Compactification of the moduli of abelian varieties and

Morphisms of $SQ_{g,K}$ to Alexeev’s Moduli

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1 Hesse cubic curves

\[ C(\mu) : x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \]

\[ (\mu \in \mathbb{P}^1_{\mathbb{Z}[\zeta, 1/3]} ) \]
\[ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \]

if $\mu$ gets closer to $\infty$
\[ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu \in \mathbb{Z}[\zeta_3, 1/3]) \]

if \( \mu \) gets much closer to \( \infty \)
\[ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \quad (\mu^3 = 1 \text{ or } \infty) \]

It degenerates into 3 copies of \( \mathbb{P}^1 \)
2 Moduli of cubic curves

**Thm 1** (classical form over $\mathbb{C}$) (Hesse 1849)

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\text{isom.}$$

$$\simeq \mathbb{C} \setminus \{1, \zeta_3, \zeta_3^2\} \simeq \mathbb{H}/\Gamma(3) \ (\mathbb{H} : \text{upper half plane})$$

$$SQ_{1,3} := \overline{A_{1,3}}$$

$$= \{\text{stable cubics with 9 inflection pts}\}/\text{isom} = \{\text{Hesse cubics}\}/\text{isom = id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \text{ or } \infty \right\} \simeq \mathbb{P}^1$$

$$= \{\text{moduli of compact objects}\}$$
We wish to extend this to arbitrary dimension

1. over $\mathbb{Z}[\zeta_N, 1/N]$ (Today) or over $\mathbb{Z}[\zeta_N]$

2. to define a representable functor of compact obj. $F := SQ_{g,K}$ (fine moduli)

3. to relate $SQ_{g,K}$ to GIT stability, (This is new)

4. GIT stable objects = our model PSQASes: Projectively Stable Quasi Abelian Scheme

5. to relate 3 known compactif. $SQ_{g,K}$, $SQ_{g,K}^{\text{toric}}$ Alexeev’s moduli $\overline{A}_{g,d}$
3  Moduli over $\mathbb{Z}[\zeta_N, 1/N]$

(Thm 2) (a new version of the theorem of Hesse)

$$SQ_{1,3} = P^1_{\mathbb{Z}[\zeta_3, 1/3]},$$

the projective fine moduli

(1) The universal cubic curve

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0$$

where $(\mu_0, \mu_1) \in SQ_{1,3} = P^1$.

(2) when $k$ is alg. closed and char. $k \neq 3$
\[ SQ_{1,3}(k) = \begin{cases} 
\text{closed orbit cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom.} \\
= \begin{cases} 
\text{Hesse cubics} \\
\text{with level 3-str. } /k 
\end{cases} /\text{isom. = id.} \\
A_{1,3}(k) = \begin{cases} 
\text{closed orbit nonsing. cubics} \\
\text{with level 3-str. } /k 
\end{cases} /\text{isom.} \\
= \begin{cases} 
\text{nonsing. Hesse cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom. = id.} \]
Thm 3  (N. 1999) There exists the fine moduli $SQ_{g,K}$
projective over $\mathbb{Z}[\zeta_N, 1/N], N = \sqrt{|K|}$, For $k$ closed

$$SQ_{g,K}(k) = \begin{cases} 
\text{closed orb. deg. abelian sch. } /k \\
\text{with level } G(K)\text{-structure} \\
G(K)\text{-invariant PSQAS } /k \\
\text{with level } G(K)\text{-structure}
\end{cases} /\text{isom.}$$

$$Ag,K(k) = \begin{cases} 
\text{(nonsingular) abelian schemes } /k \\
\text{with level } G(K)\text{-structure} \\
G(K)\text{-inv. abelian schemes } /k \\
\text{with level } G(K)\text{-structure}
\end{cases} /\text{isom.}$$
4 Comparison of three compactifications

Summary \( N = \sqrt{|K|}, \mathcal{O}_N = \mathbb{Z}[\zeta_N, 1/N], \ d > 0. \)

1. \( SQ_{g,K} \) is a proj. fine moduli over \( \mathcal{O}_N \) [N99],
2. \( SQ_{g,K}^{\text{toric}} \) is a proj. coarse mod. over \( \mathcal{O}_N \) [N01] [N10],
3. \( AP_{g,d} = \{(P, G, D)\} \) is a proper separated coarse moduli over \( \mathbb{Z} \) [Alexeev02],
4. \( \text{dim} \ SQ_{g,K} = \text{dim} \ SQ_{g,K}^{\text{toric}} = g(g + 1)/2, \)
5. \( \text{dim} \ AP_{g,d} = g(g + 1)/2 + d - 1, \)
6. \( \exists \ a \ bij. \ mor. \ sq : SQ_{g,K}^{\text{toric}} \to SQ_{g,K}[N10] \)

\[
(SQ_{g,K}^{\text{toric}})_{\text{norm}} \simeq SQ_{g,K}^{\text{norm}} \quad (1)
\]
$SQ_{g,K,1/N} := SQ_{g,K}$ : proj. over $Z[\zeta_N, 1/N]$ (1999)

$\overline{AP}_{g,N} :$ by Alexeev, over $Z$, dim. excessive by $N - 1$ (2002)

$\overline{A}_{g,N} :$ by Olsson, over $Z$, proper separated (2008)
Thm 4 \[ \exists \text{ a finite Galois morph. over } \mathcal{O}_N, \ N = \sqrt{|K|}, \]

sqap : \( SQ_{g,K}^{\text{toric}} \times (P^{N-1} \setminus H_{g,K}) \to \overline{AP}_{g,N} \otimes \mathcal{O}_N \)

\((P, \phi, \tau) \times v \mapsto (P, \text{Aut}^\dagger_0(P), \text{Div}(\phi^*(v)))\)

such that for any fixed \( v \in P^{N-1} \setminus H_{g,K} \)

