COINVARIANT ALGEBRAS OF FINITE SUBGROUPS OF SL(3,C)

YASUSHI GOMI, IKU NAKAMURA AND KEN-ICHI SHINODA

ABSTRACT. For most of the finite subgroups of $\mathrm{SL}(3, \mathbf{C})$, we give explicit formulae for the Molien series of the coinvariant algebras, generalizing McKay's formulae [McKay99] for subgroups of SU(2). We also study the G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^3)$ for any finite subgroup G of SO(3), which is known to be a minimal (crepant) resolution of the orbit space $\mathbf{C}^3/G$. In this case the fiber over the origin of the Hilbert-Chow morphism from $\mathrm{Hilb}^G(\mathbf{C}^3)$ to $\mathbf{C}^3/G$ consists of finitely many smooth rational curves, whose planar dual graph is identified with a certain subgraph of the representation graph of G. This is an SO(3) version of the McKay correspondence in the SU(2) case.

0. Introduction

Let G be a finite subgroup of $\mathrm{SL}(n, \mathbf{C})$, S_G the coinvariant algebra of G, and $(S_G)_i$ the subspace of S_G of homogeneous degree i respectively. For each irreducible representation ρ of G, let $\langle \rho, (S_G)_i \rangle_G$ be the multiplicity of ρ in $(S_G)_i$ and define the Molien series $P_{S_G,\rho}(t)$ of S_G for ρ to be

$$P_{S_G,\rho}(t) = \sum \langle \rho, (S_G)_i \rangle_G t^i.$$

Since S_G is finite-dimensional, $P_{S_G,\rho}(t)$ is a polynomial of t. One can define similarly the Molien series $P_{M,\rho}(t)$ for an arbitrary graded G-module M with finite dimensional graded pieces. If M is the polynomial algebra S in two variables and if G is a subgroup of SU(2), then the Molien series $P_{S,\rho}(t)$ of S is a rational function of t by [Springer87] and it is well understood as is the connection with the Dynkin diagram corresponding to G (cf. [Springer87] and [McKay99]). In these cases the Molien series $P_{S_G,\rho}(t)$ of S_G is easily derived from the formula for $P_{S,\rho}(t)$.

The first purpose of this paper is to give an explicit formula for $P_{S_G,\rho}$ when G is one of the exceptional finite subgroups of $\mathrm{SL}(3, \mathbf{C})$ of type from (E) to (L) in the notation of [YY93]. Using the Koszul complex with G-action, we derive a certain system of equations analogous to the SU(2) case [McKay99] satisfied by the Molien series $P_{S,\rho}$. The equations are obtained just by taking alternating sums of componentwise generating functions of G-modules in the Koszul complex. They are given explicitly in terms of irreducible decompositions of tensor products with

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the natural representation ρ_{nat} and its second exterior product $\overset{2}{\wedge}\rho_{nat}$. This will be discussed in Section 2. The consequence of this section enables us to compute $P_{S,\rho}$ explicitly later. However the calculation of $P_{S_G,\rho}$ in the exceptional cases (E)-(L) is much harder, which will be discussed in Sections 4 and 5. This study of the Molien series $P_{S_G,\rho}$ was in fact motivated by the study of the G-orbit Hilbert scheme explained below, in particular by the study of $\pi^{-1}(0)$.

For a positive integer N, $\mathrm{Hilb}^N(\mathbf{C}^n)$ is the universal scheme which parametrizes all zero-dimensional subschemes of $\mathbf{C}^3$ of length N. For a finite subgroup G of $\mathrm{GL}(n,\mathbf{C})$, we choose $N = |G|$, the order of G. Then the group G acts in the natural manner on $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$. The G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^n)$ is by definition the unique irreducible component of the G-invariant part of $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$ dominating $\mathbf{C}^n/G$, the G-invariant part of the corresponding Chow scheme of $|G|$ points. In other words, $\mathrm{Hilb}^G(\mathbf{C}^n)$ is the universal subscheme of the Hilbert scheme $\mathrm{Hilb}^{|G|}(\mathbf{C}^n)$ which parametrizes all smoothable scheme-theoretic G-orbits of length $|G|$. The G-orbit Hilbert scheme $\mathrm{Hilb}^G(\mathbf{C}^n)$ is a fairly natural algebro-geometric object which incorporates all representation-theoretic information about G as a subgroup of $\mathrm{GL}(n,\mathbf{C})$. It has already been studied in detail in the SU(2) case [IN99] and in the case where G is a noncommutative simple subgroup A_5 or $\mathrm{PSL}(2,7)$ of $\mathrm{SL}(3,\mathbf{C})$ [GNS00]. The scheme $\mathrm{Hilb}^N(\mathbf{C}^n)$ is known to be very singular if $n \geq 3$. However for a finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$, $\mathrm{Hilb}^G(\mathbf{C}^3)$ is known to be nonsingular by [N01] in the abelian case and by [BKR01] in the general case.

The second purpose of the article is to study $\mathrm{Hilb}^G(\mathbf{C}^3)$, among other things, the fiber $\pi^{-1}(0)$ of the Hilbert-Chow morphism $\pi : \mathrm{Hilb}^G(\mathbf{C}^3) \to \mathbf{C}^3/G$ when G is a finite subgroup of SO(3). This will be discussed in Section 3.

It is well known that there is a surjective homomorphism from SU(2) onto SO(3) having ± 1 as its kernel, by which non-abelian subgroups of SU(2) and SO(3) correspond bijectively. For a subgroup G of SO(3) we define the representation graph $R(G)$ of G by using the irreducible decompositions of tensor products with ρ_{nat} in the same manner as in the SU(2) case. First we observe that $\pi^{-1}(0)$ is a union of finitely many smooth rational curves. So we define in the same way as in the SU(2) case the planar dual graph $\overline{R}(G)$ of $\pi^{-1}(0)$ by associating a vertex to each rational curve in $\pi^{-1}(0)$, and by associating an edge connecting a pair of the vertices to each intersection point of the corresponding curves. Then it turns out that the planar dual graph $\overline{R}(G)$ is identified with a particular subgraph of $R(G)$. In other words, every irreducible rational curve in $\pi^{-1}(0)$ is labeled by one of the nontrivial irreducible representations of G and vice versa, whose intersections are described purely in terms of irreducible decompositions of tensor products with ρ_{nat} in a manner similar to the SU(2) case. Thus we have a complete description of $\pi^{-1}(0)$ in the SO(3) case. However in almost all cases other than (A), (H) and (I) in the notation of [YY93] the precise structure of $\pi^{-1}(0)$ is yet to be determined.

This paper is organized as follows. In Section 1, we explain basic lemmas necessary for computing $P_{S_G,\rho}$. In Section 2, we first recall the Koszul complex over S and show that any alternating sum of componentwise generating functions of the G-modules in the Koszul complex is equal to zero, which yields a Springer-McKay type identity of $P_{S,\rho}$. In Section 3, we describe $\pi^{-1}(0)$ completely when G is a subgroup of SO(3).

In Sections 4 and 5 we give tables of $P_{S_G,\rho}$ for every finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$ of type from (E) to (L) and every non-trivial representation ρ of G.

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1. The coinvariant algebra for a finite subgroup G of $\mathrm{SL}(3,\mathbf{C})$

1.1. **The Molien series.** Let V be an n-dimensional complex vector space, $V^\vee$ the dual of V and G a finite subgroup of $\mathrm{GL}(V)$. We denote by ρ the matrix representation of G afforded by the natural inclusion of G into $\mathrm{GL}(V)$ and by $\rho^\vee$ its contragredient representation. As usual we call ρ the natural representation of G. We use the same notation as in [GNS00]; in particular we denote by $S = S(V^\vee)$, $\mathfrak{m} = S_+$, S^G and S^G_+ respectively the symmetric algebra of $V^\vee$ over $\mathbf{C}$, the maximal ideal of S of the origin, the invariant algebra of G, and the maximal ideal of S^G of the origin. Let $\mathfrak{n}$ be the ideal of S generated by S^G_+ and $S_G := S/\mathfrak{n}$ the coinvariant algebra of G. Since $\mathfrak{n}$ is a graded ideal of S, S_G is a graded algebra, too.

By the Noether normalization lemma, we can take a minimal system of homogeneous parameters $f_1, f_2, \ldots, f_n$ of S^G so that S^G is a finite module over $\mathbf{C}[f_1, ..., f_n]$. Extending them we choose a minimal system of homogeneous generators $f_1, f_2, \ldots, f_r$ of S^G and fix them once for all. The ideal $\mathfrak{n}$ of S is generated by $f_1, f_2, \ldots, f_r$.

Let $\hat{G} = \{\rho_0 = 1, \rho_1, \ldots, \rho_s\}$ be the set of representatives of equivalence classes of all irreducible representations of G and χ_i the character of ρ_i for $0 \le i \le s$. For an arbitrary graded $\mathbf{C}G$-module $M = \oplus_{i\ge 0} M_i$ with $\dim M_i < \infty$, we define the Molien series of M for ρ_j by

$$P_{M,\rho_j}(t) = \sum_{i\ge 0} \langle M_i, \rho_j\rangle_G \, t^i,$$

where

$$\langle M_i, \rho_j\rangle_G = \dim \mathrm{Hom}_G(\rho_j, M_i) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_j(g)}\ \mathrm{Tr}_{M_i}(g).$$

The following is derived easily from the formula in [Bourbaki, Lemme 2, p. 110]

$$P_{S,\rho_j}(t) = \frac{1}{|G|}\sum_{g\in G} \frac{\overline{\chi_j(g)}}{\det(1-\rho^\vee(g)t)}. \tag{1}$$

Now we recall from [Stanley79, (4.9)].

Theorem 1.2. *Let $f_1, f_2, \ldots, f_r$ be homogeneous generators of S^G chosen as above, $d_i = \deg f_i$, $(f_1, f_2, \ldots, f_n)$ the ideal of S generated by $f_1, f_2, \ldots, f_n$, and let $R = S/(f_1, f_2, \ldots, f_n)$. Then as $\mathbf{C}G$-modules we have*

$$S \simeq R \otimes \mathbf{C}[f_1, f_2, \ldots, f_n] \ \text{ and } \ R \simeq (\mathbf{C}G)^e$$

where $e = |G|^{-1} d_1 d_2 \cdots d_n$.

Proposition 1.3. *Keeping the notations as above, we have*

(i)

$$P_{R,\rho_j}(t) = \frac{\prod_{i=1}^n (1-t^{d_i})}{|G|}\sum_{g\in G} \frac{\overline{\chi_j(g)}}{\det(1-\rho^\vee(g)t)}.$$

(ii) $P_{R,\rho_j}(t) - P_{S_G,\rho_j}(t)$ *is a polynomial with non-negative integer coefficients.*
(iii)

$$\sum_{j=0}^{s}(\deg\rho_j)P_{S_G,\rho_j}(t) = \sum_{j\geq 0}\dim(S_G)_i t^i.$$

Proof. (i) It follows from Theorem 1.2 that $P_{S,\rho_j}(t) = P_{R,\rho_j}(t)/\prod_{i=1}^n(1-t^{d_i})$. From Molien's formula (1), we infer (i).
(ii) Since we have a canonical surjection from R to S_G, $P_{R,\rho_j}(t) - P_{S_G,\rho_j}(t)$ has non-negative integer coefficients.
(iii) Let $S_G = \oplus_{j=0}^s (S_G)_{\rho_j}$ be the decomposition into homogeneous components, namely ρ_j-factors $(S_G)_{\rho_j}$ of S_G. Since $\dim(S_G)_{\rho_j} = (\deg\rho_j)\langle S_G, \rho_j\rangle_G$, the above equation is clear from the definition of $P_{S_G,\rho_j}(t)$. □

We note that if there exists a complex reflection group $\tilde{G}$ of $\mathrm{GL}(V)$ containing G with $[\tilde{G} : G] = 2$, then it is easier to calculate $P_{S_G,\rho_j}(t)$ by using the following

Theorem 1.4. ([Bourbaki] or [GNS00, 1.6]) Assume that there exists a complex reflection subgroup $\tilde{G}$ of $\mathrm{GL}(V)$ containing G with $[\tilde{G} : G] = 2$.

(i) There exist n homogeneous $\tilde{G}$-invariants $f_1, f_2, \ldots, f_n$ such that as $\mathbf{C}\tilde{G}$-modules $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, \ldots, f_n]$ and $S_{\tilde{G}} = S/(f_1, f_2, \ldots, f_n) \simeq \mathbf{C}\tilde{G}$.
(ii) Let $f_{n+1} = \mathrm{Jac}(f_1, f_2, \ldots, f_n)$. Then we have

$$S^G = \mathbf{C}[f_1, f_2, \ldots, f_n, f_{n+1}] \text{ and } S_{\tilde{G}} \simeq S_G \oplus \mathbf{C}f_{n+1}.$$

Moreover

$$(S_{\tilde{G}})_k \simeq \begin{cases} (S_G)_k, & \text{if } k < d_{n+1}, \\ \mathbf{C}f_{n+1}, & \text{if } k = d_{n+1}, \\ 0, & \text{if } k > d_{n+1}, \end{cases}$$

where $d_{n+1} = \deg f_{n+1} = \sum_{i=1}^n (d_i - 1)$.

