Hesse cubics and GIT stability

Iku Nakamura

(Hokkaido University)

2014 Feb. 17, Lakeside Lecture

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Aim

- **To compatify the moduli space of abelian var. by $SQ_{g,K}$**

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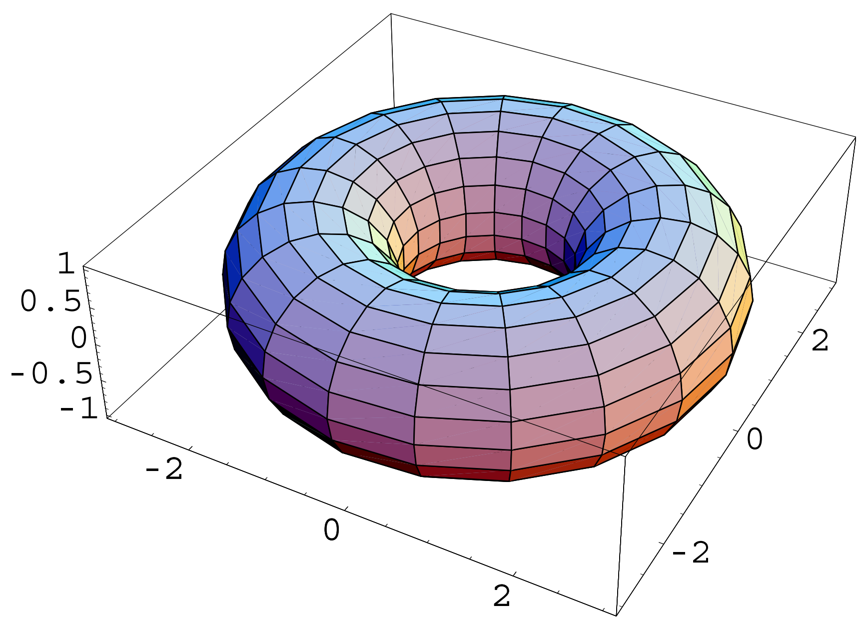
1 Hesse cubic curves

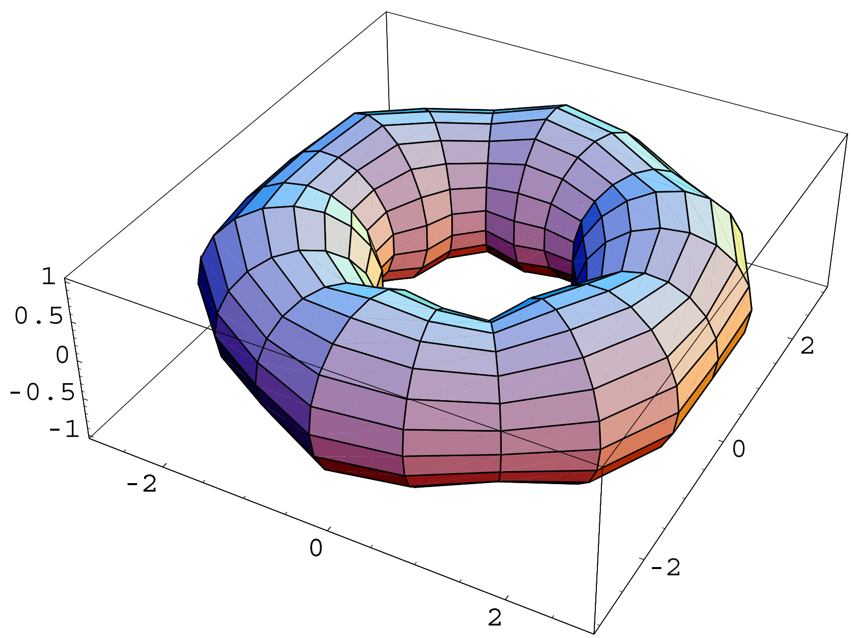
$$C(\mu) : x_0^3 + x_1^3 + x_2^3 - 3\mu\, x_0 x_1 x_2 = 0$$

$$(\mu \in \mathbf{P}^1_{\mathbf{C}})$$



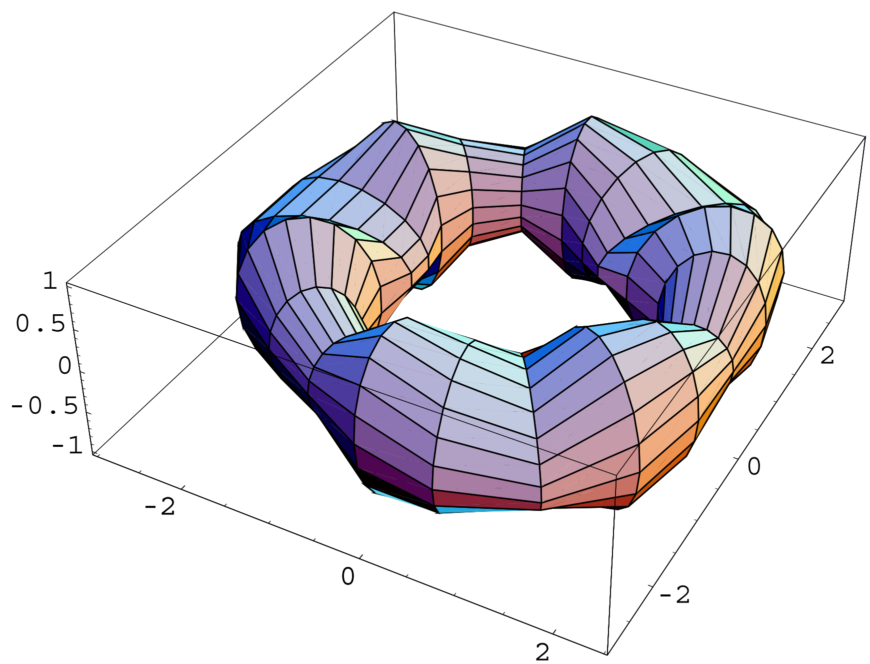
$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0$$