\((P, \phi, \tau) \mapsto (P, \text{Aut}^\dagger_0(P), \text{Div}(\phi^*(v)))\)

is a closed immersion of \( SQ_{g,K}^{\text{toric}} \).
5 Tate curve and PSQAS

\[ R:\text{DVR}, \, L = \text{Frac}(R) = R[1/q], \, q \text{ uniformizer.} \]

Tate curve : \[ G_m(L)/w \rightarrow qw \]

Hesse cubics at \( \infty \) : \[ G_m(L)/w \rightarrow q^3w \]

Rewrite Tate curve as \[ G_m(L)/w^n \rightarrow q^{mn}w^n \ (n \in \mathbb{Z}) \]

Hesse cubics at \( \infty \) : \[ G_m(L)/w^n \rightarrow q^{3mn}w^n \ (n \in \mathbb{Z}) \]

The general case : \( B \) pos. def. symmetric

\[ G_m(L)^g/w^x \rightarrow q^{B(x,y)}b(x, y)w^x, \]

\[ b(x, y) \in L^\times \ (x \in X, \, y \in Y) \]
The usual Tate curve over CDVR $R$

$$X : x_0x_2^2 = x_1^3 - x_0x_1^2 + qx_0^3$$

Or $$X : y^2 = x^3 - x^2 + q$$

The generic fibre $$X_\eta : y^2 = x^3 - x^2 + q \quad (q \neq 0)$$

The fibre $$X_0 : y^2 = x^2(x - 1)$$ for $q = 0$ : a limit of $X_q$

$$X_0 \setminus \{0, 0\} = \mathbb{G}_m,$$

To compactify the moduli, need to find all nice limits !!
The general case: \( B \) pos. def. symmetric

The generic fibre:

\[
G_m(L)^g / w^x \mapsto q^{B(x,y)} b(x, y) w^x,
\]

\[
b(x, y) \in L^\times \quad (x \in X, y \in Y)
\]

PSQAS is the closed fibre of it
6 Review of Theta functions

An elliptic curve, \( w = e^{2\pi i z}, \ q = e^{2\pi i \tau} / 6 \)

\[
E(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) = \mathbb{C}^*/w \rightarrow wq^6, \quad q = e^{2\pi i \tau} / 6
\]

Theta function \( \theta_k(\tau, z) = \sum_{m \in \mathbb{Z}} q^{(k+3m)^2}w^{k+3m} \).

The map \( \Theta \) embeds \( E(\tau) \) into \( \mathbb{P}^2 \).

\[
\Theta : E(\tau) \ni z \mapsto [x_0, x_1, x_2] = [\theta_0, \theta_1, \theta_2] \in \mathbb{P}^2
\]

To compactify the moduli

we find the limit of the image of \( \Theta \) as \( q \to 0 \)

General case will lead us to the next definition
Before it, recall again \( w = e^{2\pi iz}, \quad q = e^{2\pi i\tau/6} \)

\[
\theta_k(\tau, z + \frac{1}{3}) = \zeta^k_3 \theta_k(\tau, z),
\]

\[
\theta_k(\tau, z + \frac{\tau}{3}) = (qw)^{-1}\theta_{k+1}(\tau, z),
\]

\[
[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)
\]

\( \sigma, \tau \) are the liftings to \( \text{GL}(3) \),

\( z \mapsto z + \frac{1}{3} \) is lifted to \( \sigma(\theta_k) = \zeta^k_3 \theta_k \)

\( z \mapsto z + \frac{\tau}{3} \) is lifted to \( \tau(\theta_k) = \theta_{k+1} \)

\( G(3) := \text{the group } \langle \sigma, \tau \rangle \)

The image of \( \Theta \) is a Hesse cubic.
Heisenberg groups $G(K), G(3)$

$G(3) = \langle \sigma, \tau \rangle$ acts on $V$, order $|G(3)| = 27,$

$$V = Rx_0 + Rx_1 + Rx_2,$$

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathbb{Z}/3\mathbb{Z})$$

$\zeta_3$ is a primitive cube root of 1, $R \ni \zeta_3, 1/3$

- $x_0^3 + x_1^3 + x_2^3, x_0x_1x_2 \in S^3V$ only are $G(3)$-invariant
- $G(3)$ determines $x_i$ "uniquely" ($\because V : G(3)$-irred,)
- $x_i$ are classical theta over $\mathbb{C}$

General case will lead us to the next definition
In terms of theta, \( w = e^{2\pi i z} \), \( q = e^{2\pi i \tau / 6} \)

\[
\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),
\]

\[
\theta_k(\tau, z + \frac{\tau}{3}) = (qw)^{-1} \theta_{k+1}(\tau, z),
\]

\[
[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)
\]

\( \sigma, \tau \) are the liftings to \( \text{GL}(3) \),

\( z \mapsto z + \frac{1}{3} \) is lifted to \( \sigma(\theta_k) = \zeta_3^k \theta_k \)

\( z \mapsto z + \frac{\tau}{3} \) is lifted to \( \tau(\theta_k) = \theta_{k+1} \)

\( G(3) := \text{the group } \langle \sigma, \tau \rangle \)
8 Definition of PSQAS

$R$: DVR, $q$ a uniformizer of $R$,

$k(0) = R/m$, $k(\eta) = R[1/q]$ : the fraction field of $R$

Suppose $(G_\eta, L_\eta)$: abelian variety over $k(\eta)$

$(G, L)$ is the (connected) Néron model of $(G_\eta, L_\eta)$

Let $\lambda(L_\eta): G_\eta \to tG_\eta = \text{Pic}^0(G_\eta)$

$(tG_\eta, tL_\eta)$ dual AV, $tG_\eta = \text{Pic}^0(G_\eta)$.