Corollary 1.5. *Under the same assumptions in Theorem 1.4*

$$P_{S_G,\rho_j}(t) = P_{S_{\tilde{G}},\rho_j}(t) = \prod_{i=1}^n (1-t^{d_i})P_{S,\rho_j}(t),$$

$$P_{S_G,\rho_0}(t) = P_{S_{\tilde{G}},\rho_0}(t) + t^{n+1} = \prod_{i=1}^n (1-t^{d_i})P_{S,\rho_j}(t) + t^{d_{n+1}}.$$

Proof. Immediate from Theorem 1.4. □

Remark 1.6. Let G be a finite subgroup of $\mathrm{SL}(3,\mathbf{C})$ of exceptional type (E)-(L). Then homogeneous generators of S^G are known explicitly in [YY93]. Moreover, since $(S_G)_i \simeq S_i/(\mathfrak{n})_i$ and $(\mathfrak{n})_i = V^\vee \cdot (\mathfrak{n})_{i-1} + \sum_{\deg f_j = i}\mathbf{C}f_j$, we can calculate $(\mathfrak{n})_i$ inductively. Thus all the informations of Proposition 1.3 are available, which turns out to be sufficient to determine $P_{S_G,\rho_j}(t)$ by the case-by-case examination. The results are summarized in Sections 4 and 5.

Either of the groups of type (H), (I) and (L) is a subgroup of some complex reflection group of index two, while the group of type (E), (F) or (J) is a subgroup of some complex reflection group of index 6, 3 or 12 respectively. In these cases we can apply [Steinberg64] and [Stanley79] to describe $R := S/(f_1, f_2, f_3)$ in some detail. However no group of type (G) or (K) is a subgroup of a complex reflection group. Nevertheless in any case from (E) to (L) the algebra R has a remarkable duality as in the cases of complex reflection groups. We will discuss it elsewhere.

2. Koszul complex and Springer-McKay identities of Molien series

We keep the previous notations. We start with the Koszul complex for the symmetric algebra $S = S(V^\vee)$ (cf. [Lang84, XVI §10]).

Lemma 2.1. *Let* $\overset{k}{\wedge} V^\vee$ *be the* k*-th alternating product of* $V^\vee$.

(i) *There is a unique homomorphism*

$$d_k : \overset{k}{\wedge} V^\vee \otimes S \longrightarrow \overset{k-1}{\wedge} V^\vee \otimes S$$

such that for $x_i \in V^\vee$ *and* $y \in S$

$$\begin{aligned} & d_k((x_1 \wedge x_2 \wedge \cdots \wedge x_k) \otimes y) \\ & = \sum_{i=1}^{k} (-1)^{i-1} (x_1 \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k) \otimes (x_i \cdot y). \end{aligned}$$

(ii) *There is an exact sequence with* d_k *given by* (i)

$$0 \to \overset{n}{\wedge} V^\vee \otimes S \overset{d_n}{\to} \overset{n-1}{\wedge} V^\vee \otimes S \overset{d_{n-1}}{\to} \cdots \overset{d_2}{\to} V^\vee \otimes S \overset{d_1}{\to} S \overset{d_0}{\to} \mathbf{C} \to 0.$$

(iii) *For each integer* $m \geq 1$ *we have an exact sequence*

$$0 \to \overset{n}{\wedge} V^\vee \otimes S_{m-n} \to \overset{n-1}{\wedge} V^\vee \otimes S_{m-n+1} \to \cdots \to S_m \to 0,$$

where $S_j = 0$ *for* $j < 0$.

(iv) *For each integer* $m \geq 1$ *and for each irreducible representation* ρ_j $(0 \leq j \leq s)$, *we have an exact sequence*

$$0 \to (\overset{n}{\wedge} V^\vee \otimes S_{m-n})_{\rho_j} \to (\overset{n-1}{\wedge} V^\vee \otimes S_{m-n+1})_{\rho_j} \to \cdots \to (S_m)_{\rho_j} \to 0.$$

Proof. For a proof of (i), (ii) and (iii), see [Lang84, (10.13) and (10.14)]. Since d_k is a G-homomorphism, we decompose the exact sequence of (iii) into ρ_j-components, which proves (iv). □

We denote by $\rho^{(k)}$ (resp. $\rho^{\vee(k)}$) the $\mathbf{C}G$-module $\overset{k}{\wedge} V$ (resp. $\overset{k}{\wedge} V^\vee$). Note that $\rho^{(0)} = 1, \rho^{(1)} = \rho, \rho^{(n)} = \det$, and $\rho^{\vee(k)}$ is the dual $\mathbf{C}G$-module of $\rho^{(k)}$. Define non-negative integers $a_{i,j}^{(k)}$ by

$$\rho^{(k)} \otimes \rho_i = \sum_{j=0}^{s} a_{ij}^{(k)} \rho_j, \quad \text{for } 0 \leq i \leq s \text{ and } 0 \leq k \leq n. \tag{2}$$

Theorem 2.2. *The Molien series* $P_{S,\rho_j}(t)$ *satisfy the following equations:*

$$\sum_{k=0}^{n} \sum_{j=0}^{s} (-1)^k a_{ij}^{(k)} t^k P_{S,\rho_j}(t) = \delta_{i,0} \quad \textit{for } i = 0, 1, \ldots, s.$$

Proof. We see

$$\begin{aligned}\dim(\overset{k}{\wedge}V^\vee\otimes S_{m-k})_{\rho_i} &= \deg(\rho_i)\dim\operatorname{Hom}_G(\rho_i,\ \rho^{\vee(k)}\otimes S_{m-k})\\ &= \deg(\rho_i)\dim\operatorname{Hom}_G(\rho^{(k)}\otimes\rho_i,\ S_{m-k})\\ &= \deg(\rho_i)\sum_{j=0}^{s}a_{ij}^{(k)}\dim\operatorname{Hom}_G(\rho_j,\ S_{m-k}).\end{aligned}$$

Thus we obtain

$$\sum_{m\geq 0}(\dim(\overset{k}{\wedge}V^\vee\otimes S_{m-k})_{\rho_i})t^m = \deg(\rho_i)\sum_{j=0}^{s}a_{ij}^{(k)}t^kP_{S,\rho_j}(t).$$

Hence our theorem follows from Lemma 2.1 (ii) and (iv). □

Remark 2.3. This proposition can be proved directly by using (1).

Corollary 2.4. *Keep the same notation in Theorem 2.2.*

(i) *If G is a subgroup of* $\mathrm{SL}(V)$*, then*

$$\sum_{k=1}^{n-1}\sum_{j=0}^{s}(-1)^k a_{ij}^{(k)}t^kP_{S,\rho_j}(t) = (-1-(-1)^nt^n)P_{S,\rho_i}(t)+\delta_{i,0},$$

(ii) *If G is a subgroup of* $\mathrm{SL}(2,\mathbf{C})$*, then*

$$\sum_{j=0}^{s}a_{ij}^{(1)}P_{S,\rho_j}(t) = (t+t^{-1})P_{S,\rho_i}(t)-t^{-1}\delta_{i,0},$$

(iii) *If G is a subgroup of* $\mathrm{SL}(3,\mathbf{C})$ *and if $\rho^\vee=\rho$, then*

$$\sum_{j=0}^{s}a_{ij}^{(1)}P_{S,\rho_j}(t) = (t+1+t^{-1})P_{S,\rho_i}(t)+(t^2-t)^{-1}\delta_{i,0}.$$

(The assumption in (iii) is satisfied if $G\subset\mathrm{SO}(3)$.)

Proof. If G is a finite subgroup of $\mathrm{SL}(V)$, then $\rho^{(0)}$ and $\rho^{(n)}$ are trivial. So (i) follows at once from Theorem 2.2. If $\dim V=2$, we obtain (ii) by dividing both sides of (i) by $-t$. Under the assumption of (iii), we have $\rho^{(1)}=\rho^{(2)}=\rho$. Dividing both sides of (i) by (t^2-t), we obtain (iii). □

Put $F_j(t)=P_{S,\rho_j}(t)\prod_{i=1}^n(1-t^{d_i})$ for $0\leq j\leq s$. By Theorem 1.4

$$F_j(t)=\begin{cases}1+t^{d_{n+1}} & \text{if } j=0\\ P_{S_G,\rho_j}(t) & \text{if } j\neq 0.\end{cases}$$

The next corollary is immediate from Corollary 2.4.

Corollary 2.5. *Keep the notation as above. Let $0\leq i\leq s$. Then*

(i) *If G is a finite subgroup of* $\mathrm{SL}(2,\mathbf{C})$*, then*

$$\sum_{j=0}^{s}a_{ij}^{(1)}F_j(t)=(t+t^{-1})F_i(t)-\frac{(1-t^{d_1})(1-t^{d_2})}{t}\delta_{i,0}.$$

(ii) *If G is a finite subgroup of* SO(3), *then*

$$\sum_{j=0}^{s} a_{ij}^{(1)} F_j(t) = (t+1+t^{-1})F_i(t) + \frac{(1-t^{d_1})(1-t^{d_2})(1-t^{d_3})}{(t^2-t)}\delta_{i,0}.$$

Remark 2.6. The system of equations in Corollary 2.4 (ii) were given in [Springer87] and [McKay99] by using corresponding Coxeter-Dynkin diagrams, or McKay's semi-affine graphs. Corollary 2.4 (i) claims, roughly speaking, that one can calculate all the Molien series once one knows $a_{ij}^{(k)}$, in particular only $a_{ij}^{(1)}$ when $G \subset \mathrm{SL}(2, \mathbf{C})$ or $G \subset \mathrm{SO}(3)$. In this sense the representation graph (or rather the indices $a_{ij}^{(1)}$) of a subgroup G of SO(3) plays the same role in calculating Molien series as the Coxeter-Dynkin diagram for a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$.

2.7. **Complex reflection groups.** If G is a finite subgroup of $\mathrm{SL}(2, \mathbf{C})$ or SO(3), there exists a complex reflection group $\tilde{G}$ containing G with $[\tilde{G} : G] = 2$. We list all such pairs G and $\tilde{G}$ in Table 1 and Table 2. We use the notation in [Cohen76]; the group G_i is the complex reflection group with Shephard-Todd number i. The symbol $W(A)$ stands for the Weyl group of type A. The integer d_i in the tables is the degree of f_i defined in Theorem 1.4.

G in $\mathrm{SL}(2, \mathbf{C})$	order	$\tilde{G}$	d_1, d_2
cyclic	l	$W(I_2^{(l)})$	$2, l$
binary dihedral	$4l$	$G(2l, l, 2)$	$4, 2l$
binary tetrahedral	24	G_{12}	$6, 8$
binary octahedral	48	G_{13}	$8, 12$
binary icosahedral	120	G_{22}	$12, 20$

TABLE 1. Subgroups of $\mathrm{SL}(2, \mathbf{C})$

G in SO(3)	order	$\tilde{G}$	d_1, d_2, d_3
cyclic	l	$W(I_2^{(l)})$	$1, 2, l$
dihedral	$2l$	$W(I_2^{(l)} \times A_1)$	$2, 2, l$
tetrahedral ($\simeq A_4$)	12	$W(A_3)$	$2, 3, 4$
octahedral ($\simeq S_4$)	24	$W(B_3)$	$2, 4, 6$
icosahedral ($\simeq A_5$)	60	$W(H_3)$	$2, 6, 10$

TABLE 2. Subgroups of SO(3)

3. Geometric McKay correspondence for subgroups of SO(3)

Let $\pi : \mathrm{Hilb}^G(\mathbf{C}^3) \to \mathbf{C}^3/G$ be the Hilbert-Chow morphism for $G \subset \mathrm{SO}(3)$.

Theorem 3.1. *Let G be a finite subgroup of* SO(3). *For $I \in \mathrm{Hilb}^G(\mathbf{C}^3)$ with $I \subset \mathfrak{m}$, we define $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$. For $1 \le i \le s$, we define $C_j = \{I \in \mathrm{Hilb}^G(\mathbf{C}^3); V(I) \supset \rho_j, I \subset \mathfrak{m}\}$. Then*

(i) *$C_j \simeq \mathbf{P}^1$ and $\pi^{-1}(0) = \cup_{j=1}^s C_j$.*
(ii) *If $I \in C_j$ and $I \notin C_i$ for any $j \neq i$, then $V(I) \simeq \rho_j$ as G-modules.*
(iii) *If only two rational curves C_i and C_j meet at $I \in \pi^{-1}(0)$, then C_i and C_j intersect at I transversally and $V(I) \simeq \rho_i + \rho_j$.*
(iv) *If G is either cyclic, A_4 or D_{4m+2}, then there are no three rational curves meeting at a point of $\pi^{-1}(0)$.*
(v) *If $G = D_{4m}, S_4$ or A_5, then there is a unique $I \in \pi^{-1}(0)$ such that $\{I\} = C_i \cap C_j \cap C_k$ for $\rho_i, \rho_j, \rho_k \in \hat{G}$ all distinct. In this case $V(I) \simeq \rho_i + \rho_j + \rho_k$ and the curves C_i, C_j, C_k meet transversally at I as coordinate axes of $(\mathbf{C}^3, 0)$.*
(vi) *No four rational curves C_i meet at a point of $\pi^{-1}(0)$.*

Our proof of Theorem 3.1 is carried out by the case by case examination. When G is abelian, our theorem is proved by the same argument as in the two dimensional case. When G is isomorphic to the alternating group A_4 or A_5, our theorem has been proved in [GNS00]. So we only need to prove our theorem when G is a dihedral group or $G = S_4$. We will give a proof of it in the subsections 3.4, 3.5 and 3.6.