if μ gets closer to ∞



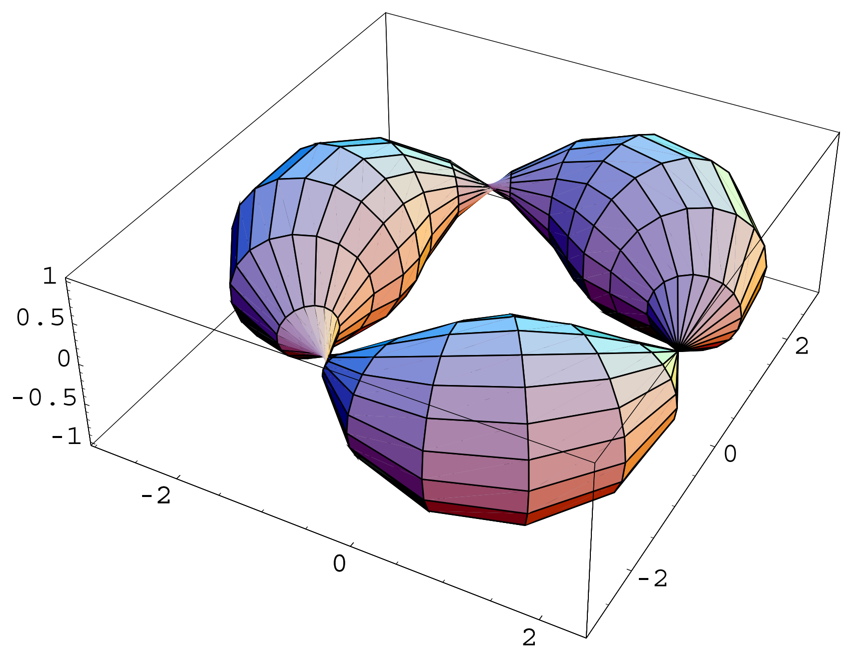
$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu \in \mathrm{C})$$

if μ gets much closer to ∞



$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu^3 = 1 \ \text{or} \ \infty)$$

It degenerates into 3 copies of P^1 ($= S^2$)



2 Moduli of cubic curves

Thm 1 (Hesse 1849)

(1) Any nonsing. cubic curve is transformed

into $C(\mu)$ under SL(3), $(\mu^3 \neq 1, \infty)$

(2) $C(\mu)$ has 9 flexes

$[1 : -\beta : 0]$, $[0 : 1 : -\beta]$,$[-\beta : 0 : 1]$ $(\beta \in \{1, \zeta_3, \zeta_3^2\})$

(3) $C(\mu)$ and $C(\mu')$ are isomorphic with 9 points fixed

if and only if $\mu = \mu'$

Thm 2 (classical form) (Hesse 1849)

$$A_{1,3} := \{\text{nonsing. cubics with 9 flexes}\}/\ \text{isom.}$$

$$= \left\{C(\mu); \mu^3 \neq 1, \infty\right\} \simeq \mathbf{C} \setminus \{1, \zeta_3, \zeta_3^2\}$$

$$SQ_{1,3} := \overline{A_{1,3}}$$

$$= \{\text{stable cubics with 9 flexes}\}/\ \text{isom.}$$

$$= \{\text{Hesse cubics } C(\mu)\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \text{ or } \infty\right\} \simeq \mathbf{P}^1$$

$$= \{\text{moduli of compact objects}\}$$

We wish to extend this to aribitrary dimension

1. over $\mathbf{Z}[\zeta_N, 1/N]$, ζ_N:N-th root of 1
2. to construct a projective fine moduli $SQ_{g,K}$ of compact objects PSQASes,
3. GIT stable objects = PSQASes:

   Projectively **S**table **Q**uasi **A**belian **S**cheme

3 Tate curve and PSQAS

The Tate curve over CDVR R (e.g. $k[[q]]$, $\mathbf{Z}_p$)

$$X : y^2 = x^3 - x^2 + q$$

The fibre $X_0 : y^2 = x^2(x-1)$ for $q = 0$

$$X_0 \setminus \{\text{sing. pt}\} = \mathbf{C}^*$$

Hesse cubics over CDVR R

$$Y_q : q(x_0^3 + x_1^3 + x_2^3) = x_0 x_1 x_2$$

The fibre $Y_0 : x_0 x_1 x_2 = 0$ for $q = 0$,

$$Y_0 \setminus \{\text{sing. pts}\} = \mathbf{C}^* \times (\mathbf{Z}/3\mathbf{Z})$$

R:CDVR, $L = \mathrm{Frac}(R) = R[1/q]$, q uniformizer.