$(tG, tL)$: the (connected) Néron model of $(tG_\eta, tL_\eta)$

Suppose $G_0$ a split torus over $k(0)$,

Then $(tG_0, tL_0)$ is a split torus over $k(0)$
For the Tate curve over CDVR $R$

The generic fibre $G_\eta : y^2 = x^3 - x^2 + q$ \quad (q \neq 0)

The fibre $X_0 : y^2 = x^2(x - 1)$ for $q = 0$ : a limit of $X_q$

$X_0 \setminus \{0, 0\} = G_m,$

This is the key assumption $G_0$ a split torus
\[ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu^3 = 1 \text{ or } \infty) \]

It degenerates into 3 copies of \( \mathbb{P}^1 \)

\[ \mu = \infty, \ x_0 x_1 x_2 = 0 \text{ contains } G_m \times \mathbb{Z}/3\mathbb{Z} \]

This is the key assumption \( G_0 \text{ a split torus} \)
Definition of PSQAS

$R$ : DVR, $q$ a uniformizer of $R$,

$k(0) = R/m$, $k(\eta) = R[1/q]$ : the fraction field of $R$

Suppose $(G_\eta, L_\eta)$ : abelian variety over $k(\eta)$

$(G, L)$ is the (connected) Néron model of $(G_\eta, L_\eta)$

Let $\lambda(L_\eta) : G_\eta \to tG_\eta = \text{Pic}^0(G_\eta)$

$(tG_\eta, tL_\eta)$ dual AV, $tG_\eta = \text{Pic}^0(G_\eta)$.

$(tG, tL) :$ the (connected) Néron model of $(tG_\eta, tL_\eta)$

Suppose $G_0$ a split torus over $k(0)$,

Then $(tG_0, tL_0)$ is a split torus over $k(0)$
Let \( X = \text{Hom}(G_0, G_m) \), \( Y = \text{Hom}(tG_0, G_m) \).

Hence \( X \cong \mathbb{Z}^g \), \( Y \cong \mathbb{Z}^g \),

\( \lambda(L_\eta) \) extends, \( \exists \) a surjection \( G_0 \rightarrow tG_0 \)

Hence \( Y : \) a sublattice of \( X \), \([X : Y] < \infty\). 

\[ K_\eta := \ker \lambda(L_\eta), \quad N := |K_\eta|. \]

\( K := \) the closure of \( K_\eta \). May assume \( \text{Over } \mathbb{Z}[\zeta_N, 1/N] \)

\( K \cong (X/Y) \oplus (X/Y)^\vee \),

This finite group helps us to take up the necessary data
From $G$ and $K$ we can construct

- $G(K)$: Heisenberg group scheme

\[
1 \to \mu_N \to G(K) \to K \to 0 \text{ (exact)}
\]

\[
(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),
\]

- $R[X/Y] = \bigoplus_{x \in X/Y} R v(x)$ (group alg. of $X/Y$)

\[
v(0) = 1, \quad v(x + y) = v(x)v(y)
\]

- $G(K)$ acts on $R[X/Y]$ by

\[
(a, z, \alpha) \cdot v(x) = a\alpha(x)v(z + x)
\]

$a, b \in \mu_N; \quad z, x \in (X/Y); \quad \alpha, \beta \in (X/Y)^\vee$
Facts. \( G \) : conn. Néron model of \( G_\eta \),

\[
K_\eta := \ker(\lambda(L_\eta)) \cong (X/Y) \oplus (X/Y)^\lor,
\]

- \( V := H^0(G, L) \): finite \( R \)-free, \( G(K) \)-irreducible
- \( V = H^0(G, L) \cong R[X/Y] \) as \( G(K) \)-module
- \( H^0(G, L) \ni \exists \theta_x \xleftarrow{G(K)\text{-isom}} v(x) \in R[X/Y] \) gp alg

\( \theta_x \) can be thought as ”classical theta”

Idea: Find the limit of the image \([\theta_x]_{x \in X/Y}\)
Let $G_{\text{for}}$ : the formal completion of $G$ along $G_0$

Key Fact:

\[ G_{\text{for}} \cong (G_{m,R}^g)_{\text{for}} \]

Fourier expansion of $\theta_x$ ($x \in X/Y$) on $G_{\text{for}}$:

\[ \theta_x = \sum_{y \in Y} a(x + y)w^{x+y} \]

$a(x + y)$ : Fourier coeff. of $\theta_x$

called Faltings-Chai’s degeneration data of $(G, L)$

$\bullet$ $B(x, y) := \text{val}_q(a(x + y)a(x)^{-1}a(y)^{-1})$ is pos. def.
generalized Tate curves

The general case: $B$ pos. def. symmetric

The generic fibre:

$$G_m(L)^g/w^x \mapsto q^B(x,y)b_0(x, y)w^x,$$

$$b_0(x, y) \in L^\times \quad (x \in X, y \in Y)$$

PSQAS is the closed fibre of a gener. Tate curve
We construct a canonical generalization of Tate curves.

\[ \tilde{R} := R[a(x)w^x \vartheta, x \in X], \quad \vartheta: \text{deg one} \]

\[ \text{Proj}(\tilde{R}) : \text{locally of finite type over } R \]

\[ \mathfrak{X} : \text{the formal completion of } \text{Proj}(\tilde{R}) \]

The Quotient \( \mathfrak{X}/Y \) is a degenerating family of AV 

\( (\mathfrak{X}/Y, O_{\mathfrak{X}/Y}(1)) \) is a generalization of Tate curves
Grothendieck (EGA) guarantees

$\exists$ a projective $R$-scheme $(Z, O_Z(1))$

s.t. the formal completion $Z_{\text{for}}$ of $Z$

$Z_{\text{for}} \simeq X/Y$, $(Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$

(the stable reduction theorem)