3.2. **Graphs of G.** Here we define three graphs for a finite subgroup G of SO(3).

First we define the planar dual graph $\overline{R}(G)$ of $\pi^{-1}(0)$ as follows: the set of vertices of $\overline{R}(G)$ is $\{C_j\}_{1\le j\le s}$; C_i and C_j are joined by a single edge if and only if $C_i \cap C_j \neq \phi$. We note that in Theorem 3.1 there are three rational curves C_i, C_j and C_k in $\pi^{-1}(0)$ meeting at a point, for which we define a planar triangle in $\overline{R}(G)$ with three vertices C_i, C_j and C_k instead of a two cell. See Table 3.

Next we define the (unoriented) representation graph $R(G)$ of G as follows: the set of vertices is $\hat{G}$; let $a_{i,j}^{(1)}$ be the integer defined in (2); ρ_i and ρ_j are joined by an edge of multiplicity $a_{i,j}^{(1)}$ if $a_{i,j}^{(1)} \neq 0$, where if $i = j$ the edge joining ρ_i with itself is understood as a loop of multiplicity $a_{i,i}^{(1)}$. We note $a_{i,j}^{(1)} = 0$ or 1 for $i \neq j$, while $a_{i,i}^{(1)} = 0, 1,$ or 2. We also note that $a_{i,j}^{(1)} = a_{i,j}^{(2)}$ for any finite subgroup G of SO(3).

Finally we define a subgraph $R_0(G)$ of $R(G)$ as follows: the set of vertices is $\{\rho_j\}_{1\le j\le s}$ and ρ_i and ρ_j are joined by a single edge if and only if $i \neq j$ and $a_{i,j}^{(1)} \neq 0$. In other words, $R_0(G)$ is the subgraph of $R(G)$ obtained from $R(G)$ by removing the vertex ρ_0, all the edges starting from ρ_0 and all the loops in $R(G)$.

The following theorem is a corollary to the proof of Theorem 3.1 once we calculate the representation graph $R(G)$.

Theorem 3.3. *$\overline{R}(G)$ is isomorphic to $R_0(G)$ under the map $C_i \mapsto \rho_i$ $(1 \le i \le s)$. The graphs $\overline{R}(G)$ and $R(G)$ are given in Table 3.*

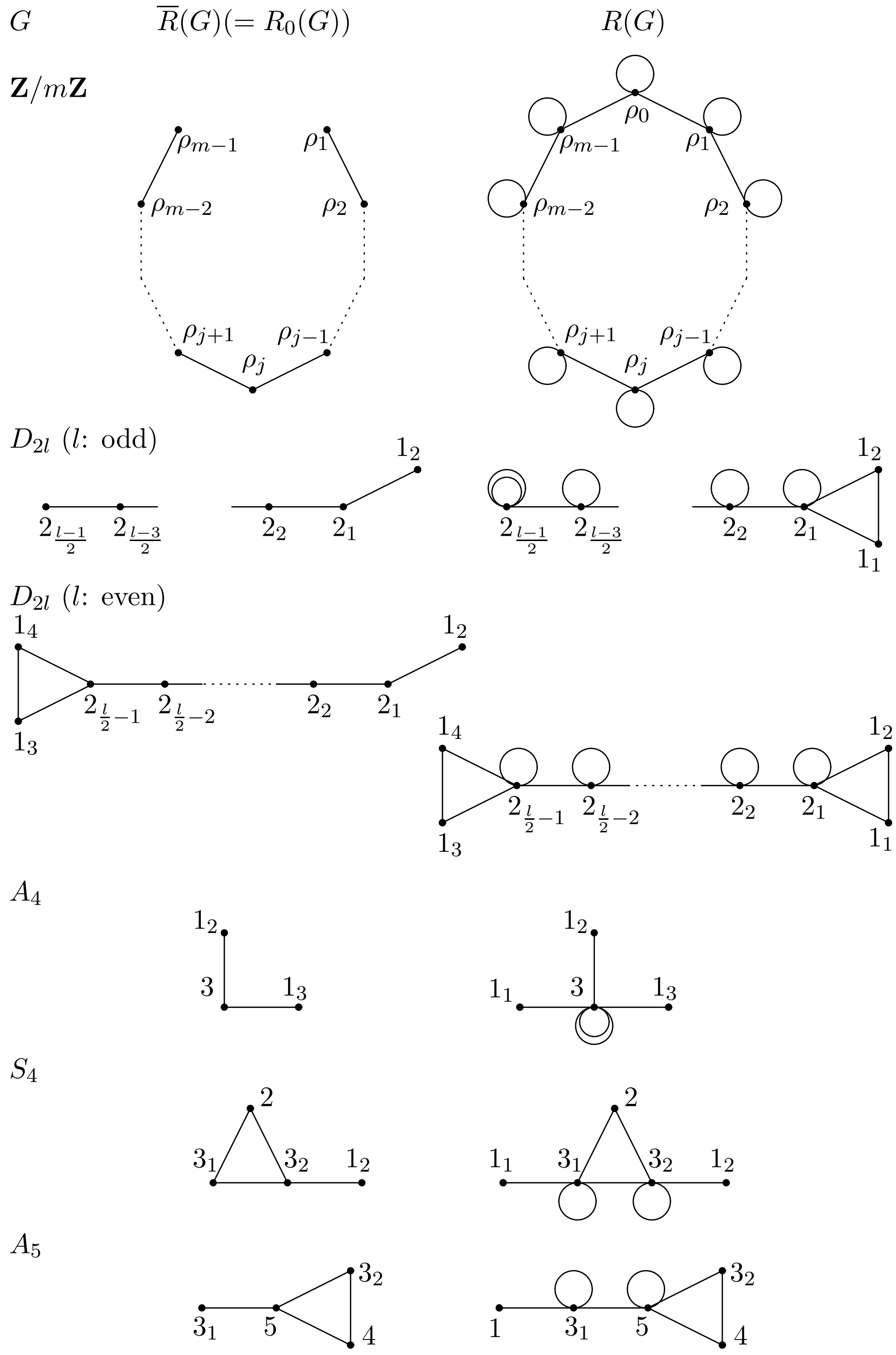


TABLE 3. Graphs of subgroups of SO(3)

In the rest of this section we give proofs of Theorem 3.1 in the cases where G is a dihedral group or G is isomorphic to S_4.

3.4. **Proof of Theorem 3.1 — the dihedral group of order** $2\ell = 4m$**.** Let G be the dihedral group of order 2ℓ:

$$G = \langle \sigma = \begin{pmatrix} \varepsilon^{-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle, \quad \text{where } \varepsilon = e^{2\pi i/\ell}.$$

We define

$$f_1 = z^2,\ f_2 = xy,\ f_3 = x^\ell + y^\ell,\ f_4 = z(x^\ell - y^\ell).$$

Then we see $\{f_1, f_2, f_3, f_4\}$ is a system of generators of S^G which satisfies

$$f_4^2 - f_1 f_3^2 + 4 f_1 f_2^\ell = 0.$$

regardless of the parity of ℓ.

First in this subsection we consider the case where ℓ is even. So we write $\ell = 2m$, $|G| = 4m$. The character table of G is as follows.

c. c	1	-1	τ	$\tau\sigma$	σ^i
age	0	1	1	1	1
$\sharp$	1	1	m	m	2
1_1	1	1	1	1	1
1_2	1	1	-1	-1	1
1_3	1	$(-1)^m$	1	-1	$(-1)^i$
1_4	1	$(-1)^m$	-1	1	$(-1)^i$
2_j	2	$(-1)^j 2$	0	0	$\varepsilon^{ij} + \varepsilon^{-ij}$
					$(1 \le i, j \le m-1)$

TABLE 4. Characters of $G(D_{2\ell})$, $\ell = 2m$:even

The coinvariant algebra S_G splits into irreducible components as in Table 5. Using Table 5 we define ideals in $\mathrm{Hilb}^G(\mathbf{C}^3)$ $[a:b] \in \mathbf{P}^1$ as in [GNS00].

$$\begin{aligned}
&I([a:b]_{1_2}) = (az + b(x^{2m} - y^{2m}), xz, yz) + \mathfrak{n},\\
&I([a:b]_{1_3}) = (a(x^m + y^m) + b(x^m - y^m)z, x^{m+1}, y^{m+1}, (x^m + y^m)z) + \mathfrak{n},\\
&I([a:b]_{1_4}) = (a(x^m - y^m) + b(x^m + y^m)z, x^{m+1}, y^{m+1}, (x^m - y^m)z) + \mathfrak{n},\\
&I([a:b]_{2_j}) = S[G] \cdot (ax^j z + by^{2m-j}, x^{j+1}z, x^{2m-j+1}) + \mathfrak{n}.\\
&\hfill (i = 1, 2, \dots, m-1)
\end{aligned}$$

It is clear that $V(I([a:b]_\rho)) \simeq \rho$ as G-modules. We note that the following exhaust all the possible cases of coincidence between $I([a:b]_\rho)$.

$$\begin{aligned}
&I([0:1]_{1_2}) = I([1:0]_{2_1}),\\
&I([0:1]_{2_j}) = I([1:0]_{2_{j+1}}), \quad \text{for } j = 1, 2, \dots, m-2,\\
&I([0:1]_{2_{m-1}}) = I([0:1]_{1_3}) = I([0:1]_{1_4}).
\end{aligned}$$

degree	$(S_G)_j$	irred. factors
1	$\langle x, y\rangle \oplus \langle z\rangle$	$2_1 + 1_2$
$2 \le j \le m-1$	$\langle x^j, y^j\rangle \oplus \langle x^{j-1}z, -y^{j-1}z\rangle$	$2_j + 2_{j-1}$
m	$\langle x^m + y^m\rangle \oplus \langle x^m - y^m\rangle$	
	$\oplus\langle x^{m-1}z, -y^{m-1}z\rangle$	$1_3 + 1_4 + 2_{m-1}$
$m+1$	$\langle y^{m+1}, x^{m+1}\rangle \oplus \langle (x^m - y^m)z\rangle$	
	$\oplus\langle (x^m + y^m)z\rangle$	$2_{m-1} + 1_3 + 1_4$
$m+2 \le j \le 2m-1$	$\langle y^j, x^j\rangle \oplus \langle y^{j-1}z, -x^{j-1}z\rangle$	$2_{2m-j} + 2_{2m-j+1}$
$2m$	$\langle x^{2m} - y^{2m}\rangle \oplus \langle y^{2m-1}z, -x^{2m-1}z\rangle$	$1_2 + 2_1$

TABLE 5. The coinvariant algebra of $G(D_{2\ell})$, $\ell = 2m$: even

Now we prove

$$\pi^{-1}(0) = \cup_{\rho\in\hat{G}\setminus\{1_1\}} I([a:b]_\rho).$$

It is immediate from the definition and the Diagram D_{4m} (see 3.7) that $I([a:b]_\rho)$ are contained in $\pi^{-1}(0)$. Conversely let I be an ideal contained in $\pi^{-1}(0)$, that is, $\mathfrak{n} \subset I \subset \mathfrak{m}$ and $S/I \simeq \mathbf{C}[G]$. By the Diagram D_{4m} , it is easy to see that $x^{2m-j}z, y^{2m-j}z \in I$ for all $j = 1, 2, \ldots, m-1$ and that $x^j + ax^jz + by^{2m-j} \notin I$ for any $a, b \in \mathbf{C}$ and $j = 1, 2, \ldots, m-1$. If $x^jz + by^{2m-j} \in I$ for some $b \neq 0$ and some $j = 1, 2, \ldots, m-1$, then we have $I([1:b]_{2_j}) \subset I$ which implies $I([1:b]_{2_j}) = I$.

Now we assume the contrary, that is, that $x^jz + by^{2m-j} \notin I$ for any nonzero b and any $j = 1, \cdots, m-1$. Then by the condition $S/I \simeq \mathbf{C}[G]$ we have either $x^jz \in I$ or $y^{2m-j} \in I$. If there is $j \ge 2$ such that $x^jz \in I$, $x^{j-1}z \notin I$, then $y^{2m-j+1} \in I$. It follows that $I = I([1:0]_{2_j})$. If $xz \in I$, then $I = I([a:b]_{1_2})$.