(e.g. $R = k[[q]]$, $L = k((q))$)

Tate curve : $\mathrm{G}_m(L)/w \mapsto qw$

Hesse cubics at ∞ : $\mathrm{G}_m(L)/w \mapsto q^3 w$

Rewrite Tate curve as $\mathrm{G}_m(L)/w^n \mapsto q^{mn} w^n$ $(n \in \mathrm{Z})$

Hesse cubics at ∞ : $\mathrm{G}_m(L)/w^n \mapsto q^{3mn} w^n$ $(n \in \mathrm{Z})$

The general case : B pos. def. symmetric

$$\mathrm{G}_m(L)^g / w^x \mapsto q^{B(x,y)} b(x,y) w^x,$$

$$b(x,y) \in L^\times \quad (x \in X = \mathrm{Z}^g, y \in Y = \mathrm{Z}^g)$$

The general case : B pos. def. symmetric

The generic fibre:

$$\mathrm{G}_m(L)^g/w^x \mapsto q^{B(x,y)}b(x,y)w^x,$$

$$b(x,y) \in L^\times \quad (x \in X = \mathbf{Z}^g, y \in Y = \mathbf{Z}^g)$$

PSQAS is the closed fibre of it

Projectively Stable Quasi Abelian Scheme

This is a generalization of Hesse cubics.

What do they look like ?

4 The shape of PSQASes — Delaunay decompositions

"Limits of theta functions are described by

Delaunay decomp of B."

PSQAS is a geometric limit of thetas

PSQAS is a generalization of 3-gons.

The general case : B pos. def. symmetric

$$G_m(R)^g / w^x \mapsto q^{B(x,y)} b(x,y) w^x,$$

$$b(x,y) \in R^\times \quad (x \in X = \mathbb{Z}^g, y \in Y = \mathbb{Z}^g)$$

PSQAS is the closed fiber of it.

Let $X = \mathrm{Z}^g$, B a positive symmetric on $X \times X$.

$$\|x\| = \sqrt{B(x,x)} : \text{a distance of } X \otimes \mathrm{R} \text{ (fixed)}$$

Def 3 Let $\alpha \in X_{\mathrm{R}}$.

$D(\alpha)$: a Delaunay cell :=

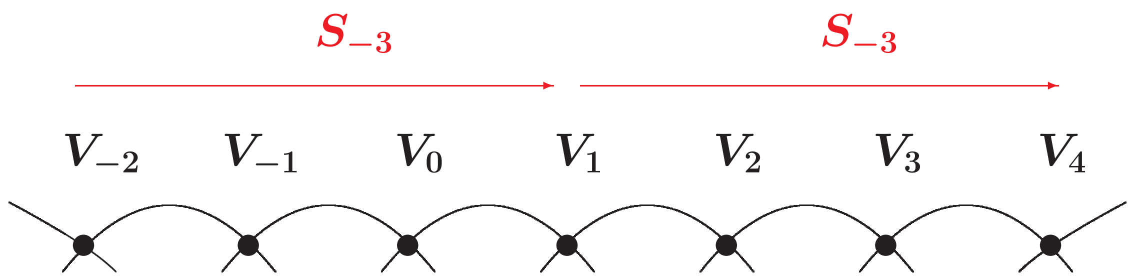
the convex closure of points of X closest to α.

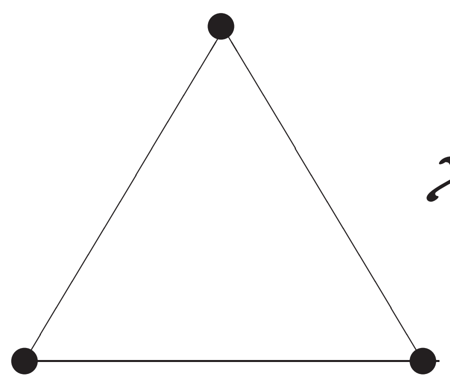
Exam 4 1-dim. $B(x,y) = 2xy$, $X = \mathrm{Z}$, $Y = n\mathrm{Z}$,