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

Projectively Stable Quasi Abelian Scheme

$G(K)$ acts on $(Z, O_Z(1))$
Summary

Let $R$ be CDVR over $\mathbb{Z}[\zeta_N, 1/N]$

- There is a natural choice of $\theta_x \in H^0(G, L)$
- $a(x + y)$, $y \in Y$ is Fourier coeff of $\theta_x$, $x \in X/Y$
- all $a(x)$ recover the given $G_\eta$ over $k(\eta) := \text{Frac}(R)$
- There is an extension $X/Y$ of $G_\eta$ to $R$ so that
  (a) it is a canonical generalization of Tate curves,
  (b) $G(K)$ acts on $(X/Y, O_{X/Y}(1))$
  (c) hence $G(K)$ acts on $(Z, O_Z(1))$
  (d) the closed fibre $(Z_0, O_{Z_0}(1))$ is a PSQAS.
Exam 1

\[ g = 1, \ X = Z, \ Y = 3Z. \]

\[ \chi = \text{Proj}(\tilde{R}), \quad a(x) = q^{x^2}, \ (x \in X) \]

\[ S_{-3} \]

\[ V_{-2}, V_{-1}, V_0, V_1, V_2, V_3, V_4 \]

\[ \chi_0/Y \]
Recall

**Thm 5** Over $\mathbb{Z}[\zeta_3, 1/3]$

\[
A_{1,3} := \{ \text{nonsing. cubics with 9 inflection pts} \} / \text{isom.}
\]

\[
SQ_{1,3} := \overline{A_{1,3}}
\]

\[
= \{ \text{stable cubics with 9 inflection pts} \} / \text{isom} = \text{id}
\]

\[
= A_{1,3} \cup \left\{ C(\mu); \mu^3 = 1 \text{ or } \infty \right\} \cong \mathbb{P}^1.
\]

Hesse cubics are PSQASes in dimension one, level 3.
We wish to extend this to arbitrary dimension

1. over $\mathbb{Z}[\zeta_N, 1/N]$ or over $\mathbb{Z}[\zeta_N]$

2. to define a representable functor of compact obj.

$$F := SQ_{g,K}$$ (fine moduli)

3. to relate to GIT stability, that is,

to aim at $F(k) =$GIT stable objects for $k$ alg. closed
\[ SQ_{g,K,1/N} := SQ_{g,K} : \text{proj. over } \mathbb{Z}[\zeta_N, 1/N] \] (1999)

\[ \overline{AP}_{g,N} : \text{over } \mathbb{Z}, \text{ dim. excessive by } N - 1 \] (2002)

Olsson : over \( \mathbb{Z} \), nonseparated nonproper stack (2008)

Olsson uses the same model as ours (Alexeev-Nakamura’s model)

We prefer to separated moduli.

It is easy to construct nonseparated stack moduli.
9 Separatedness of the moduli

There are difficulties never seen in dimension one

- Classical level structure = base of $n$-division points,
- Singular limits of Abelian varieties are very reducible
- Classical level str. gives non-separated moduli
- We need to prove in any dimension,

**Lemma. (Valuative Lemma for Separatedness)**

Given $R$ : DVR, $L = \text{Frac}(R)$, $X, Y \in F(R)$.

If $X_L \simeq Y_L$, then $X \simeq Y$. In other words,

Isom. over $L$ implies isom. over $R$. 

• separated = Hausdorff,  (e.g. if $X$ projective, then separated)

• $X$: non-separated = non Hausdorff,

• If non-Hausdorff, then $\exists P_n \in X \ (n = 1, 2, \cdots)$,

$$P = \lim P_n, \ Q = \lim P_n.$$ But $P \neq Q$

• This really happens in geometry.
Example  \( R : \) DVR,  \( q : \) uniformizer of  \( R, \ L = R[1/q], \)

\( E, \ E' : \) elliptic curves over  \( R \)

\[
E : y^2 = x^3 - q^6, \quad E' : Y^2 = X^3 - 1
\]

Let us consider \( P_n := E_L, \ Q_n := E'_L \)

\( P_n = Q_n, \ i.e. \ E_L \simeq E'_L \)

because

\[
E_L : (y/q^3)^2 = (x/q)^3 - 1,
\]

\[
E'_L : Y^2 = X^3 - 1
\]
Example \( R : \text{DVR}, q : \text{uniformizer of } R, L = R[1/q], \)
\( E, E' : \text{elliptic curves over } R \)

\[ E : y^2 = x^3 - q^6, \quad E' : Y^2 = X^3 - 1 \]

Let us consider \( P_n := E_L, Q_n := E'_L \)

\( P := E_0 = \lim E_L, Q := E'_0 = \lim E'_L \)

\( P_n = Q_n, \text{ i.e. } E_L \simeq E'_L \) \quad But \( P \neq Q \)

\( P := E_0 : y^2 = x^3, \quad Q := E'_0 : Y^2 = X^3 - 1 \)
To overcome the difficulty of level str/n-div. pts:

- **Non-abelian Heisenberg gp.** \( G := G(K) \)
- **New level str.** = Framing of irred. reps. of \( G \)
- **To prove Val. Lemma for Separatedness, we use**

Schur’s Lemma over \( R \):

Let \(|G| = N, R : \text{a ring over } \mathbb{Z}[\zeta_N, 1/N], V : \text{free } R\text{-mod.} \)

\( V : \text{irr. } G\text{-mod. of wt one}, (\Rightarrow G \subset GL(V \otimes R)) \)

Let \( h \in GL(V \otimes R) \). If \( gh = hg \) for \( \forall g \in G \), then \( h \) is scalar.
Summary

• Separatedness of the moduli

follows from $G(K)$-Irreducibility of $V = H^0(X, L)$, 
$(X, L) = (Z_0, O_{Z_0}(1))$ : any PSQAS, level $N \geq 3$
if $K \simeq \ker(\lambda(L) : G_\eta \rightarrow G^t_\eta \ (\text{dual})).$
We re-start with

**Thm 6** Over $\mathbb{Z}[\zeta_3, 1/3]$

\[ A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\} / \text{isom.} \]

\[ \overline{A}_{1,3} := \{\text{stable cubics with 9 inflection pts}\} / \text{isom.} \]

\[ = \{\text{Hesse cubics}\} / \text{isom=id} \]

\[ = A_{1,3} \cup \left\{ C(\mu); \mu^3 = 1 \text{ or } \infty \right\} \simeq \mathbb{P}^1. \]

