,$$

because

$$S_1\cdot V_4'(\rho_3) = V_5(\rho_2)\oplus V_5(\rho_4)\oplus V_5(\rho_5).$$

Since $S_1\cdot V_4(\rho_0) = 0$ in $\mathrm{Gen}(I_y)$, we draw the arrows from $V_4(\rho_3)$ to $V_5(\rho_j)$ $(j = 2,4,5)$. The other arrows from or to $V_4(\rho_3)$ are also drawn by (the local structure near) the points $P(\rho_3,\rho_4)$ and $P(\rho_3,\rho_5)$.

The point $z = P(\rho_1,\rho_2)$ is given by the ideal I_z with $\mathrm{Gen}(I_z)$ as below:

$$\begin{aligned} I_z &= V_6(\rho_1)S + S_1\cdot V_2(\rho_1)S,\\ \mathcal{V}^\dagger\otimes_{O_Y} O_z &= \mathrm{Gen}(I_z) = V_6(\rho_1)\oplus V_3(\rho_2)\oplus V_6(\rho_0). \end{aligned}$$

Since $S_1 \cdot V_2(\rho_1) = V_3(\rho_2)$, we draw an arrow from $V_2(\rho_1)$ to $V_3(\rho_2)$. Meanwhile the ideal I_z is also given by

$$I_z = V_3(\rho_2)S + S_1 \cdot V_5(\rho_2)S,$$

whence we draw the arrows from $V_5(\rho_2)$ to $V_6(\rho_1)$ and $V_6(\rho_0)$. Note that $S_1 \cdot V_5(\rho_2) = V_6(\rho_0) \oplus V_6(\rho_1) \oplus V_6(\rho_3)$, where $V_6(\rho_3) = 0$ in $\mathrm{Gen}(I_z)$ because $V_6(\rho_3) \subset V_3(\rho_2)S$. Therefore we draw no arrows from $V_5(\rho_2)$ to $V_6(\rho_3)$ in Figure 1.

Remark 6.5. The similar quiver structure for the other rational double singularity is observed by using [15, Tables 7, 10, 13, 17] where we note that the invariant ring in the E_8-case is generated by polynomials of degree 12, 20 and 30 [15, p. 223]. See also [20].

6.6. **The cup product at $P(\rho_2, \rho_3)$.** Now we apply Subsection 4.4 to explain the above process, by taking up $y = P(\rho_2, \rho_3)$ especially. We recall

$$\begin{aligned}
\mathrm{Soc}(O_{Z_y}) &= V_3(\rho_2) \oplus V_4'(\rho_3),\\
I_y &= V_5(\rho_2)S + S_1 \cdot V_3(\rho_2)S,\\
\mathrm{Gen}(I_y) &= V_5(\rho_2) \oplus V_4''(\rho_3) \oplus V_4(\rho_0).
\end{aligned}$$

At the point $y = P(\rho_2, \rho_3)$, we saw

$$S_1 \cdot V_3(\rho_2) = V_4''(\rho_3) \oplus V_4(\rho_0),$$

where $\dim_k S_1V_3(\rho_2) = 3$ in contrast with $\dim(\rho_{\mathrm{nat}} \otimes_k \rho_2) = 4$. This explains the two arrows from $V_3(\rho_2)$ to $V_4(\rho_0)$ and $V_4(\rho_3)$ in Figure 1, and the nonexistence of arrows from $V_3(\rho_2)$ to $V_5(\rho_2) \subset \mathrm{Gen}(I_y)$.

Now we shall explain this fact using Theorem 4.11. In what follows, for $\rho \in \mathrm{Irr}(G)$ we denote $O_0 \otimes_k \rho \in D_c^G(U)$ simply by ρ if no confusion is possible. We also use the following notation:

$$\begin{aligned}
\mathrm{Ext}^k(A, B) &:= \mathrm{Hom}^k_{D_c(Y)}(A, B),\\
\mathrm{Ext}^k(C, D) &:= \mathrm{Hom}^k_{D_c^G(U)}(C, D),
\end{aligned}$$

for $A, B \in D_c(Y)$ and $C, D \in D_c^G(U)$.

Following Subsec. 4.4, we consider the cup products

$$\begin{aligned}
\theta_{2,2} &: \mathrm{Ext}^0(\rho_2, O_{Z_y}) \times \mathrm{Ext}^1(O_{Z_y}, \rho_2) \longrightarrow \mathrm{Ext}^1(\rho_2, \rho_2)\\
\theta_{2,3} &: \mathrm{Ext}^0(\rho_2, O_{Z_y}) \times \mathrm{Ext}^1(O_{Z_y}, \rho_3) \longrightarrow \mathrm{Ext}^1(\rho_2, \rho_3)\\
\theta_{2,0} &: \mathrm{Ext}^0(\rho_2, O_{Z_y}) \times \mathrm{Ext}^1(O_{Z_y}, \rho_0) \longrightarrow \mathrm{Ext}^1(\rho_2, \rho_0)
\end{aligned}$$

Let $F_i := O_{E(\rho_i)}(-1)$. These are translated into the cup product as in Subsec. 4.4:

$$\begin{aligned}
\theta_{2,2} &: \mathrm{Ext}^1(O_y, F_2) \times \mathrm{Ext}^0(F_2, O_y) \longrightarrow \mathrm{Ext}^1(F_2, F_2)\\
\theta_{2,3} &: \mathrm{Ext}^1(O_y, F_2) \times \mathrm{Ext}^0(F_3, O_y) \longrightarrow \mathrm{Ext}^1(F_3, F_2)\\
\theta_{2,0} &: \mathrm{Ext}^1(O_y, F_2) \times \mathrm{Ext}^1(O_F, O_y) \longrightarrow \mathrm{Ext}^2(O_F, F_2)
\end{aligned}$$