then PSQAS Z_0 is an n-gon of P^1

Exam 5 $g = 1,\ X = \mathbf{Z},\ Y = 3\mathbf{Z}.$

$$\mathcal{X} = \mathrm{Proj}(\widetilde{R}), \quad a(x) = q^{x^2},\ (x \in X)$$



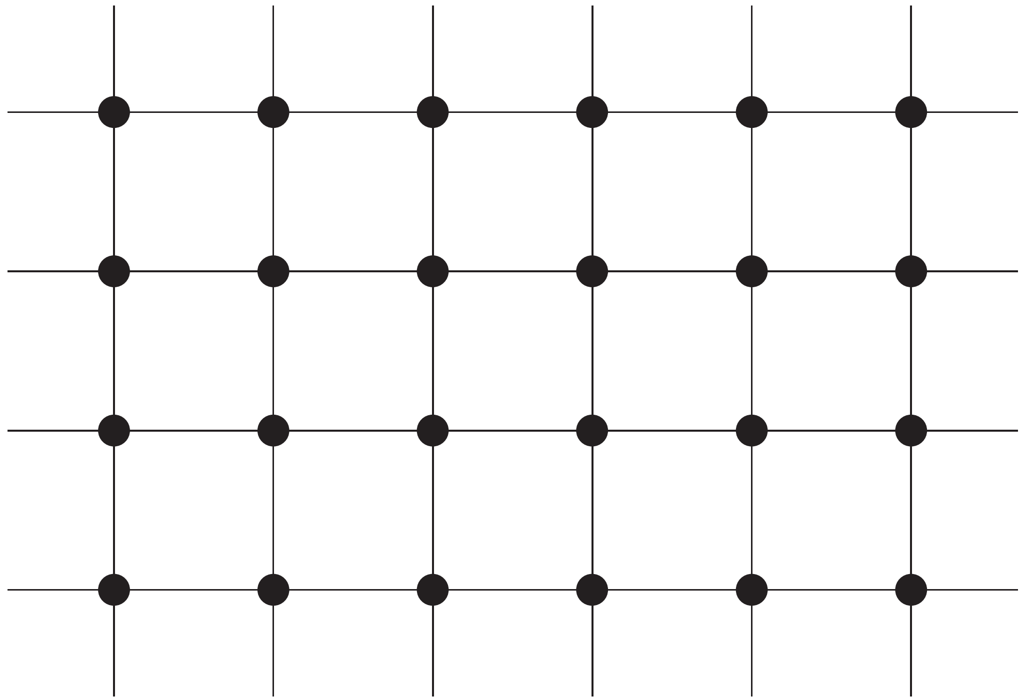


$\mathcal{X}_0/Y$

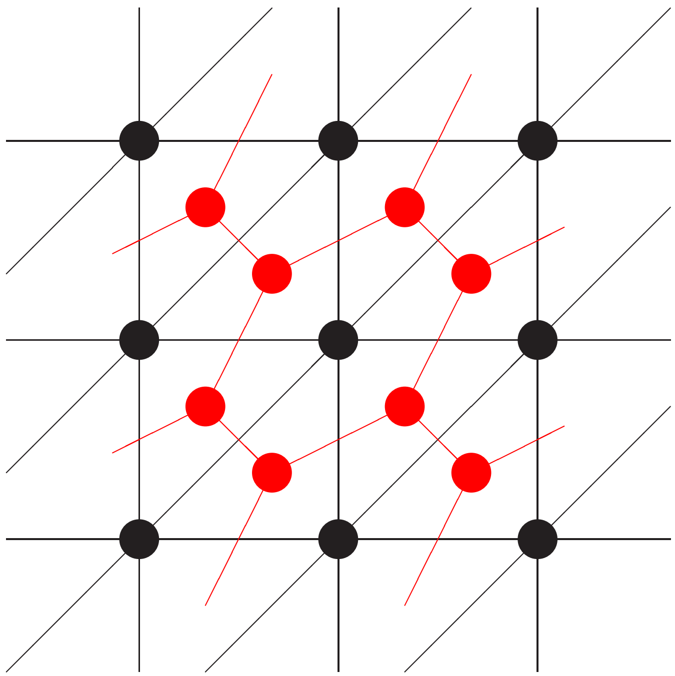
- **Each PSQAS (its scheme struture) and its decomposition into torus orbits (its stratification) are described by Delaunay decomp.**
- **Each pos. symm. B defines a Delaunay decomp.**
- **Different B can yield the same Delaunay decomp. and the same PSQAS.**

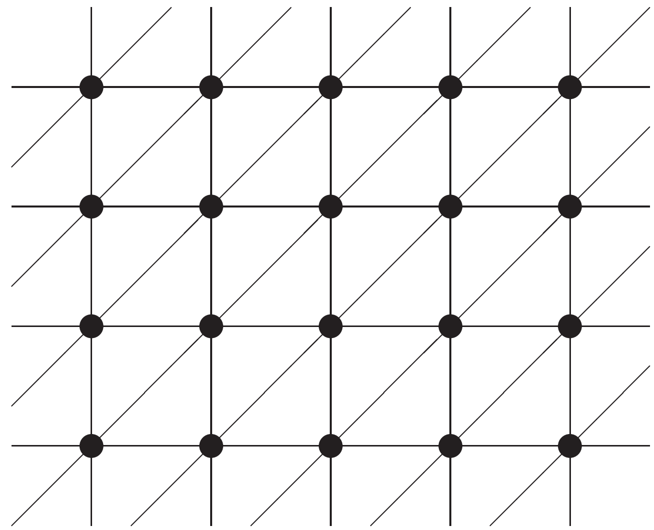
Exam 6 $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Z_0 := \mathcal{X}_0/Y$ is a union of $\mathrm{P}^1 \times \mathrm{P}^1$



Exam 7 $\quad B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$





- **This (mod Y) is a PSQAS.**

  It is a union of P^2, each triangle stands for P^2,

- **each line segment is a P^1, two P^2 intersect along P^1**

- **six P^2 meet at a point,**

  locally $k[x_1, \cdots, x_6]/(x_i x_j, |i - j| \geq 2)$

We re-start with

Thm 8 **Over $\mathbf{Z}[\zeta_3, 1/3]$**

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$\overline{A_{1,3}} := \{\text{stable cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \text{ or } \infty\right\} \simeq \mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}.$$