It remains to consider the case where there is no j such that $x^jz \in I$. Hence $y^{m+1} \in I$. If $x^m + y^m + b(x^m - y^m)z \in I$ (resp. $x^m - y^m + b(x^m + y^m)z \in I$) for some $b \in \mathbf{C}$, then $I = I([1:b]_{1_3})$(resp. $I([1:b]_{1_4})$). Otherwise I contains $(x^m - y^m)z$ and $(x^m + y^m)z$ and then we have $I = I([0:1]_{1_3})$. Thus we complete the proof of Theorem 3.1 when G is a dihedral group of order $4m$.

3.5. **Proof of Theorem 3.1 — the dihedral group of order** $4m+2$**.** Now we consider the second case where G is a dihedral group of order $2\ell = 4m+2$. Table 6 is the character table of G. The coinvariant algebra S_G splits into irreducible components as in Table 7.

We define

$$\begin{aligned}
I([a:b]_{1_2}) &= (az + b(x^{2m+1} - y^{2m+1}), xz, yz) + \mathfrak{n},\\
I([a:b]_{2_j}) &= S[G]\cdot(ax^jz + by^{2m-j+1}, x^{j+1}z, x^{2m-j+2}) + \mathfrak{n},\\
&\qquad j = 1, 2, \ldots, m,
\end{aligned}$$

where

$$\begin{aligned}
I([0:1]_{1_2}) &= I([1:0]_{2_1}),\\
I([0:1]_{2_j}) &= I([1:0]_{2_{j+1}}), \quad \text{for } j = 1, 2, \ldots, m-1.
\end{aligned}$$

c. c	1	τ	σ^i
age	0	1	1
$\sharp$	1	$2m+1$	2
1_1	1	1	1
1_2	1	-1	1
2_j	2	0	$\varepsilon^{ij}+\varepsilon^{-ij}$
			$(1 \le i,j \le m)$

TABLE 6. Characters of $G(D_{2\ell})$, $\ell = 2m+1$:odd

degree	$(S_G)_j$	irred. factors
1	$\langle x,y\rangle \oplus \langle z\rangle$	2_1+1_2
j	$\langle x^j,y^j\rangle \oplus \langle x^{j-1}z,-y^{j-1}z\rangle$	2_j+2_{j-1}
		$(2 \le j \le m-1)$
m	$\langle x^m,y^m\rangle \oplus \langle x^{m-1}z,-y^{m-1}z\rangle$	2_m+2_{m-1}
$m+1$	$\langle y^{m+1},x^{m+1}\rangle \oplus \langle x^mz,-y^mz\rangle$	2_m+2_m
$m+2$	$\langle y^{m+2},x^{m+2}\rangle \oplus \langle x^{m+1}z,-y^{m+1}z\rangle$	$2_{m-1}+2_m$
j	$\langle y^j,x^j\rangle \oplus \langle y^{j-1}z,-x^{j-1}z\rangle$	$2_{2m-j+1}+2_{2m-j+2}$
		$(m+3 \le j \le 2m)$
$2m+1$	$\langle x^{2m+1}-y^{2m+1}\rangle \oplus \langle y^{2m}z,-x^{2m}z\rangle$	1_2+2_1

TABLE 7. The coinvariant algebra of $G(D_{2\ell})$, $\ell = 2m+1$: odd

We see $\pi^{-1}(0) = \cup_{\rho\in\hat{G}\setminus\{1_1\}} I([a:b]_\rho)$ in the same manner as in the case of even ℓ. As before we see that $x^{2m-j}z, y^{2m-j}z \in I$ for any $j = 1,2,\dots,m-1$ and that $x^j + ax^jz + by^{2m-j} \notin I$ for any $a,b \in \mathbf{C}$ and $j = 1,2,\dots,m-1$. If $x^jz + by^{2m-j} \in I$ for some $b \neq 0$ and some $j = 1,2,\dots,m-1$, then $I = I([1:b]_{2_j})$. If there is $j \ge 2$ such that $x^jz \in I$, $x^{j-1}z \notin I$, then $I = I([1:0]_{2_j})$. If $xz \in I$, then $I = I([a:b]_{1_2})$. If $x^mz \notin I$, then $y^{m+1}, x^{m+1} \in I$ so that $I = I([0:1]_{2_m})$.

3.6. **Proof of Theorem 3.1 — the symmetry group $G = S_4$.** Let

$$G = \langle \sigma = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle.$$

We define

$$f_1 = xyz,\ f_2 = x^2+y^2+z^2,\ f_3 = x^4+y^4+z^4,$$
$$f_4 = (x^2-y^2)(y^2-z^2)(z^2-x^2).$$

Then $\{f_1^2, f_2, f_3, f_1f_4\}$ is a system of generators of S^G which satisfies

$$\begin{aligned}4(f_1f_4)^2 + 108f_1^6 - 20f_1^4f_2^3 + 36f_1^4f_2f_3\\ + f_1^2f_2^6 - 4f_1^2f_2^4f_3 + 5f_1^2f_2^2f_3^2 - 2f_1^2f_3^3 = 0.\end{aligned}$$

The following is the character table of G.

c.c	1	σ^2	τ	σ	$\sigma\tau\sigma^2$
age	0	1	1	1	1
$\sharp$	1	3	8	6	6
1_1	1	1	1	1	1
1_2	1	1	1	-1	-1
2	2	2	-1	0	0
3_1	3	-1	0	1	-1
3_2	3	-1	0	-1	1

TABLE 8. Characters of S_4

The decomposition of the coinvariant algebra S_G into irreducible components is given in Table 9 where

$$g = x^2 + \omega y^2 + \omega^2 z^2, \quad \bar{g} = x^2 + \omega^2 y^2 + \omega z^2, \quad \omega = e^{2\pi\sqrt{-1}/3}.$$

d	$(S_G)_d$	irred. factors
1	$\langle x, y, z\rangle$	3_1
2	$\langle g, \bar{g}\rangle \oplus \langle yz, zx, xy\rangle$	$2 + 3_2$
3	$\langle f_1\rangle \oplus \langle x^3, y^3, z^3\rangle$	
	$\oplus\langle (y^2 - z^2)x, (z^2 - x^2)y, (x^2 - y^2)z\rangle$	$1_2 + 3_1 + 3_2$
4	$\langle \bar{g}^2, g^2\rangle \oplus \langle (y^2 - z^2)yz, (z^2 - x^2)zx, (x^2 - y^2)xy\rangle$	
	$\oplus\langle f_1x, f_1y, f_1z\rangle$	$2 + 3_1 + 3_2$
5	$\langle f_1g, -f_1\bar{g}\rangle \oplus \langle f_1yz, f_1zx, f_1xy\rangle$	
	$\oplus\langle (y^2 - z^2)x^3, (z^2 - x^2)y^3, (x^2 - y^2)z^3\rangle$	$2 + 3_1 + 3_2$
6	$\langle f_4\rangle \oplus \langle f_1(y^2 - z^2)x, f_1(z^2 - x^2)y, f_1(x^2 - y^2)z\rangle$	
	$\oplus\langle f_1x^3, f_1y^3, f_1z^3\rangle$	$1_2 + 3_1 + 3_2$
7	$\langle f_1\bar{g}^2, -f_1g^2\rangle$	
	$\oplus\langle f_1(y^2 - z^2)yz, f_1(z^2 - x^2)zx, f_1(x^2 - y^2)xy\rangle$	$2 + 3_2$
8	$\langle f_1(y^2 - z^2)x^3, f_1(z^2 - x^2)y^3, f_1(x^2 - y^2)z^3\rangle$	3_1

TABLE 9. The coinvariant algebra of S_4

We define

$$\begin{aligned}
&I([a:b]_{1_2}) = S\cdot(af_1+bf_4, f_1x, f_1y, f_1z)+\mathfrak{n},\\
&I([a:b]_{2}) = S[G]\cdot(ag^2+bf_1\bar{g}, (y^2-z^2)x^3, f_1yz)+\mathfrak{n},\\
&I([a:b]_{3_1}) = S[G]\cdot(a(y^2-z^2)yz+bf_1yz, f_1g, (y^2-z^2)x^3)+\mathfrak{n}.\\
&I([a:b]_{3_2}) = S[G]\cdot(af_1x+b(y^2-z^2)x^3, f_1g, f_1yz)+\mathfrak{n}.
\end{aligned}$$

Let $\bar{S}_d=(S_G)_d$, the degree d part of S_G. Let $I\in \mathrm{Hilb}^G(\mathbf{C}^3)$ such that $\mathfrak{n}\subset I\subset \mathfrak{m}$. First we note by using the quiver diagram of S_4 as before that I does not contain the elements whose projections to $\bar{S}_1\oplus\bar{S}_2$ (the degree one and two parts of S_G) are nonzero. We note also that I contains $\bar{S}_7\oplus\bar{S}_8$.

Assume that I contains an element af_1+bf_4 for $a\neq 0$. Then by the quiver diagram of S_4, we see easily that $I=I([a:b]_{1_2})$.

Now we consider the case I contains no element af_1+bf_4 for $a\neq 0$. Since $S_G/I=\mathbf{C}[G]$, $f_4\in I$, that is $\bar{S}_6(1_2)\subset I$. If I contains an element $af_1x+b(y^2-z^2)x^3$ for $a\neq 0$, then $I=I([a:b]_{3_2})$. If I contains an element $a(y^2-z^2)yz+bf_1yz$ for $a\neq 0$, then $I=I([a:b]_{3_1})$.

Now we consider the remaining cases. By the quiver diagram of S_4, we see $\bar{S}_5(3_1)\oplus \bar{S}_5(3_2)\subset I$ and $\bar{S}_6\subset I$. If $a\bar{g}^2+bf_1g\in I$ for $a\neq 0$, then $I=I([a:b]_2)$. If I contains no element $a\bar{g}^2+bf_1g$ for $a\neq 0$, then $f_1g\in\bar{S}_5(2)\subset I$ because I contains no elements with nonzero projections to $\bar{S}_1\oplus\bar{S}_2$. Hence $\bar{S}_5\subset I$, and $I=I([0:1]_2)=I([0:1]_{3_1})=I([0:1]_{3_2})$.

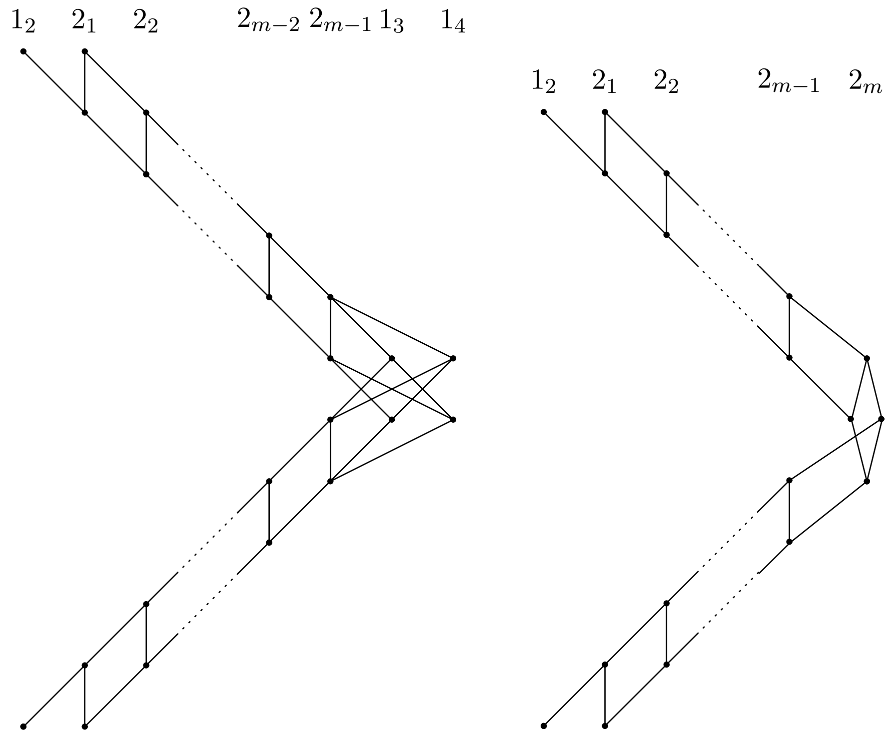
The following exhaust all the possible cases of coincidence between $I([a:b]_\rho)$.

$$\begin{aligned}
&I([0:1]_{1_2}) = I([1:0]_{3_1}),\\
&I([0:1]_{2}) = I([0:1]_{3_1}) = I([0:1]_{3_2}).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

3.7. **Quiver diagrams.** The following diagrams are drawn in the same manner as in [GNS00]. They express the quiver structure of S_G, that is the decomposition of $S_1\cdot((S_G)_d)_{\rho_j}$. The rows are indexed by degrees and the columns by irreducible representations. Each irreducible factor ρ_j of $(S_G)_d$ has multiplicity one except when $G=D_{4m+2}$, $d=m+1$, $\rho_j=2_m$ and $(S_G)_{m+1}=\langle y^{m+1},x^{m+1}\rangle\oplus\langle x^mz,-y^mz\rangle = 2\cdot 2_m$. Each vertex in the diagram stands for nonzero $((S_G)_d)_{\rho_j}$ and we join $((S_G)_d)_{\rho_j}$ and $((S_G)_{d+1})_{\rho_k}$ with an edge when nonzero $((S_G)_{d+1})_{\rho_k}$ appears in $S_1\cdot((S_G)_d)_{\rho_j}$. In the unique exceptional case where $G=D_{4m+2}$, the diagram shows

$$S_1\cdot((S_G)_m)_{2_{m-1}}=\langle x^mz,-y^mz\rangle,\ S_1\cdot((S_G)_m)_{2_m}=(S_G)_{m+1}.$$



($\ell = 2m$: even) ($\ell = 2m + 1$: odd.)