We convert it into $G(3)$-equivariant theory

$G(3)$: Heisenberg group of level 3
10 Heisenberg groups $G(K), G(3)$

$G(3) = \langle \sigma, \tau \rangle$ acts on $V$, order $|G(3)| = 27,$

\[ V = Rx_0 + Rx_1 + Rx_2, \]

\[ \sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathbb{Z}/3\mathbb{Z}) \]

$\zeta_3$ is a primitive cube root of 1, $R \ni \zeta_3, 1/3$

Fact

- $x_0^3 + x_1^3 + x_2^3$, $x_0x_1x_2 \in S^3V$ only are $G(3)$-invariant
- $G(3)$ determines $x_i$ "uniquely" ($\therefore V:G(3)$-irred,)
- $x_i$ are classical theta over $\mathbb{C}$
Summary

$G(K) : \text{Heisenberg gp. e.g. } G(3)$

- $G(K)$ chooses a basis of $V = H^0(X, L)$, $X: \text{PSQAS}$
- $G(K)$ chooses a basis of $H^0(G, L)$, $G: \text{Néron model}$
- $G(K)$ determines Faltings-Chai degeneration data
- $G(K)$ extends $G_\eta$ to define $(Z, O_Z(1))$, $Z = \mathcal{X}/\mathcal{Y}$
- Separatedness of the moduli
  
  follows from $G(K)$-Irreducibility of $V = H^0(X, L)$, $X : \text{any PSQAS, level } N \geq 3$
## 11 The space of closed orbits

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>the set of geometric objects</td>
</tr>
<tr>
<td>$G$</td>
<td>the group of isomorphisms</td>
</tr>
<tr>
<td>$x, x'$ are isom.</td>
<td>$G$-orbits are the same $O(x) = O(x')$</td>
</tr>
<tr>
<td>$X_{ps}$</td>
<td>the set of properly-stable objects</td>
</tr>
<tr>
<td>$X_{ss}$</td>
<td>the set of semistable objects</td>
</tr>
<tr>
<td>$X_{ss} // G$</td>
<td>&quot;compact moduli&quot;</td>
</tr>
</tbody>
</table>
Action on $C^2$ of $G = G_m(= C^*)$,

$$C^2 \ni (x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in G_m)$$

What is the quotient of $C^2$ by $G$?

- Simple answer: the set of $G$-orbits (×)
- Answer: $\text{Spec}(\text{the ring of all } G\text{-invariant poly.})(\square)$
- $t := xy$ is the unique $G$-inv. !

$$C^2//G := \text{Spec } C[t] = \{t \in C\}$$

But this is different from "the set of $G$-orbits".

- $C^2//G = \{t \in C\}$ is the set of all closed orbits.
\[ x y = 0 \]

\[ O(d, 1) \quad O(c, 1) \]

\[ \cong \{ t \in \mathbb{C} \} \]

\[ (c > 0, d < 0) \]

- \( t = 0 \) is a point of \( C = \mathbb{C}^2 // G = \text{Spec} \mathbb{C}[t] \).
- But \( \{ x y = 0 \} \) consists of three \( G \)-orbits
  
  \[ C^* \times \{ 0 \}, \quad \{ 0 \} \times C^*, \quad \{ (0, 0) \} \]

- \( \{ (0, 0) \} \) is the only closed orbit in \( \{ x y = 0 \} \).
Def 7 The same notation as before. Let \( p \in X \).

(1) **semistable** if \( \exists \) \( G \)-inv. homog. poly. \( F \), \( F(p) \neq 0 \),

(2) **Kempf-stable** (= closed orbit)

if the orbit \( O(p) \) is closed in \( X_{ss} \),

(3) **properly-stable** if (2) and \( \text{Stab}(p) \) finite.

Rem stable \( \Rightarrow \) closed orbit \( \Rightarrow \) semistable
Thm 8  (Seshadri,Mumford) $G$ : reductive, acting on a scheme $X$, (e.g. $G = G_m$). Let $X_{ss}$ = the set of semistable points. Then

- $X_{ss} // G := \text{Spec}(\text{all } G\text{-inv.}) = \text{the set of closed orbits}$.
- $X_{ss} // G$ is a scheme, $X_{ps} // G$ is also a scheme,
- $X_{ss} // G$ compactifies $X_{ps} // G$.

Rem  The set of points with closed orbits is not an algebraic subscheme.
Thus we consider only those objects with closed orbits.

As its consequence we will see

- Abelian varieties have closed orbits (Kempf), and
- our PSQASes have closed orbits,

Conversely

- Any degenerate abelian scheme with closed orbit is one of our PSQASes
- There is a simple characterization of our PSQASes,
- This characterization enables us to compactify the moduli of abelian varieties.
Def 9  \( C \) is a stable curve of a genus \( g \) if

1. connected projective reduced with finite autom.,
2. the singularities of \( C \) are like \( xy = 0 \)
3. \( \dim H^1(O_C) = g \)

Let \( \overline{M}_g \) : moduli of stable curves of genus \( g \),

\( M_g \) : moduli of nonsing. curves of genus \( g \).

Thm 10  \( \overline{M}_g \) compactifies \( M_g \)

(Deligne-Mumford 1969)
The definition of stable curves is irrelevant to GIT stability

Nevertheless

Theorem 11: The following are equivalent

1. $C$ is a stable curve (moduli-stable)
2. any Hilbert point of $\Phi_{\lfloor mK \rfloor}(C)$ is GIT-stable
3. any Chow point of $\Phi_{\lfloor mK \rfloor}(C)$ is GIT-stable

(1)$\Leftrightarrow$(2) Gieseker 1982 (before Mumford 1977)
(1)$\Leftrightarrow$(3) Mumford 1977 (suggested by Gieseker 1982)
### 13 Stability of cubic curves

<table>
<thead>
<tr>
<th>CUBIC CURVES</th>
<th>STABILITY</th>
<th>STAB GP.</th>
</tr>
</thead>
<tbody>
<tr>
<td>smooth elliptic</td>
<td>stable</td>
<td>finite</td>
</tr>
<tr>
<td>3-gon</td>
<td>closed orbit</td>
<td>2-dim</td>
</tr>
<tr>
<td>a line + a conic (transv.)</td>
<td>semistable</td>
<td>1-dim</td>
</tr>
<tr>
<td>irreducible with a node</td>
<td>semistable</td>
<td>finite</td>
</tr>
<tr>
<td>others</td>
<td>unstable</td>
<td>1-dim</td>
</tr>
</tbody>
</table>
Thm 12 For a cubic $C$, the following cond. are equiv.