To construct moduli, consider $G(3)$-equiv. theory

$G(3)$: **Heisenberg group of level 3**

5 Heisenberg group $G(3)$

$G(3) = \langle \sigma, \tau \rangle$ acts on V, order $|G(3)| = 27$,

$$V = Rx_0 + Rx_1 + Rx_2,$$

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathbf{Z}/3\mathbf{Z})$$

ζ_3 is a primitive cube root of 1, $R \ni \zeta_3, 1/3$

Fact

- $x_0^3 + x_1^3 + x_2^3$, $x_0x_1x_2 \in S^3V$ only are $G(3)$-invariant
- $G(3)$ determines x_i "uniquely" ($\because$ V:$G(3)$-irred,)
- x_i are classical theta over $R = \mathbf{C}$

6 (Classical) Theta functions

$E(\tau)$: an elliptic curve $/\mathbb{C}$

$$E(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

This is the same as

$$E(\tau) = \mathbb{C}^*/w \mapsto wq^6,$$

$$(\text{set } q = e^{2\pi i\tau/6}, \quad w = e^{2\pi i z})$$

6 (Classical) Theta functions

$E(\tau)$: an elliptic curve /C

$$E(\tau) = \mathrm{C}^*/w \mapsto wq^6, \quad w = e^{2\pi i z}, q = e^{2\pi i \tau/6}$$

Def 9 Theta functions $(k = 0, 1, 2)$

$$\theta_k(\tau, z) = \sum_{m \in \mathrm{Z}} q^{(k+3m)^2} w^{k+3m}.$$

The following Θ embedds $E(\tau)$ into P^2.

$$\Theta : E(\tau) \ni z \mapsto [x_0, x_1, x_2] = [\theta_0, \theta_1, \theta_2] \in \mathrm{P}^2$$

where $[\theta_0, \theta_1, \theta_2]$ is the ratio of θ_k.

Recall again $w = e^{2\pi i z}$, $q = e^{2\pi i \tau/6}$

$$\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$

$$\theta_k(\tau, z + \frac{\tau}{3}) = (qw)^{-1}\theta_{k+1}(\tau, z),$$

$$[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)$$

σ, τ are their liftings to GL(3),

$z \mapsto z + \frac{1}{3}$ is lifted to $\sigma(\theta_k) = \zeta_3^k \theta_k$

$z \mapsto z + \frac{\tau}{3}$ is lifted to $\tau(\theta_k) = \theta_{k+1}$

$G(3) :=$ the group $\langle \sigma, \tau \rangle$

$$[\boldsymbol{\sigma}, \boldsymbol{\tau}] = \boldsymbol{\sigma\tau\sigma}^{-1}\boldsymbol{\tau}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}.$$

$G(3) = \langle \sigma, \tau \rangle$ is not commutative.

7 The space of closed orbits

Often in algebraic geometry,

$$\text{moduli} = X/G$$

where X a scheme, $G = \mathrm{PGL}(V)$

To be more preicse

X	the set (scheme) of geometric objects
G	the group of isomorphisms
x, x' are isom.	G-orbits are the same $O(x) = O(x')$
X_s	the set of stable objects
X_{ss}	the set of semistable objects
$X_{ss}//G$	"compact moduli"

Exam 10 Action on $\mathbb{C}^2$ of $G = \mathbb{G}_m (= \mathbb{C}^*)$,

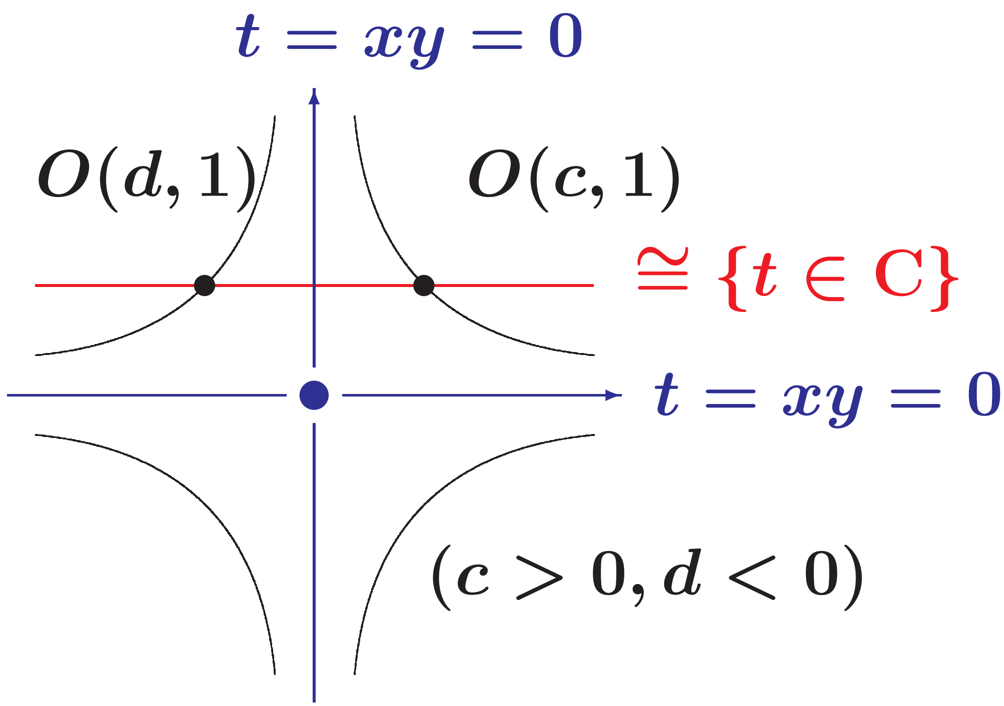
$$\mathbb{C}^2 \ni (x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbb{G}_m)$$

What is the quotient of $\mathbb{C}^2$ by G ?