Diagram $D_{2\ell}$

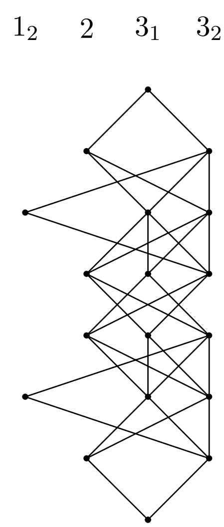


Diagram S_4

4. The Molien series P_{S_G,ρ_j} — the case (E)

Finite subgroups of $\mathrm{SL}(3,\mathbf{C})$ are classified in [Blichfeldt17]. With the notation in [YY93], there are exactly 4 infinite series labeled by (A), (B), (C), (D), and 8 exceptional cases labeled by (E) through (L). Homogeneous generators of the invariant rings for the exceptional 8 groups, together with explicit descriptions of these groups, are given in [YY93], which we shall follow. [1] Since the character tables of these groups can be obtained by using, for example, GAP, we omitted them; instead we give short descriptions of irreducible characters. In what follows we denote by 1_0 the trivial character (or representation) of G.

In this and the next section we calculate P_{R,ρ_j} and P_{S_G,ρ_j} explicitly for (E)-(L). See also [GNS00]. In this section we discuss the case (E) in some detail as a prototype for all the other cases. In what follows in order to save space we will not explain the customary notation.

Let

$$G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \rangle,$$

where $\omega = \mathrm{e}^{2\pi i/3}$. Then we have $|G| = 108$, and
$\hat{G} = \{1_0, 1_1, 1_2, 1_3, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 4_1, 4_2\}$,
where $1_1(V) = \sqrt{-1}$, $1_2 = 1_1^2$, $1_3 = 1_1^3$, $3_1 = \rho$, $3_2 = 1_1\rho$, $3_3 = 1_2\rho$, $3_4 = 1_3\rho$, $3_5 = \rho^\vee$, $3_6 = 1_1\rho^\vee$, $3_7 = 1_2\rho^\vee$, $3_8 = 1_3\rho^\vee$, $4_1(T) = 1$ and $4_2(T) = -2$.

The decompositions of $\rho_i \otimes \rho$ are given in Appendix.

We also have $S^G = \mathbf{C}[f_1, f_2, f_3, f_4, f_5]$ with $\deg f_1 = 6$, $\deg f_2 = 6$, $\deg f_3 = 12$, $\deg f_4 = 12$, and $\deg f_5 = 9$.

Put $R = S/(f_1, f_2, f_3)$. Then we can easily compute $P_{R,\rho_j}(t)$ by applying Proposition 1.3. Thus we see R splits into irreducible representations as in Table 10.

Next we calculate the Molien series $P_{S_G}(t)$ by the repeated use of the trivial relation $(\mathfrak{n})_i = V^\vee \cdot (\mathfrak{n})_{i-1} + (S^G)_i$ for any i. In the case (E) we need to compute only for $i \le 21$. What we do is not more than elementary linear algebra, so we omit the details of the computation. We see

$$\begin{aligned} P_R(t) &= \frac{(1-t^6)^2(1-t^{12})}{(1-t)^3} \\ &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 \\ &\qquad + 33t^8 + 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} \\ &\qquad + 26t^{15} + 21t^{16} + 15t^{17} + 10t^{18} + 6t^{19} + 3t^{20} + t^{21}, \\ P_{S_G}(t) &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 \\ &\qquad + 33t^8 + 34t^9 + 33t^{10} + 30t^{11} + 24t^{12} + 15t^{13} + 6t^{14}. \end{aligned}$$

Then in view of Proposition 1.3 we can compute the Molien series $P_{S_G,\rho_j}(t)$. Summarizing the computation we see S_G splits as in Table 11.

[1] Since our results use the results in [YY93], we mention here some of their misprints:
page 34, line 1, $\frac{1}{\sqrt{-7}}$ should be $\frac{-1}{\sqrt{-7}}$,
page 80, line 2, $(15+5\sqrt{15}i)x^3y^3$ should be $(15+5\sqrt{15}i)y^3z^3$.

deg	1_0	1_1	1_2	1_3	3_1	3_2	3_3	3_4	3_5	3_6	3_7	3_8	4_1	4_2	$\dim R_d$
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	3
2	0	0	0	0	0	1	0	1	0	0	0	0	0	0	6
3	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
4	0	0	0	0	0	0	0	0	1	2	0	2	0	0	15
5	0	0	0	0	3	1	2	1	0	0	0	0	0	0	21
6	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
7	0	0	0	0	0	0	0	0	2	2	4	2	0	0	30
8	0	0	0	0	3	3	2	3	0	0	0	0	0	0	33
9	1	1	0	1	0	0	0	0	0	0	0	0	4	4	35
10	0	0	0	0	0	0	0	0	2	3	4	3	0	0	36
11	0	0	0	0	2	3	4	3	0	0	0	0	0	0	36
12	1	1	0	1	0	0	0	0	0	0	0	0	4	4	35
13	0	0	0	0	0	0	0	0	3	3	2	3	0	0	33
14	0	0	0	0	2	2	4	2	0	0	0	0	0	0	30
15	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
16	0	0	0	0	0	0	0	0	3	1	2	1	0	0	21
17	0	0	0	0	1	2	0	2	0	0	0	0	0	0	15
18	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
19	0	0	0	0	0	0	0	0	0	1	0	1	0	0	6
20	0	0	0	0	1	0	0	0	0	0	0	0	0	0	3
21	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 10. The decomposition of R of type (E)

deg	1_0	1_1	1_2	1_3	3_1	3_2	3_3	3_4	3_5	3_6	3_7	3_8	4_1	4_2	$\dim(S_G)_d$
1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	3
2	0	0	0	0	0	1	0	1	0	0	0	0	0	0	6
3	0	1	0	1	0	0	0	0	0	0	0	0	1	1	10
4	0	0	0	0	0	0	0	0	1	2	0	2	0	0	15
5	0	0	0	0	3	1	2	1	0	0	0	0	0	0	21
6	0	0	2	0	0	0	0	0	0	0	0	0	3	3	26
7	0	0	0	0	0	0	0	0	2	2	4	2	0	0	30
8	0	0	0	0	3	3	2	3	0	0	0	0	0	0	33
9	0	1	0	1	0	0	0	0	0	0	0	0	4	4	34
10	0	0	0	0	0	0	0	0	1	3	4	3	0	0	33
11	0	0	0	0	2	2	4	2	0	0	0	0	0	0	30
12	0	0	0	0	0	0	0	0	0	0	0	0	3	3	24
13	0	0	0	0	0	0	0	0	1	1	2	1	0	0	15
14	0	0	0	0	0	0	2	0	0	0	0	0	0	0	6

Table 11. The decomposition of S_G of type (E)

In other words,

$$\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= t^3+t^9,\\
P_{S_G,1_2}(t) &= 2t^6,\\
P_{S_G,1_3}(t) &= t^3+t^9,\\
P_{S_G,3_1}(t) &= 3t^5+3t^8+2t^{11},\\
P_{S_G,3_2}(t) &= t^2+t^5+3t^8+2t^{11},\\
P_{S_G,3_3}(t) &= 2t^5+2t^8+4t^{11}+2t^{14},\\
P_{S_G,3_4}(t) &= t^2+t^5+3t^8+2t^{11},\\
P_{S_G,3_5}(t) &= t+t^4+2t^7+t^{10}+t^{13},\\
P_{S_G,3_6}(t) &= 2t^4+2t^7+3t^{10}+t^{13},\\
P_{S_G,3_7}(t) &= 4t^7+4t^{10}+2t^{13},\\
P_{S_G,3_8}(t) &= 2t^4+2t^7+3t^{10}+t^{13},\\
P_{S_G,4_1}(t) &= t^3+3t^6+4t^9+3t^{12},\\
P_{S_G,4_2}(t) &= t^3+3t^6+4t^9+3t^{12}.
\end{aligned}$$

As a consequence we see

$$P_{S_G,\rho_j}(t) = [(1-t^9)(1-t^{12})P_{R,\rho_j}(t)]_+$$

where $[f(t)]_+ = \sum_{d=0}^{21} \max\{a_d, 0\}t^d$ for $f(t) = \sum a_d t^d \in \mathbf{Z}[t]$. Note that this formula does not imply a similar formula for ρ_{S_G}.

5. The Molien series P_{S_G,ρ_j}

In this section we report the results for the other types (F)-(L). For the sake of reader's convenience we list the decompositions of $\rho_i \otimes \rho$ in Appendix.

5.1. **The group of type (F).**

$$G = \langle \begin{pmatrix} 1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega \end{pmatrix}, \frac{1}{\sqrt{-3}}\begin{pmatrix} 1 & 1 & \omega^2\\ 1 & \omega & \omega\\ \omega & 1 & \omega \end{pmatrix} \rangle,$$

where $\omega = e^{2\pi i/3}$.
$|G| = 216$.
$\hat{G} = \{1_0, 1_1, 1_2, 1_3, 2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 6_1, 6_2, 8\}$.
where $1_3 = 1_1 1_2$, $3_1 = \rho$, $3_2 = 1_1\rho$, $3_3 = 1_2\rho$, $3_4 = 1_3\rho$, $3_5 = \rho^\vee$, $3_6 = 1_1\rho^\vee$, $3_7 = 1_2\rho^\vee$, $3_8 = 1_3\rho^\vee$, $6_1 = \rho^2 - \rho^\vee$, $6_2 = \rho^{\vee 2} - \rho$.
$S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$,
with $\deg f_1 = 6, \deg f_2 = 9, \deg f_3 = 12, \deg f_4 = 12$.
Let $R = S/(f_1, f_2, f_3)$. Then we have

$$\begin{aligned}
P_{R,1_0}(t) &= 1 + t^{12} + t^{24},\\
P_{R,1_1}(t) &= P_{R,1_2} = P_{R,1_3} = t^6 + t^{12} + t^{18},
\end{aligned}$$

$$
\begin{aligned}
P_{R,2}(t) &= t^3+2t^9+2t^{15}+t^{21},\\
P_{R,3_1}(t) &= 2t^5+2t^8+2t^{11}+t^{17}+t^{20}+t^{23},\\
P_{R,3_2}(t) &= P_{R,3_3}=P_{R,3_4}=t^5+t^8+3t^{11}+2t^{14}+2t^{17},\\
P_{R,3_5}(t) &= t+t^4+t^7+2t^{13}+2t^{16}+2t^{19},\\
P_{R,3_6}(t) &= P_{R,3_7}=P_{R,3_8}=2t^7+2t^{10}+3t^{13}+t^{16}+t^{19},\\
P_{R,6_1}(t) &= 2t^4+2t^7+5t^{10}+3t^{13}+4t^{16}+t^{19}+t^{22},\\
P_{R,6_2}(t) &= t^2+t^5+4t^8+3t^{11}+5t^{14}+2t^{17}+2t^{20},\\
P_{R,8}(t) &= t^3+3t^6+5t^9+6t^{12}+5t^{15}+3t^{18}+t^{21}.
\end{aligned}
$$

We see

$$
\begin{aligned}
P_{S_G}(t) =& 1+3t+6t^2+10t^3+15t^4+21t^5+27t^6+33t^7+39t^8+44t^9+48t^{10}\\
&+51t^{11}+51t^{12}+48t^{13}+42t^{14}+34t^{15}+24t^{16}+15t^{17}+8t^{18}+3t^{19},
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= P_{S_G,1_2}=P_{S_G,1_3}=t^6+t^{12},\\
P_{S_G,2}(t) &= t^3+2t^9+t^{15},\\
P_{S_G,3_1}(t) &= 2t^5+2t^8+2t^{11},\\
P_{S_G,3_2}(t) &= P_{S_G,3_3}=P_{S_G,3_4}=t^5+t^8+3t^{11}+2t^{14}+t^{17},\\
P_{S_G,3_5}(t) &= t+t^4+t^7+t^{13}+t^{16}+t^{19},\\
P_{S_G,3_6}(t) &= P_{S_G,3_7}=P_{S_G,3_8}=2t^7+2t^{10}+3t^{13}+t^{16},\\
P_{S_G,6_1}(t) &= 2t^4+2t^7+5t^{10}+3t^{13}+2t^{16},\\
P_{S_G,6_2}(t) &= t^2+t^5+4t^8+3t^{11}+4t^{14}+t^{17},\\
P_{S_G,8}(t) &= t^3+3t^6+5t^9+6t^{12}+4t^{15}+t^{18}.
\end{aligned}
$$