Then by Lemmas 4.10 and 4.9, we see

$$\begin{aligned}\mathrm{Ext}^1(\rho_2,\rho_2) &\simeq \mathrm{Ext}^1(F_2,F_2)=0,\\ \mathrm{Ext}^1(\rho_2,\rho_3) &\simeq \mathrm{Ext}^1(F_3,F_2)=k,\\ \mathrm{Ext}^1(\rho_2,\rho_0) &\simeq \mathrm{Ext}^2(O_F,F_2)=k,\end{aligned}$$

and

$$\theta_{2,2}=0, \theta_{2,3}\neq 0, \theta_{2,0}\neq 0.$$

By Theorem 4.11, this is equivalent to the following relation in $O_U/\mathfrak{m}I_y$:

$$S_1\cdot V_3(\rho_2)\cap V_5(\rho_2)=\{0\},\ S_1\cdot V_3(\rho_2)=V_4''(\rho_3)\oplus V_4(\rho_0).$$

where $V_4''(\rho_3)=\mathrm{Gen}(I_y)[\rho_3]$, the ρ_3-part of $\mathrm{Gen}(I_y)$, which is disjoint from $V_4'(\rho_3)=\mathrm{Soc}(O_{Z_y})[\rho_3]$, the ρ_3-part of $\mathrm{Soc}(O_{Z_y})$.

References

[1] M. Artin, J.-L. Verdier, Reflexive modules over rational double points, Math. Ann., **270** (1985) 79-82.
[2] T. Bridgeland, Equivalences of triangulated categories and Fourier-Mukai transforms, Bull. London Math. Soc. **31** (1999) 25–34.
[3] T. Bridgeland, A. King, M. Reid, The McKay correspondence as an equivalence of derived categories, Jour. Amer. Math. Soc., **14** (2001) 535–554.
[4] S. Cautis, A. Craw, T. Logvinenko, Derived Reid's recipe for abelian subgroups of $SL_3(\mathbf{C})$, arXiv:1205.3110v2, math.AG, 26 Sep. 2014.
[5] H. Cartan, S. Eilenberg, Homological algebra, Oxford university press (London) and Princeton university press (Princeton), 1973.
[6] H. Esnault, Reflexive modules on quotient surface singularities, Jour. reine angew. Math., **362** (1985) 63–71.
[7] H. Esnault, H. Knörrer, Reflexive modules over rational double points, Math. Ann., **272** (1985) 545-548.
[8] A. Grothendieck, Éléments de géométrie algébrique, I,II,III,IV, Publ. Math. IHES, **4** (1960), **8** (1961) , **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).
[9] A. Grothendieck, Local cohomology, Lecture Notes in Math., **41**, Springer-Verlag, Berlin Heidelberg New York, 1967.
[10] G. Gonzalez-Sprinberg, J. Verdier, Construction géométrique de la correspondence de McKay, Ann. scient. Éc. Norm. Sup. **16** (1983) 409–449.
[11] R. Hartshorne, Residue and Duality, Lecture Notes in Mathematics, **20**, Springer Verlag, Berlin Heidelberg New York, 1966.
[12] R. Hartshorne, Algebraic Geometry, Graduate texts in mathematics, **52**, Springer Verlag, Berlin Heidelberg New York, 1977.
[13] R. Hartshorne, Stable reflexive sheaves, Math. Ann., **254**, (1980) 121-176.
[14] A. Ishii, On the McKay correspondence for a finite small subgroup of $\mathrm{GL}(2,\mathbf{C})$, Jour. reine angew. Math., **549** (2002) 221–233.
[15] Y. Ito and I. Nakamura, Hilbert schemes and simple singularities, New trends in algebraic geometry, Proceedings of European Math. Conference 1996, London Mathematical Society Lecture Note Series **264**, Cambridge University Press (1999) 151-233.
[16] M. Kapranov, E. Vasserot, Kleinian singularities, derived categories and Hall algebras, Math. Ann., **316**, (2000) 565–576.
[17] Y. Kawamata, Derived categories of toric varieities, arXiv:1412.8040v1, [math:AG] 27 Dec 2014.
[18] J. Lipman, Rational singularities, Publ. Math. Inst. Hautes. Etudes Sci. **36**, (1969) 195–279.

[19] J. McKay, Graphs, singularities, and finite groups, in Santa Cruz, conference on finite groups (Santa Cruz, 1979), Proc. Symp. Pure Math., AMS **37**, 1980, pp. 183–186.

[20] I. Nakamura, McKay correspondence, Centre de Recherches Mathématiques CRM Proceedings and Lecture Notes, Volume 47, 2009, pp. 267-298.

[21] E. Viehweg, Rational singularities of higher dimensional schemes, Proc. Amer. Math. Soc. **63** (1997) 6–8.

[22] J. Wunram, Reflexive modules on quotient surfaces singularities, Math. Ann., **279** (1988) 583–598.

Division of Mathematical and Information Sciences, Hiroshima University, 1-7-1 Kagamiyama, Higashi-Hiroshima, 739-8521, Japan

E-mail address: akira141@hiroshima-u.ac.jp

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan

E-mail address: nakamura@math.sci.hokudai.ac.jp