- Simple answer : the set of G-orbits (×)
- Answer : Spec(the ring of all G-invariant poly.)(○)
- $t := xy$ is the unique G-inv. !

$$\mathbb{C}^2 // G := \operatorname{Spec} \mathbb{C}[t] = \{t \in \mathbb{C}\}$$

But this is different from "the set of G-orbits".



- $t = 0$ is a point of $\mathbb{C} = \mathbb{C}^2//G = \operatorname{Spec} \mathbb{C}[t]$.
- But $\{xy = 0\}$ consists of three G-orbits

$$\mathbb{C}^* \times \{0\}, \quad \{0\} \times \mathbb{C}^*, \quad \{(0,0)\}$$

- $\{(0,0)\}$ is the only closed orbit in $\{xy = 0\}$

Thm 11 $\mathbb{C}^2//G = \{t \in \mathbb{C}\}$ is the set of all closed orbits.

Thm 12 (Seshadri,Mumford)

G : reductive, acting on a scheme X, (e.g. $G = \mathbb{G}_m$).

Let X_{ss} = the set of semistable points. Then

$$X_{ss}//G := \mathrm{Spec}(\text{all } G\text{-invariants})$$
$$= \text{the set of closed orbits.}$$

Closed means that the orbit is closed in X_{ss}.

The most natual choice is objects with closed orbits.

Def 13 The same notation as before. Let $p \in X$.

(1) semistable if $\exists$ G-inv. homog. poly. F, $F(p) \neq 0$,

(2) Kempf-stable if the orbit $O(p)$ is closed in X_{ss},

(3) properly-stable if (2) and Stab(p) finite.

Rem stable $\Longrightarrow$ closed orbit $\Longrightarrow$ semistable

The general theory suggests us to consider

only those objects with closed orbits

We will see

- Abelian varieties have closed orbits (Kempf), and
- our PSQASes have closed orbits,

  Conversely
- Any degenerate abelian scheme with closed orbit is one of our PSQASes
- This enables us to compactify the moduli of abelian varieties.

8 Stable curves of Deligne-Mumford

Def 14 C is a stable curve of a genus g if

(1) connected projective reduced with finite autom.,

(2) the singularities of C are like $xy = 0$

(3) $\dim H^1(O_C) = g$

Let $\overline{M_g}$: moduli of stable curves of genus g,

M_g : moduli of nonsing. curves of genus g.

Thm 15 $\overline{M_g}$ compactifies M_g

(Deligne-Mumford 1969)

Definition of stable curves is irrelevant to GIT stability

Nevertheless

Thm 16 The following are equivalent

(1) C is a stable curve (moduli-stable)

(2) any Hilbert point of $\Phi_{|mK|}(C)$ is GIT-stable

(3) any Chow point of $\Phi_{|mK|}(C)$ is GIT-stable

---

(1)⇔(2) Gieseker 1982 (before Mumford 1977)

(1)⇔(3) Mumford 1977 (suggested by Gieseker 1982)

9 Stability of cubic curves

Cubic cuves	Stability	Stab gp.
smooth elliptic	stable	finite
3-gon	closed orbit	2-dim
a line+a conic (transv.)	semistable	1-dim
irred. with a node	semistable	finite
others	unstable	1-dim

Thm 17 **For a cubic C, the following cond. are equiv.**

(1) C has a closed SL(3)-orbit in $(S^3V)_{ss}$

(2) C is a Hesse cubic curve, that is, $G(3)$-invariant

(3) C is either smooth elliptic or a 3-gon

10 Stability in higher-dim.

Thm 18 (N.1999)

Assume (X, L) is a limit of abelian varieties A with $\ker(\lambda(L)) = K$, $\lambda(L) : A \to A^t$ (dual)

Then the following are equivalent:

(1) X has a closed SL(V)-orbit in Hilb$_{ss}$ (GIT-stable)

(2) X is invariant under $G(K)$ ($G(K)$-stable)

(3) X is one of our PSQASes (moduli-stable)

Thm 19 For cubics the following are equiv:

(1) it has a closed SL(3)-orbit (GIT-stable)

(2) it is a Hesse cubic, that is, $G(3)$-inv. ($G(3)$-stable)

(3) it is smooth ell. or a 3-gon. (moduli-stable)

Thm 20 Let X be a degenerate AV.

The following are equiv. under natural assump.:

(1) it has a closed SL(V)-orbit (GIT-stable)

(2) X is $G(K)$-inv ($G(K)$-stable)

(3) it is a PSQAS (moduli-stable)

11 Moduli over $\mathbf{Z}[\zeta_N, 1/N]$

Thm 21 (a new version of the theorem of Hesse)

$$SQ_{1,3} = \mathrm{P}^1_{\mathbf{Z}[\zeta_3, 1/3]},$$

the projective fine moduli

(1) The universal cubic curve

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$$

where $(\mu_0, \mu_1) \in SQ_{1,3} = \mathrm{P}^1$.