5.2. **The group of type (G).**

$$
G=\langle \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, U=\begin{pmatrix}\varepsilon^2&0&0\\0&\varepsilon^2&0\\0&0&\varepsilon^5\end{pmatrix}, \frac{1}{\sqrt{-3}}\begin{pmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{pmatrix}\rangle,
$$

where where $\varepsilon=\mathrm{e}^{2\pi i/9}, \omega=\mathrm{e}^{2\pi i/3}$.
$|G|=648$.
$\hat{G}=\{1_0,1_1,1_2,2_1,2_2,2_3,3_1,3_2,3_3,3_4,3_5,3_6,3_7,6_1,6_2,6_3,6_4,6_5,6_6,8_1,8_2,8_3,9_1,9_2\}$, where 2_1 and 3_7 are rational valued characters and $1_1(U)=\omega$, $1_2=1_1^2$, $2_2=1_12_1$, $2_3=1_22_1$, $3_1=\rho$, $3_2=1_13_1$, $3_3=1_23_1$, $3_4=\rho^\vee$, $3_5=1_13_4$, $3_6=1_23_4$, $6_1=\rho^2-\rho^\vee$, $6_2=1_16_1$, $6_3=1_26_1$, $6_4=\rho^{\vee 2}-\rho$, $6_5=1_16_4$, $6_6=1_26_4$, $8_1=\rho\rho^\vee-1_0$, $8_2=1_18_1$, $8_3=1_28_1$, $9_1=3_7\rho$, $9_2=3_7\rho^\vee$.
$S^G=\mathbf{C}[f_1,f_2,f_3,f_4]$,

with $\deg f_1 = 9, \deg f_2 = 12, \deg f_3 = 18, \deg f_4 = 18$.
$R = S/(f_1, f_2, f_3)$. Then we have

$$
\begin{aligned}
P_{R,1_0}(t) &= 1 + t^{18} + t^{36},\\
P_{R,1_1}(t) &= 2t^{12} + t^{30},\\
P_{R,1_2}(t) &= t^6 + 2t^{24},\\
P_{R,2_1}(t) &= 3t^{15} + 3t^{21},\\
P_{R,2_2}(t) &= 2t^9 + 2t^{15} + t^{27} + t^{33},\\
P_{R,2_3}(t) &= t^3 + t^9 + 2t^{21} + 2t^{27},\\
P_{R,3_1}(t) &= t^8 + 2t^{11} + 3t^{17} + t^{20} + t^{26} + t^{35},\\
P_{R,3_2}(t) &= t^5 + 2t^{11} + t^{14} + 2t^{20} + t^{23} + 2t^{29},\\
P_{R,3_3}(t) &= t^5 + t^8 + t^{14} + 2t^{17} + 3t^{23} + t^{32},\\
P_{R,3_4}(t) &= t + t^{10} + t^{16} + 3t^{19} + 2t^{25} + t^{28},\\
P_{R,3_5}(t) &= 2t^7 + t^{13} + 2t^{16} + t^{22} + 2t^{25} + t^{31},\\
P_{R,3_6}(t) &= t^4 + 3t^{13} + 2t^{19} + t^{22} + t^{28} + t^{31},\\
P_{R,3_7}(t) &= t^6 + 2t^{12} + 3t^{18} + 2t^{24} + t^{30},\\
P_{R,6_1}(t) &= t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 5t^{22} + t^{25} + t^{28} + t^{31},\\
P_{R,6_2}(t) &= t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 3t^{22} + t^{25} + 3t^{28},\\
P_{R,6_3}(t) &= t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + 2t^{25} + t^{28} + t^{34},\\
P_{R,6_4}(t) &= t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + 2t^{23} + t^{26} + t^{29} + t^{32},\\
P_{R,6_5}(t) &= t^2 + t^8 + 2t^{11} + 2t^{14} + 2t^{17} + 5t^{20} + t^{23} + 3t^{26} + t^{29},\\
P_{R,6_6}(t) &= 3t^8 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23} + 3t^{26} + t^{32},\\
P_{R,8_1}(t) &= 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24} + 3t^{27},\\
P_{R,8_2}(t) &= t^3 + t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 5t^{21} + 2t^{24} + 2t^{27} + 2t^{30},\\
P_{R,8_3}(t) &= 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 4t^{24} + t^{27} + t^{30} + t^{33},\\
P_{R,9_1}(t) &= t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 5t^{23} + 2t^{26} + 2t^{29},\\
P_{R,9_2}(t) &= 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 4t^{25} + t^{28} + t^{31}.
\end{aligned}
$$

We see

$$
\begin{aligned}
P_{S_G}(t) =& 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + 45t^8 + 54t^9\\
&+ 63t^{10} + 72t^{11} + 80t^{12} + 87t^{13} + 93t^{14} + 98t^{15} + 102t^{16}\\
&+ 105t^{17} + 105t^{18} + 102t^{19} + 96t^{20} + 88t^{21} + 78t^{22}\\
&+ 66t^{23} + 52t^{24} + 36t^{25} + 21t^{26} + 10t^{27} + 3t^{28}, \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= 2t^{12},
\end{aligned}
$$

$$
\begin{aligned}
P_{S_G,1_2}(t) &= t^6 + t^{24},\\
P_{S_G,2_1}(t) &= 3t^{15} + 3t^{21},\\
P_{S_G,2_2}(t) &= 2t^9 + 2t^{15},\\
P_{S_G,2_3}(t) &= t^3 + t^9 + t^{21} + t^{27},\\
P_{S_G,3_1}(t) &= t^8 + 2t^{11} + 3t^{17} + t^{20},\\
P_{S_G,3_2}(t) &= t^5 + 2t^{11} + t^{14} + 2t^{20},\\
P_{S_G,3_3}(t) &= t^5 + t^8 + t^{14} + 2t^{17} + 2t^{23},\\
P_{S_G,3_4}(t) &= t + t^{10} + t^{16} + 2t^{19} + 2t^{25},\\
P_{S_G,3_5}(t) &= 2t^7 + t^{13} + 2t^{16} + t^{22},\\
P_{S_G,3_6}(t) &= t^4 + 3t^{13} + 2t^{19} + t^{28},\\
P_{S_G,3_7}(t) &= t^6 + 2t^{12} + 3t^{18} + t^{24},\\
P_{S_G,6_1}(t) &= t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 4t^{22},\\
P_{S_G,6_2}(t) &= t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 2t^{22} + t^{25},\\
P_{S_G,6_3}(t) &= t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + t^{25},\\
P_{S_G,6_4}(t) &= t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + t^{23},\\
P_{S_G,6_5}(t) &= t^2 + t^8 + 2t^{11} + 2t^{14} + 2t^{17} + 4t^{20} + t^{23} + 2t^{26},\\
P_{S_G,6_6}(t) &= 3t^8 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23},\\
P_{S_G,8_1}(t) &= 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24},\\
P_{S_G,8_2}(t) &= t^3 + t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 4t^{21} + t^{24} + t^{27},\\
P_{S_G,8_3}(t) &= 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 2t^{24},\\
P_{S_G,9_1}(t) &= t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 4t^{23} + t^{26},\\
P_{S_G,9_2}(t) &= 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 2t^{25}.
\end{aligned}
$$

5.3. **The group of type (H).**

$$
G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix} \rangle,
$$

where $\varepsilon = e^{2\pi i/5}, \omega = e^{2\pi i/3}, s = \varepsilon^2 + \varepsilon^3$ and $t = \varepsilon + \varepsilon^5$.
$|G| = 60$.
$\hat{G} = \{1_0, 3_1 = \rho = \rho^\vee, 3_2, 4, 5\}$,
Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a Coxeter group of type H_3 and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 2, \deg f_2 = 6, \deg f_3 = 10$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$. Hence we have

$$
\begin{aligned}
P_{S_G,1_0} &= 1,\\
P_{S_G,3_1} &= t^3 + t^5 + t^7 + t^8 + t^{10} + t^{12},
\end{aligned}
$$

$$P_{S_G,3_2} = t + t^5 + t^6 + t^9 + t^{10} + t^{14},$$
$$P_{S_G,4} = t^3 + t^4 + t^6 + t^7 + t^8 + t^9 + t^{11} + t^{12},$$
$$P_{S_G,5} = t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{13}.$$

5.4. **The group of type (I).**

$$G = \langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \frac{-1}{\sqrt{-7}} \begin{pmatrix} \varepsilon^4-\varepsilon^3 & \varepsilon^2-\varepsilon^5 & \varepsilon-\varepsilon^6 \\ \varepsilon^2-\varepsilon^5 & \varepsilon-\varepsilon^6 & \varepsilon^4-\varepsilon^3 \\ \varepsilon-\varepsilon^6 & \varepsilon^4-\varepsilon^3 & \varepsilon^2-\varepsilon^5 \end{pmatrix} \rangle,$$

where $\varepsilon = e^{2\pi i/7}$.
$|G| = 168$.
$\hat{G} = \{1_0, 3_1 = \rho, 3_2 = \rho^\vee, 6, 7, 8\}$,
Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a complex reflection group of type $J_3(4)$ (c.f. [Cohen76]) and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 4, \deg f_2 = 6, \deg f_3 = 14$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$.
Hence we have

$$P_{S_G,1_0} = 1,$$
$$P_{S_G,3_1} = t^3 + t^5 + t^{10} + t^{12} + t^{13} + t^{20},$$
$$P_{S_G,3_2} = t + t^8 + t^9 + t^{11} + t^{16} + t^{18},$$
$$P_{S_G,6} = t^2 + t^4 + t^6 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{15} + t^{17} + t^{19},$$
$$P_{S_G,7} = t^3 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{18},$$
$$P_{S_G,8} = t^4 + t^5 + t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + 2t^{14} + t^{15} + t^{16} + t^{17}.$$

5.5. **The group of type (J).**

$$G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix}, W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \rangle,$$

where $\varepsilon = e^{2\pi i/5}, \omega = e^{2\pi i/3}, s = \varepsilon^2 + \varepsilon^3$, and $t = \varepsilon + \varepsilon^4$.
$|G| = 180$.
$\hat{G} = \{1_0, 1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 4_1, 4_2, 4_3, 5_1, 5_2, 5_3\}$,
where 4_1 and 5_1 are rational valued characters and $1_1(W) = \omega$, $1_2 = 1_1^2$, $3_1 = \rho$, $3_2 = \rho^\vee = 1_1 3_1$, $3_3 = 1_2 3_1$, $3_4(x) = 3_1(x^7)$, $\forall x \in G$, $3_5 = 1_1 3_4$, $3_6 = 1_2 3_4$, $4_2 = 1_1 4_1$, $4_3 = 1_2 4_1$, $5_2 = 1_1 5_1$ and $5_3 = 1_2 5_1$.
$S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$,
with $\deg f_1 = 6, \deg f_2 = 6, \deg f_3 = 15, \deg f_4 = 12$.
Put $R = S/(f_1, f_2, f_3)$. Then we have

$$P_{R,1_0}(t) = 1 + t^{12} + t^{24},$$
$$P_{R,1_1}(t) = t^2 + t^{14} + t^{20},$$
$$P_{R,1_2}(t) = t^4 + t^{10} + t^{22},$$
$$P_{R,3_1}(t) = 2t^5 + t^8 + 2t^{11} + 2t^{14} + t^{17} + t^{23},$$

$$
\begin{aligned}
P_{R,3_2}(t) &= t + t^7 + 2t^{10} + 2t^{13} + t^{16} + 2t^{19},\\
P_{R,3_3}(t) &= t^3 + t^6 + 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{21},\\
P_{R,3_4}(t) &= 2t^5 + t^8 + t^{11} + 2t^{14} + 3t^{17},\\
P_{R,3_5}(t) &= 3t^7 + 2t^{10} + t^{13} + t^{16} + 2t^{19},\\
P_{R,3_6}(t) &= t^3 + 2t^9 + 3t^{12} + 2t^{15} + t^{21},\\
P_{R,4_1}(t) &= t^3 + 2t^6 + 2t^9 + 2t^{12} + 2t^{15} + 2t^{18} + t^{21},\\
P_{R,4_2}(t) &= t^5 + 3t^8 + 3t^{11} + 2t^{14} + 2t^{17} + t^{20},\\
P_{R,4_3}(t) &= t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + t^{19},\\
P_{R,5_1}(t) &= 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + 3t^{18},\\
P_{R,5_2}(t) &= t^2 + t^5 + 3t^8 + 3t^{11} + 3t^{14} + 2t^{17} + 2t^{20},\\
P_{R,5_3}(t) &= 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + 3t^{16} + t^{19} + t^{22}.
\end{aligned}
$$