(1) $C$ has a closed $\text{SL}(3)$-orbit in $(S^3V)_{ss}$
(2) $C$ is a Hesse cubic curve, that is, $G(3)$-invariant
(3) $C$ is either smooth elliptic or a 3-gon
14 Stability in higher-dim.

**Thm 13** (Kempf) \((A, L)\) an abelian variety, 
\(V = H^0(A, L)\) very ample, \(w:=\)Hilbert point of \((A, L)\). Then \(\text{SL}(V)w\) is closed in \(P_{ss}\) : the semistable locus of a big proj. space.

**Thm 14** (N.1999) 
\((X, L) : \) PSQAS of level \(G(K)\), 
\(V = H^0(X, L) \text{ very ample}\). Then any Hilbert point of \((X, L)\) has a closed \(\text{SL}(V)\)-orbit.
Thm 15  (N.1999)

Assume $(X, L)$ is a limit of abelian varieties $A$ with $\ker(\lambda(L)) = K$, $\lambda(L): A \to A^t$ (dual)

Then the following are equivalent:

(1) $X$ has a closed $\text{SL}(V)$-orbit (GIT-stable)
(2) $X$ is invariant under $G(K)$ ($G(K)$-stable)
(3) $X$ is one of our PSQASes (moduli-stable)
To be more precise,

**Thm 16** (N.1999)

Assume \((X, L)\) is a limit of AV A’s with \(\ker(\lambda(L)) = K\)

Then the following are equivalent:

1. The \(m\)-th Hilbert point of \(X\) has a closed \(\text{SL}(V)\)-orbit in \(\mathbf{P}(\wedge M S^m V)_{ss}\) (GIT-stable)
2. \(X\) is invariant under \(G(K)\) (\(G(K)\)-stable)
3. \(X\) is one of our PSQASes (moduli-stable)

where \(M := \dim H^0(X, mL)\).
**Thm 17**  For **cubics** the following are equiv:

1. it has a closed SL(3)-orbit  \((\text{GIT-stable})\)
2. it is a Hesse cubic, that is , \(G(3)\)-inv.  \((G(3)\)-stable\)
3. it is smooth ell. or a 3-gon.  \((\text{moduli-stable})\)

**Thm 18**  Let \(X\) be a degenerate AV. The following are equiv. under natural assump.:

1. it has a closed SL(\(V\))-orbit  \((\text{GIT-stable})\)
2. \(X\) is \(G(K)\)-inv  \((G(K)\)-stable\)
3. it is a PSQAS \((\text{p.20})\)  \((\text{moduli-stable})\)
Thus we see

• Abelian varieties have closed orbits (Kempf), and
• our PSQASes have closed orbits,

Conversely

• Any degenerate abelian scheme with closed orbit is one of our PSQASes

• \( X \) is our PSQAS iff \( X \) is \( G(K) \)-stable,

• This characterization will compactify the moduli of abelian varieties.
The characterization of PSQASes will compactify the moduli of abelian varieties. We recall ”Closed orbit” is not a Zariski open/closed condition.

Exam 3

Let $G := \{(s, t, u) \in (G_m)^3; stu = 1\}$

$$C_{a,b,c} : ax_0^3 + bx_1^3 + cx_2^3 - x_0x_1x_2 = 0.$$  

$G$ acts on $A^3 : (a, b, c) \mapsto (sa, tb, uc)A^3$

Closed $(G_m)^2$-orbit iff $abc \neq 0$ or $(a, b, c) = (0, 0, 0)$. 
Thm 19  (a new version of the theorem of Hesse)

\[ SQ_{1,3} = \mathbb{P}_{\mathbb{Z}[\zeta_3, 1/3]}^{1}, \]

the projective fine moduli

(1) The universal cubic curve

\[ \mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0 \]

where \((\mu_0, \mu_1) \in SQ_{1,3} = \mathbb{P}^{1}.\)

(2) when \(k\) is alg. closed and char. \(k \neq 3\)
\[ SQ_{1,3}(k) = \begin{cases} 
\text{closed orbit cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom.} \]

\[ = \begin{cases} 
\text{Hesse cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom.} = \text{id.} \]

\[ A_{1,3}(k) = \begin{cases} 
\text{closed orbit nonsing. cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom.} \]

\[ = \begin{cases} 
n\text{onsing. Hesse cubics} \\
\text{with level 3-structure } /k 
\end{cases} /\text{isom.} = \text{id.} \]
Thm 20 (N. 1999) There exists the fine moduli $SQ_{g,K}$ projective over $\mathbb{Z}[\zeta_N, 1/N]$, $N = \sqrt{|K|}$, For $k$ closed

$$SQ_{g,K}(k) = \begin{cases} 
\text{closed orb. deg. abelian sch. } /k \\
\text{with level } G(K)\text{-structure} \\
G(K)\text{-invariant PSQAS } /k \\
\text{with level } G(K)\text{-structure}
\end{cases} / \text{isom.}$$

$$Ag_{g,K}(k) = \begin{cases} 
\text{(nonsingular) abelian schemes } /k \\
\text{with level } G(K)\text{-structure} \\
G(K)\text{-inv. abelian schemes } /k \\
\text{with level } G(K)\text{-structure}
\end{cases} / \text{isom.}$$
Summary \( G(K) : \) Heisenberg gp. \( e.g. \ G(3) \)