(2) when k is alg. closed and char. $k \neq 3$

$$\boldsymbol{SQ_{1,3}(k)} = \left\{ \begin{array}{l} \textbf{closed orbit cubics} \\ \textbf{with level 3-structure } /\boldsymbol{k} \end{array} \right\} /\textbf{isom.}$$

$$= \left\{ \begin{array}{l} \textbf{Hesse cubics} \\ \textbf{with level 3-str. } /\boldsymbol{k} \end{array} \right\} /\textbf{isom.=id.}$$

$$\boldsymbol{A_{1,3}(k)} = \left\{ \begin{array}{l} \textbf{closed orbit nonsing. cubics} \\ \textbf{with level 3-str. } /\boldsymbol{k} \end{array} \right\} /\textbf{isom.}$$

$$= \left\{ \begin{array}{l} \textbf{nonsing. Hesse cubics} \\ \textbf{with level 3-structure } /\boldsymbol{k} \end{array} \right\} /\textbf{isom.=id.}$$

Thm 22 (N. 1999) There exists the fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$, $N = \sqrt{|K|}$, For k closed

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{degenerate abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

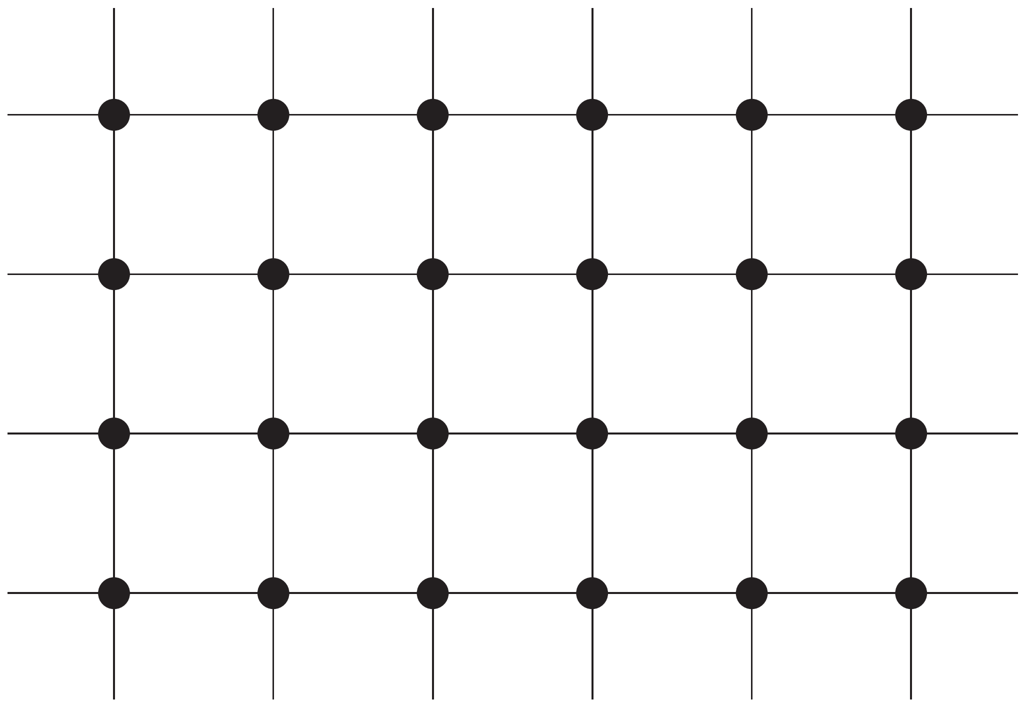
$$= \left\{ \begin{array}{l} G(K)\text{-invariant PSQAS } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\},$$

$$A_{g,K}(k) = \left\{ \begin{array}{l} \text{(nonsingular) abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\} /\text{isom.}$$

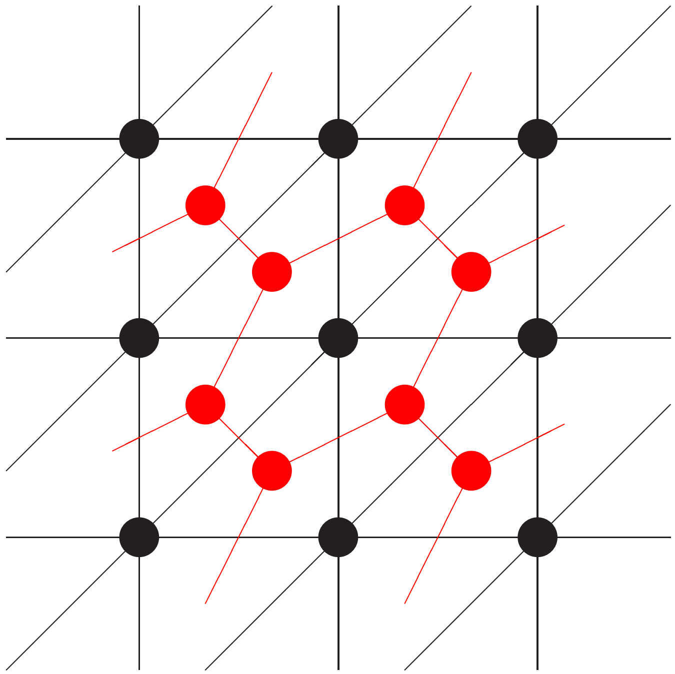
$$= \left\{ \begin{array}{l} G(K)\text{-invariant (nonsingular) AS} \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