We see

$$
\begin{aligned}
P_{S_G}(t) = &1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 + 33t^8\\
&+ 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} + 25t^{15}\\
&+ 19t^{16} + 12t^{17} + 5t^{18} + 3t^{19} + t^{20}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= t^2 + t^{20},\\
P_{S_G,1_2}(t) &= t^4 + t^{10},\\
P_{S_G,3_1}(t) &= 2t^5 + t^8 + 2t^{11} + 2t^{14},\\
P_{S_G,3_2}(t) &= t + t^7 + 2t^{10} + t^{13} + t^{16} + t^{19},\\
P_{S_G,3_3}(t) &= t^3 + t^6 + 2t^9 + t^{12} + t^{15},\\
P_{S_G,3_4}(t) &= 2t^5 + t^8 + t^{11} + 2t^{14} + t^{17},\\
P_{S_G,3_5}(t) &= 3t^7 + 2t^{10} + t^{13} + t^{16},\\
P_{S_G,3_6}(t) &= t^3 + 2t^9 + 3t^{12} + t^{15},\\
P_{S_G,4_1}(t) &= t^3 + 2t^6 + 2t^9 + 2t^{12} + t^{15},\\
P_{S_G,4_2}(t) &= t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},\\
P_{S_G,4_3}(t) &= t^4 + 2t^7 + 2t^{10} + 3t^{13} + 2t^{16},\\
P_{S_G,5_1}(t) &= 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + t^{18},\\
P_{S_G,5_2}(t) &= t^2 + t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},\\
P_{S_G,5_3}(t) &= 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + t^{16}.
\end{aligned}
$$

5.6. **The group of type (K).**

$$G = \langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \frac{1}{\sqrt{-7}} \begin{pmatrix} \varepsilon^4-\varepsilon^3 & \varepsilon^2-\varepsilon^5 & \varepsilon-\varepsilon^6 \\ \varepsilon^2-\varepsilon^5 & \varepsilon-\varepsilon^6 & \varepsilon^4-\varepsilon^3 \\ \varepsilon-\varepsilon^6 & \varepsilon^4-\varepsilon^3 & \varepsilon^2-\varepsilon^5 \end{pmatrix}, W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \rangle,$$

where $\varepsilon = e^{2\pi i/7}$ and $\omega = e^{2\pi i/3}$.
$|G| = 504$
$\hat{G} = \{1_0, 1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 6_1, 6_2, 6_3, 7_1, 7_2, 7_3, 8_1, 8_2, 8_3\}$,
where 6_1, 7_1 and 8_1 are rational valued characters and $1_1(W) = \omega$, $1_2 = 1_1^2$, $3_1 = \rho$, $3_2 = 1_1 3_1$, $3_3 = 1_2 3_1$, $3_4 = \rho^\vee$, $3_5 = 1_1 3_4$, $3_6 = 1_2 3_4$, $6_2 = 1_1 6_1$, $6_3 = 1_2 6_1$, $7_2 = 1_1 7_1$, $7_3 = 1_2 7_1$, $8_2 = 1_1 8_1$ and $8_3 = 1_2 8_1$.
$S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$,
with $\deg f_1 = 6, \deg f_2 = 12, \deg f_3 = 21, \deg f_4 = 18$.
Put $R = S/(f_1, f_2, f_3)$. Then we have

$$\begin{aligned}
P_{R,1_0}(t) &= 1 + t^{18} + t^{36}, \\
P_{R,1_1}(t) &= t^8 + t^{14} + t^{32}, \\
P_{R,1_2}(t) &= t^4 + t^{22} + t^{28}, \\
P_{R,3_1}(t) &= t^5 + t^{11} + t^{14} + 2t^{17} + 2t^{20} + t^{23} + t^{35}, \\
P_{R,3_2}(t) &= t^7 + t^{10} + 2t^{13} + t^{16} + t^{19} + t^{25} + t^{28} + t^{31}, \\
P_{R,3_3}(t) &= t^3 + t^9 + t^{12} + t^{18} + 2t^{21} + t^{24} + 2t^{27}, \\
P_{R,3_4}(t) &= t + t^{13} + 2t^{16} + 2t^{19} + t^{22} + t^{25} + t^{31}, \\
P_{R,3_5}(t) &= 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{24} + t^{27} + t^{33}, \\
P_{R,3_6}(t) &= t^5 + t^8 + t^{11} + t^{17} + t^{20} + 2t^{23} + t^{26} + t^{29}, \\
P_{R,6_1}(t) &= 2t^6 + t^9 + 3t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 3t^{24} + t^{27} + 2t^{30}, \\
P_{R,6_2}(t) &= t^2 + 2t^8 + t^{11} + 2t^{14} + 3t^{17} + 3t^{20} + 2t^{23} + 3t^{26} + t^{32}, \\
P_{R,6_3}(t) &= t^4 + 3t^{10} + 2t^{13} + 3t^{16} + 3t^{19} + 2t^{22} + t^{25} + 2t^{28} + t^{34}, \\
P_{R,7_1}(t) &= t^3 + t^6 + 2t^9 + 2t^{12} + 3t^{15} + 3t^{18} + 3t^{21} + 2t^{24} + 2t^{27} + t^{30} + t^{33}, \\
P_{R,7_2}(t) &= t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29}, \\
P_{R,7_3}(t) &= 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + t^{28} + t^{31}, \\
P_{R,8_1}(t) &= t^6 + 2t^9 + 3t^{12} + 4t^{15} + 4t^{18} + 4t^{21} + 3t^{24} + 2t^{27} + t^{30}, \\
P_{R,8_2}(t) &= t^5 + 2t^8 + 3t^{11} + 4t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29} + t^{32}, \\
P_{R,8_3}(t) &= t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 4t^{22} + 3t^{25} + 2t^{28} + t^{31}.
\end{aligned}$$

We see

$$\begin{aligned}
P_{S_G}(t) = &1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 45t^9 \\
&+ 51t^{10} + 57t^{11} + 62t^{12} + 66t^{13} + 69t^{14} + 71t^{15} + 72t^{16} + 72t^{17} \\
&+ 71t^{18} + 69t^{19} + 66t^{20} + 61t^{21} + 54t^{22} + 45t^{23} + 35t^{24} + 24t^{25} \\
&+ 13t^{26} + 3t^{27} + t^{28}.
\end{aligned}$$

Hence we have

$$
\begin{aligned}
P_{S_G,1_0}(t) &= 1,\\
P_{S_G,1_1}(t) &= t^8+t^{14},\\
P_{S_G,1_2}(t) &= t^4+t^{28},\\
P_{S_G,3_1}(t) &= t^5+t^{11}+t^{14}+2t^{17}+2t^{20},\\
P_{S_G,3_2}(t) &= t^7+t^{10}+2t^{13}+t^{16}+t^{19},\\
P_{S_G,3_3}(t) &= t^3+t^9+t^{12}+t^{18}+t^{21}+t^{24}+t^{27},\\
P_{S_G,3_4}(t) &= t+t^{13}+2t^{16}+t^{19}+t^{22}+t^{25},\\
P_{S_G,3_5}(t) &= 2t^9+t^{12}+2t^{15}+t^{18}+t^{24},\\
P_{S_G,3_6}(t) &= t^5+t^8+t^{11}+t^{17}+t^{20}+t^{23},\\
P_{S_G,6_1}(t) &= 2t^6+t^9+3t^{12}+2t^{15}+2t^{18}+2t^{21}+t^{24},\\
P_{S_G,6_2}(t) &= t^2+2t^8+t^{11}+2t^{14}+3t^{17}+2t^{20}+2t^{23}+t^{26},\\
P_{S_G,6_3}(t) &= t^4+3t^{10}+2t^{13}+3t^{16}+3t^{19}+t^{22}+t^{25},\\
P_{S_G,7_1}(t) &= t^3+t^6+2t^9+2t^{12}+3t^{15}+3t^{18}+2t^{21}+t^{24},\\
P_{S_G,7_2}(t) &= t^5+t^8+3t^{11}+3t^{14}+3t^{17}+3t^{20}+2t^{23}+t^{26},\\
P_{S_G,7_3}(t) &= 2t^7+2t^{10}+3t^{13}+3t^{16}+3t^{19}+3t^{22}+t^{25},\\
P_{S_G,8_1}(t) &= t^6+2t^9+3t^{12}+4t^{15}+4t^{18}+4t^{21}+2t^{24},\\
P_{S_G,8_2}(t) &= t^5+2t^8+3t^{11}+4t^{14}+3t^{17}+3t^{20}+2t^{23},\\
P_{S_G,8_3}(t) &= t^4+2t^7+2t^{10}+3t^{13}+3t^{16}+3t^{19}+3t^{22}+t^{25}.
\end{aligned}
$$

5.7. **The group of type (L).**

$$G = \langle \begin{pmatrix}1 & 0 & 0\\ 0 & \varepsilon^{-1} & 0\\ 0 & 0 & \varepsilon\end{pmatrix}, \begin{pmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 1 & 1\\ 2 & s & t\\ 2 & t & s\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & \lambda_1 & \lambda_1\\ 2\lambda_2 & s & t\\ 2\lambda_2 & t & s\end{pmatrix}\rangle,$$

where $\varepsilon = e^{2\pi i/5}, s = \varepsilon^2+\varepsilon^3, t = \varepsilon+\varepsilon^4, \lambda_1 = -\frac{1-\sqrt{-15}}{4}$ and $\lambda_2 = -\frac{1+\sqrt{-15}}{4}$. $|G| = 1080$.
$\hat{G} = \{1_0, 3_1, 3_2, 3_3, 3_4, 5_1, 5_2, 6_1, 6_2, 8_1, 8_2, 9_1, 9_2, 9_3, 10, 15_1, 15_2\}$,
where $3_1 = \rho$, $3_2 = \rho^\vee$, $3_3(x) = 3_1(x^7)$ for all $x \in G$, $3_4(x) = 3_2(x^7)$ for all $x \in G$, $6_1 = \rho^2 - \rho^\vee$, $6_2 = \rho^{\vee 2} - \rho$, $8_1 = 3_1 3_2 - 1_0$, $8_2 = 3_3 3_4 - 1_0$, $9_1 = 3_1 3_4$, $9_2 = 3_1 3_3$, $9_3 = 3_2 3_4$, $15_1 = 3_1 5_1$ and $15_2 = 3_2 5_1$.
Let $\tilde{G}$ be a group generated by G and $-I$ where I is the identity matrix of degree 3. Then $\tilde{G}$ is a complex reflection group of type $J_3(5)$ (c.f. [Cohen76]) and there exist three homogeneous invariants f_1, f_2, f_3 with $\deg f_1 = 6, \deg f_2 = 12, \deg f_3 = 30$ such that $S^{\tilde{G}} = \mathbf{C}[f_1, f_2, f_3]$ and $S^G = \mathbf{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \mathrm{Jac}(f_1, f_2, f_3)$.
Hence we have

$$
\begin{aligned}
P_{S_G,1_0} &= 1,\\
P_{S_G,3_1} &= t^5+t^{11}+t^{20}+t^{26}+t^{29}+t^{44},
\end{aligned}
$$

$$
\begin{aligned}
P_{S_G,3_2} &= t + t^{16} + t^{19} + t^{25} + t^{34} + t^{40},\\
P_{S_G,3_3} &= t^{5} + t^{17} + t^{20} + t^{23} + t^{32} + t^{38},\\
P_{S_G,3_4} &= t^{7} + t^{13} + t^{22} + t^{25} + t^{28} + t^{40},\\
P_{S_G,5_1} &= t^{6} + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39},\\
P_{S_G,5_2} &= t^{6} + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39},\\
P_{S_G,6_1} &= t^{4} + 2t^{10} + t^{16} + t^{19} + t^{22} + 2t^{25} + t^{28} + t^{31} + t^{37} + t^{43},\\
P_{S_G,6_2} &= t^{2} + t^{8} + t^{14} + t^{17} + 2t^{20} + t^{23} + t^{26} + t^{29} + 2t^{35} + t^{41},\\
P_{S_G,8_1} &= t^{6} + t^{9} + t^{12} + 2t^{15} + t^{18} + 2t^{21} + 2t^{24} + t^{27} + 2t^{30} + t^{33} + t^{36} + t^{39},\\
P_{S_G,8_2} &= t^{9} + 2t^{12} + 2t^{15} + 2t^{18} + t^{21} + t^{24} + 2t^{27} + 2t^{30} + 2t^{33} + t^{36},\\
P_{S_G,9_1} &= t^{6} + t^{9} + 2t^{12} + t^{15} + 2t^{18} + 2t^{21} + 2t^{24} + 2t^{27} + t^{30} + 2t^{33} + t^{36} + t^{39},\\
P_{S_G,9_2} &= t^{4} + t^{10} + 2t^{13} + 2t^{16} + 2t^{19} + 2t^{22} + t^{25} + 2t^{28} + 2t^{31} + t^{34} + 2t^{37},\\
P_{S_G,9_3} &= 2t^{8} + t^{11} + 2t^{14} + 2t^{17} + t^{20} + 2t^{23} + 2t^{26} + 2t^{29} + 2t^{32} + t^{35} + t^{41},\\
P_{S_G,10} &= t^{3} + 2t^{9} + t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 2t^{24} + 2t^{27} + 2t^{30} + t^{33} + 2t^{36} + t^{42},\\
P_{S_G,15_1} &= t^{5} + t^{8} + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 3t^{26} + 3t^{29} + 3t^{32} + 2t^{35} + 2t^{38},\\
P_{S_G,15_2} &= 2t^{7} + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + 3t^{28} + 3t^{31} + 3t^{34} + t^{37} + t^{40}.
\end{aligned}
$$