(A) \( H^0(X, L) \) is \( G(K) \)-irred for \( X \): PSQAS

- (A) implies Stability of \( X \) with \( L \) very ample,
- (A) implies Separatedness of the moduli,
- (A) gives a simple characterization of PSQASes,
- \( G(K) \) finds a compact separated moduli \( SQ_{g,K} \)
The Second Compactification over $\mathbb{Z}[\zeta_N, 1/N]$

Recall Grothendieck (EGA) guarantees

\[ \exists \text{ a projective } R\text{-scheme } (Z, O_Z(1)) \]

s.t. the formal completion $Z_{\text{for}}$ of $Z$

\[ Z_{\text{for}} \cong X/Y, \quad (Z_{\eta}, O_{Z_{\eta}}(1)) \cong (G_{\eta}, L_{\eta}) \]

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

The normalization $Z^{\text{norm}}$ of $Z$ with $Z_0^{\text{norm}}$ reduced gives a bit different central fiber

$(Z_0^{\text{norm}}, O_{Z_0^{\text{norm}}}(1))$, we call it TSQAS.
**Thm 21** (N. 2010) over \( \mathbb{Z}[\zeta_N, 1/N] \),

\[ \exists \text{ another cano. compactif. } \mathcal{S}Q_{g,K}^{\text{toric}} \]

coarse moduli of TSQASes with level-\( G(K) \) str.

\[ \exists \text{ cano. bij. birat. morphism } \]

\[ \text{sq} : \mathcal{S}Q_{g,K}^{\text{toric}} \rightarrow \mathcal{S}Q_{g,K} \]

\[ (P, \phi, \tau) \mapsto (Q, \phi_Q, \tau_Q), \quad Q := \text{Proj}(\text{Sym}(\phi)) \]

when any generic fibre of \( P \) is an abelian var.

**Corollary**

The normalizations of \( \mathcal{S}Q_{g,K}^{\text{toric}} \) and \( \mathcal{S}Q_{g,K} \) are isom.
Recall \((P, \phi, \tau) \in SQ_{g,K}^{\text{toric}}\)

- \(P: \text{TSQAS} = \text{modified PSQAS}\),
- \(\phi : P \to \mathbb{P}^{N-1} = \mathbb{P}(k[H\vee])\) is a finite morphism
- \(L = \phi^*(O_{\mathbb{P}^{N-1}}(1))\),
- \(H^0(P, L) \overset{\phi^*}{\simeq} k[H\vee] = H^0(O_{\mathbb{P}^{N-1}}(1))\)
- \(\tau : \text{a compatible action of } G(K) \text{ on the pair } (P, L)\)
- \(\tau \text{ on } P = \text{translation by } K \text{ when } P = \mathbb{A} : \mathbb{A}V\)
\((Q, \phi_Q, \tau_Q) \in SQ_g, K\)

- \(Q: \text{PSQAS},\)
- \(\phi_Q: Q \to P^{N-1} = P(k[H^\vee])\) is a closed immersion
- \(L_Q = \phi^*(O_{P^{N-1}}(1)),\)
- \(H^0(Q, L_Q) \simeq H^0(P, L) \overset{\phi^*}{\simeq} k[H^\vee] = H^0(O_{P^{N-1}}(1))\)
- \(\tau_Q:\) a compatible action of \(G(K)\) on the pair \((Q, L_Q)\)
- \(\tau_Q\) on \(Q = \text{translation by } K\) when \(Q = A: AV\)
Definition of sq: For \((P, L, \phi, \tau) \in SQ_{g,K}^{\text{toric}}(T)\)

Suppose \((P, L, \phi, \tau)\) is a \(T\)-TSQAS such that any generic fibre is AV.

Then let \(Q = \phi(P) := \text{Proj}(\text{Sym}(\phi))\)

Can define \((Q, L_Q, \phi_Q, \tau_Q)\) \(T\)-PSQAS, Then

the morphism sq is

\[\text{sq}(P, L, \phi, \tau) = (Q, L_Q, \phi_Q, \tau_Q) \in SQ_{g,K}(T)\]
Comparison of three compactifications

**Summary** \( N = \sqrt{|K|}, \mathcal{O}_N = \mathbb{Z} [\zeta_N, 1/N], \ d > 0. \)

1. \( SQ_{g,K} \) is a proj. fine moduli over \( \mathcal{O}_N \) [N99],
2. \( SQ_{g,K}^{\text{toric}} \) is a proj. coarse mod. over \( \mathcal{O}_N \) [N01] [N10],
3. \( \overline{AP}_{g,d} = \{(P, G, D)\} \) is a proper separated coarse moduli over \( \mathbb{Z} \) [Alexeev02],
4. \( \dim SQ_{g,K} = \dim SQ_{g,K}^{\text{toric}} = g(g + 1)/2, \)
5. \( \dim \overline{AP}_{g,d} = g(g + 1)/2 + d - 1, \)
6. \( \exists \) a canonical bij. birat. morphism [N10] \( \text{sq} : SQ_{g,K}^{\text{toric}} \rightarrow SQ_{g,K} \)
Alexeev’s moduli $\overline{AP}_{g,d} = \{(P, G, D)\}$

- $P$ is semi-normal proj. with $L$ ample line bundle
- $G$ semi-abelian acting on $P$ with extra cond.
- $D \in H^0(P, L)$ a Cartier divisor
- $D$ contains no $G$-orbits
- $\dim \overline{AP}_{g,d} = \dim A_g + d - 1$. 
$k$ alg. closed

$SQ_{1,K}, \ K = (\mathbb{Z}/3\mathbb{Z})^2$, Roughly

$SQ_{1,K}(k) = \{C \text{ a nonsing. cubic or a 3-gon cubic}\}$

__________________________

$\overline{AP}_{1,3}(k) = \{(C, G, D)\}$

$C$ nonsingular elliptic or a 3-gon,

or a conic plus a line, rational with a node

$G = C$ (elliptic) or $G_m$, $D \in H^0(C, L)$, degree $D = 3$. 
To define a morphism from $SQ_{1,K}$ to $AP_{1,3}$ is equivalent to the following