12 Delaunay/Voronoi decompositions

Exam 23 $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



Exam 24 $$B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

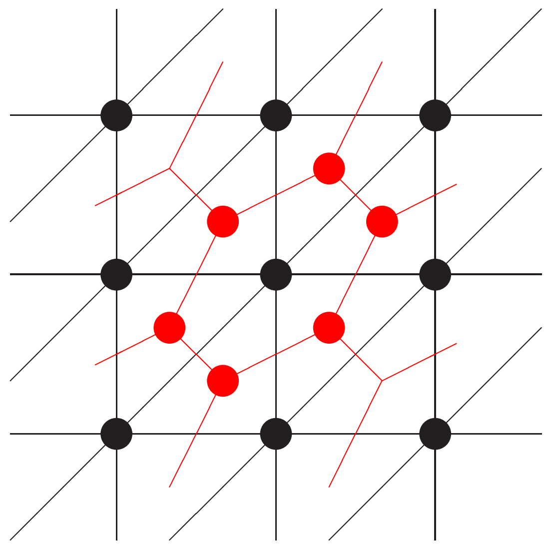


Def 25 $\quad D$: for Delaunay cells

$$V(D) := \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; D = D(\lambda)\}$$

We call it a Voronoi cell

$$\overline{V(0)} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; \|\lambda\| \leqq \|\lambda - q\|, (\forall q \in X)\}$$



This is a crystal of mica.

$$\textbf{For } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get $\overline{V(0)}$, a cube (salt),

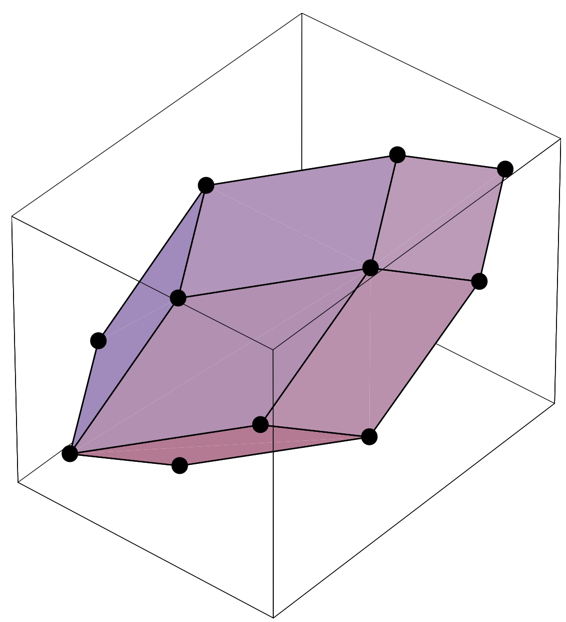
$$\textbf{For } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

then we get a hexagonal pillar (calcite),

and then

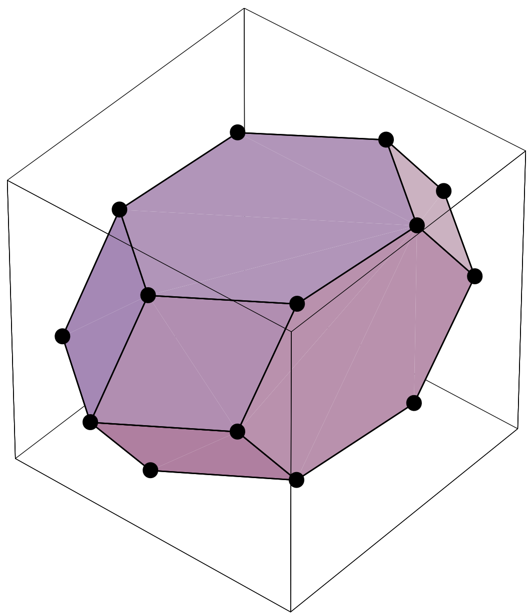
$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

A Dodecahedron (Garnet)



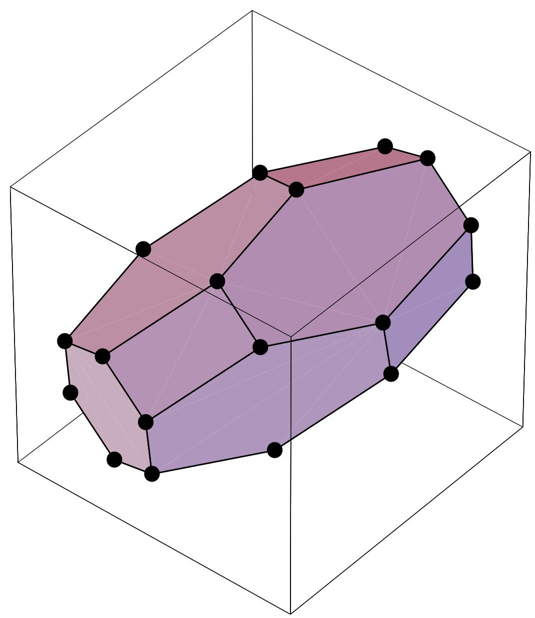
$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Apophyllite $KCa_4(Si_4O_{10})_2F \cdot 8H_2O$



$$B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

A Trunc. Octahed. — Zinc Blende ZnS



謝謝　御清聴