5.8. **Summary.** Here we list the invariants for the subgroups of type (E)-(L) where $d_{\max} = d_1 + d_2 + d_3 - 3$:

type	d_1, d_2, d_3	d_4, d_5	$d_{\max}$	$\lvert G\rvert$	e
E	$6, 6, 12$	$12, 9$	21	108	4
F	$6, 9, 12$	12	24	216	3
G	$9, 12, 18$	18	36	648	3
H	$2, 6, 10$	15	15	60	2
I	$4, 6, 14$	21	21	168	2
J	$6, 6, 15$	12	24	180	3
K	$6, 12, 21$	18	36	504	3
L	$6, 12, 30$	45	45	1080	2

Table 12. Groups (E)-(L)

Summarizing the calculation in the previous subsections we infer

Theorem 5.9. *Let G be a subgroup of* $\mathrm{SL}(3, \mathbf{C})$ *of type from (E) to (L). Let f_i be generators of the invariant ring S^G and $d_i = \deg f_i$ $(1 \le i \le n)$, $d_{\max} = d_1 + d_2 + d_3 - 3$ as in Table 12, and S_G the coinvariant algebra. Then for any irreducible* *representation ρ_j of G the Molien series P_{S_G,ρ_j} is given by the formula*

$$P_{S_G,\rho_j}(t) = [\prod_{i=4}^{n}(1-t^{d_i})P_{R,\rho_j}(t)]_+ + \begin{cases} t^{18}(\delta_{j,8}+\delta_{j,5_1}) & \text{if } G = \text{(F) or (J)}, \\ 0 & \text{otherwise}. \end{cases}$$

where $[f(t)]_+ := \sum_{d=0}^{d_{\max}} \max\{a_d, 0\}t^d$ *for* $f(t) = \sum a_d t^d \in \mathbf{Z}[t]$.

Remark 5.10. Theorem 5.9 implies the following. Suppose $j \neq 8$ if G is type (F) or $j \neq 5_1$ if G is of type (J). For any fixed irreducible representation ρ_j multiplication by f_α $(\alpha = 4, 5)$ is a homomorphism ϕ^α_{d,ρ_j} from $(R_d)_{\rho_j}$ to $(R_{d+d_\alpha})_{\rho_j}$. Then ϕ^α_{d,ρ_j} is surjective if $\dim(R_d)_{\rho_j} \geq \dim(R_{d+d_\alpha})_{\rho_j}$, while it is injective if $\dim(R_d)_{\rho_j} \leq \dim(R_{d+d_\alpha})_{\rho_j}$. In other words, $\operatorname{rank}\phi^\alpha_{d,\rho_j}$ is equal to $\min\{\dim(R_d)_{\rho_j}, \dim(R_{d+d_\alpha})_{\rho_j}\}$. Moreover $f_4R \cap f_5R = f_4f_5R \simeq f_4f_5\mathbf{C}$. In the exceptional case, for instance, of type (J) and $j = 5_1$, the nonzero coefficient of t^{19} in $P_{S_G,3_2}$ explains nonvanishing of the coefficient of t^{18} in $P_{S_G,5_1}$. We note that the above theorem does not imply $P_{S_G}(t) = [\prod_{i=4}^{n}(1-t^{d_i})P_R(t)]_+$ even in the case other than (F) and (J).

6. Appendix

In this appendix we list the decompositions of irreducible representations tensored with the natural representation ρ.

6.1. **Type (E).**

$$\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 1_1 \otimes \rho = 3_2, \\
1_2 \otimes \rho = 3_3, & 1_3 \otimes \rho = 3_4, \\
3_1 \otimes \rho = 3_5 + 3_6 + 3_8, & 3_2 \otimes \rho = 3_5 + 3_6 + 3_7, \\
3_3 \otimes \rho = 3_6 + 3_7 + 3_8, & 3_4 \otimes \rho = 3_5 + 3_7 + 3_8, \\
3_5 \otimes \rho = 1_0 + 4_1 + 4_2, & 3_6 \otimes \rho = 1_1 + 4_1 + 4_2, \\
3_7 \otimes \rho = 1_2 + 4_1 + 4_2, & 3_8 \otimes \rho = 1_3 + 4_1 + 4_2, \\
4_1 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4, & 4_2 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4.
\end{array}$$

6.2. **Type (F).**

$$\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 1_1 \otimes \rho = 3_2, \\
1_2 \otimes \rho = 3_3, & 1_3 \otimes \rho = 3_4, \\
2 \otimes \rho = 6_2, & 3_1 \otimes \rho = 3_5 + 6_1 \\
3_2 \otimes \rho = 3_6 + 6_1, & 3_3 \otimes \rho = 3_7 + 6_1, \\
3_4 \otimes \rho = 3_8 + 6_1, & 3_5 \otimes \rho = 1_0 + 8, \\
3_6 \otimes \rho = 1_1 + 8, & 3_7 \otimes \rho = 1_2 + 8, \\
3_8 \otimes \rho = 1_3 + 8, & 6_1 \otimes \rho = 2 + 2 \cdot 8, \\
6_2 \otimes \rho = 3_5 + 3_6 + 3_7 + 3_8 + 6_1, & 8 \otimes \rho = 3_1 + 3_2 + 3_3 + 3_4 + 2 \cdot 6_2.
\end{array}$$

6.3. **Type (G).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 1_1 \otimes \rho = 3_2, \\
1_2 \otimes \rho = 3_3, & 2_1 \otimes \rho = 6_5, \\
2_2 \otimes \rho = 6_6, & 2_3 \otimes \rho = 6_4, \\
3_1 \otimes \rho = 3_4 + 6_1, & 3_2 \otimes \rho = 3_5 + 6_2, \\
3_3 \otimes \rho = 3_6 + 6_3, & 3_4 \otimes \rho = 1_0 + 8_1, \\
3_5 \otimes \rho = 1_1 + 8_2, & 3_6 \otimes \rho = 1_2 + 8_3, \\
3_7 \otimes \rho = 9_1, & 6_1 \otimes \rho = 2_2 + 8_1 + 8_3, \\
6_2 \otimes \rho = 2_3 + 8_1 + 8_2, & 6_3 \otimes \rho = 2_1 + 8_2 + 8_3, \\
6_4 \otimes \rho = 3_4 + 6_2 + 9_2, & 6_5 \otimes \rho = 3_5 + 6_3 + 9_2, \\
6_6 \otimes \rho = 3_6 + 6_1 + 9_2, & 8_1 \otimes \rho = 3_1 + 6_4 + 6_6 + 9_1, \\
8_2 \otimes \rho = 3_2 + 6_4 + 6_5 + 9_1, & 8_3 \otimes \rho = 3_3 + 6_5 + 6_6 + 9_1, \\
9_1 \otimes \rho = 6_1 + 6_2 + 6_3 + 9_2, & 9_2 \otimes \rho = 3_7 + 8_1 + 8_2 + 8_3.
\end{array}
$$

6.4. **Type (H).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 3_1 \otimes \rho = 1_0 + 3_1 + 5, \\
3_2 \otimes \rho = 4 + 5, & 4 \otimes \rho = 3_2 + 4 + 5, \\
5 \otimes \rho = 3_1 + 3_2 + 4 + 5. &
\end{array}
$$

6.5. **Type (I).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 3_1 \otimes \rho = 3_2 + 6, \\
3_2 \otimes \rho = 1_0 + 8, & 6 \otimes \rho = 3_2 + 7 + 8, \\
7 \otimes \rho = 6 + 7 + 8, & 8 \otimes \rho = 3_1 + 6 + 7 + 8.
\end{array}
$$

6.6. **Type (J).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 1_1 \otimes \rho = 3_2, \\
1_2 \otimes \rho = 3_3, & 3_1 \otimes \rho = 1_2 + 3_2 + 5_3, \\
3_2 \otimes \rho = 1_0 + 3_3 + 5_1, & 3_3 \otimes \rho = 1_1 + 3_1 + 5_2, \\
3_4 \otimes \rho = 4_3 + 5_3, & 3_5 \otimes \rho = 4_1 + 5_1, \\
3_6 \otimes \rho = 4_2 + 5_2, & 4_1 \otimes \rho = 3_4 + 4_2 + 5_2, \\
4_2 \otimes \rho = 3_5 + 4_3 + 5_3, & 4_3 \otimes \rho = 3_6 + 4_1 + 5_1, \\
5_1 \otimes \rho = 3_1 + 3_4 + 4_2 + 5_2, & 5_2 \otimes \rho = 3_2 + 3_5 + 4_3 + 5_3, \\
5_3 \otimes \rho = 3_3 + 3_6 + 4_1 + 5_1. &
\end{array}
$$

6.7. **Type (K).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 1_1 \otimes \rho = 3_2, \\
1_2 \otimes \rho = 3_3, & 3_1 \otimes \rho = 3_4 + 6_3, \\
3_2 \otimes \rho = 3_5 + 6_1, & 3_3 \otimes \rho = 3_6 + 6_2, \\
3_4 \otimes \rho = 1_0 + 8_1, & 3_5 \otimes \rho = 1_1 + 8_2, \\
3_6 \otimes \rho = 1_2 + 8_3, & 6_1 \otimes \rho = 3_6 + 7_2 + 8_2, \\
6_2 \otimes \rho = 3_4 + 7_3 + 8_3, & 6_3 \otimes \rho = 3_5 + 7_1 + 8_1, \\
7_1 \otimes \rho = 6_2 + 7_2 + 8_2, & 7_2 \otimes \rho = 6_3 + 7_3 + 8_3, \\
7_3 \otimes \rho = 6_1 + 7_1 + 8_1, & 8_1 \otimes \rho = 3_1 + 6_2 + 7_2 + 8_2, \\
8_2 \otimes \rho = 3_2 + 6_3 + 7_3 + 8_3, & 8_3 \otimes \rho = 3_3 + 6_1 + 7_1 + 8_1.
\end{array}
$$

6.8. **Type (L).**

$$
\begin{array}{ll}
1_0 \otimes \rho = 3_1, & 3_1 \otimes \rho = 3_2 + 6_1, \\
3_2 \otimes \rho = 1_0 + 8_1, & 3_3 \otimes \rho = 9_2, \\
3_4 \otimes \rho = 9_1, & 5_1 \otimes \rho = 15_1, \\
5_2 \otimes \rho = 15_1, & 6_1 \otimes \rho = 8_1 + 10, \\
6_2 \otimes \rho = 3_2 + 15_2, & 8_1 \otimes \rho = 3_1 + 6_2 + 15_1, \\
8_2 \otimes \rho = 9_3 + 15_1, & 9_1 \otimes \rho = 3_3 + 9_3 + 15_1, \\
9_2 \otimes \rho = 8_2 + 9_1 + 10, & 9_3 \otimes \rho = 3_4 + 9_2 + 15_2, \\
10 \otimes \rho = 6_2 + 9_3 + 15_1, & 15_1 \otimes \rho = 6_1 + 9_2 + 2 \cdot 15_2, \\
15_2 \otimes \rho = 5_1 + 5_2 + 8_1 + 8_2 + 9_1 + 10. &
\end{array}
$$

6.9. **Adderndum.** In [GNS00, p.52, p.53] there are a few errors in notation and formulation, though harmless for the consequences in the subsequent sections of [GNS00]. As the arguments in this article are entirely independent from [GNS00] we would like to correct the errors in [GNS00] in the paper [GNS3] much closer to [GNS00].

We acknowledge Professor Li Chiang for pointing out the following errors in [GNS00] (different from the above) to us. The fourth line of [GNS00, p.57] must be replaced by

$$f^3 + \bar{f}^3 = \prod_{i=0}^{2}(f + \omega^i \bar{f}) = 27f_3^2 - 9f_2f_4 + 2f_2^3.$$

The fifth column of $S_d[\rho]$ of [GNS00, p.57, Table 2.2] must be replaced by

$$\{\bar{f}^2\} + \{f^2\} + \{yzf, \omega^2 zxf, \omega xyf\}.$$

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Department of Mathematics, Sophia University, Tokyo, 102-8554, Japan
E-mail address: gomi@mm.sophia.ac.jp

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
E-mail address: nakamura@math.sci.hokudai.ac.jp

Department of Mathematics, Sophia University, Tokyo, 102-8554, Japan
E-mail address: shinoda@mm.sophia.ac.jp