For a given a flat family over $T$

$$(C, \phi, \tau) \in SQ_{1,K}(T)$$

always! construct $(G, D)$ so that

$$(C, G, D) \in AP_{1,3}(T)$$
Problem: Construct $G$ and Find $D$

For almost all $v \in k[Z/3Z]$, 

$$(P, \phi, \tau) \times v$$

$$\mapsto (P, \text{Aut}^{\dagger 0}(P), \text{Div}(\phi^*(v)))$$

Need to prove

Any $T$-TSQAS has a flat group scheme action

This is done in general

**Thm 22** If $(P, L)$ is an $S$-flat TSQAS, then

$\text{Aut}_{S}^{\dagger 0}(P)$ is $S$-flat semi-abelian group scheme
Thm 23 \quad \exists \text{ a finite Galois morph. over } \mathcal{O}_N, N = \sqrt{|K|},

\begin{align*}
\text{sqap} : SQ_{g,K}^{\text{toric}} \times (P^{N-1} \setminus H_{g,K}) & \rightarrow \overline{AP}_{g,N} \otimes \mathcal{O}_N \\
(P, \phi, \tau) \times & \mapsto (P, \text{Aut}^{\dag 0}(P), \text{Div}(\phi^*(v)))
\end{align*}

such that for any fixed \( v \in P^{N-1} \setminus H_{g,K} \)

\( (P, \phi, \tau) \mapsto (P, \text{Aut}^{\dag 0}(P), \text{Div}(\phi^*(v))) \)

is an injective morphism of \( SQ_{g,K}^{\text{toric}} \) extending an injective immersion of \( A_{g,K}^{\text{toric}} \).

\begin{itemize}
  \item \( P^{N-1} = P(\mathcal{O}_N[H^\vee]^\vee), \; v \in \mathcal{O}_N[H^\vee] \).
  \item \( H_{g,K} \) is a hypersurf. of \( P^{N-1} \) of deg. known.
  \item \( \dim SQ_{g,K}^{\text{toric}} + N - 1 = \dim \overline{AP}_{g,N} \).
\end{itemize}
$SQ_{g,K,1/N} := SQ_{g,K} : \text{over } Z[\zeta_N, 1/N]$

$\overline{AP}_{g,N} : \text{Alexeev, over } Z, \text{no level str.}$

$\overline{A}_{g,N} : \text{Olsson, over } Z, \text{no level str.}$
"Limits of theta functions are described by the Delaunay decomposition."

PSQAS is a geometrization of limit of thetas

PSQAS is a generalization of 3-gons.

which is described by the Delaunay decomposition.
PSQAS: a generalization of Tate curve, \( R: \text{DVR} \)

Tate curve \( : \quad G_m(R)/w \mapsto qw \)

Hesse cubics at \( \infty \) \( : \quad G_m(R)/w \mapsto q^3w \)

Rewrite Tate curve as:

\[ G_m(R)/w^n \mapsto q^{mn}w^n \quad (m \in \mathbb{Z}) \]

Hesse cubics at \( \infty \) \( : \quad G_m(R)/w^n \mapsto q^{3mn}w^n \quad (m \in \mathbb{Z}) \)

The general case: \( B \) pos. def. symmetric

\[ G_m(R)^g/w^x \mapsto q^{B(x,y)}b(x,y)w^x, \]

\[ b(x, y) \in R^\times \quad (x \in X, y \in Y) \]
Let $X = Z^g$, $B$ a positive symmetric on $X \times X$.

$$\|x\| = \sqrt{B(x, x)} : \text{a distance of } X \otimes \mathbb{R} \text{ (fixed)}$$

**Def 24** Let $\alpha \in X_\mathbb{R}$. a Delaunay cell $D(\alpha)$: the convex closure of points of $X$ closest to $\alpha$.

**Exam 4** 1-dim. $B(x, y) = 2xy$, $X/Y = Z/nZ$, then PSQAS $Z_0$ is an $n$-gon of $\mathbb{P}^1$
• All Delaunay cells for a $B$ form a Delaunay decomp.
• Each PSQAS (its scheme structure) and its decomposition into torus orbits (its stratification) are described by Delaunay decomp.
• Each pos. symm. $B$ defines a Delaunay decomp.
• Different $B$ can yield the same Delaunay decomp. and the same PSQAS.
Exam 5

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Z_0 := \mathcal{X}_0/Y$$ is a union of $\mathbb{P}^1 \times \mathbb{P}^1$
Exam 6

\[ B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \]
1. This (mod $Y$) is a PSQAS.

   It is a union of $\mathbb{P}^2$, each triangle stands for $\mathbb{P}^2$,
2. each line segment is a $\mathbb{P}^1$, two $\mathbb{P}^2$ intersect along $\mathbb{P}^1$
3. six $\mathbb{P}^2$ meet at a point,
   locally $k[x_1, \cdots , x_6]/(x_i x_j, |i - j| \geq 2)$
Red one is the decomp. dual to the Delaunay decomp. called Voronoi decomp.
Voronoi decomposition
Def 25 \( D : \) for Delaunay cells

\[
V(D) := \{ \lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; D = D(\lambda) \}
\]

We call it a Voronoi cell

\[
\overline{V(0)} = \{ \lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; \|\lambda\| \leq \|\lambda - q\|, (\forall q \in X) \}
\]

This is a crystal of mica.
For $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We get $\overline{V(0)}$, a cube (salt),

For $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

then we get a hexagonal pillar (calcite),

and then
\[ B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \]

A Dodecahedron (Garnet)
\[ B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \]

Apophyllite \( KCa_4(Si_4O_{10})_2F \cdot 8H_2O \)
$B = \begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{pmatrix}$

A Trunc. Octahed. — Zinc Blende $ZnS$