COINVARIANT ALGEBRAS
OF FINITE SUBGROUPS OF SL(3,C)

YASUSHI GOMI, IKU NAKAMURA AND KEN-ICHI SHINODA

Abstract. For most of the finite subgroups of SL(3, C), we give explicit formulae for the Molien series of the coinvariant algebras, generalizing McKay’s formulae [McKay99] for subgroups of SU(2). We also study the G-orbit Hilbert scheme Hilb^G(C^3) for any finite subgroup G of SO(3), which is known to be a minimal (crepant) resolution of the orbit space C^3/G. In this case the fiber over the origin of the Hilbert-Chow morphism from Hilb^G(C^3) to C^3/G consists of finitely many smooth rational curves, whose planar dual graph is identified with a certain subgraph of the representation graph of G. This is an SO(3) version of the McKay correspondence in the SU(2) case.

0. Introduction

Let G be a finite subgroup of SL(n, C), S_G the coinvariant algebra of G, and (S_G)_i the subspace of S_G of homogeneous degree i respectively. For each irreducible representation \rho of G, let \langle \rho, (S_G)_i \rangle_G be the multiplicity of \rho in (S_G)_i and define the Molien series P_{S_G, \rho}(t) of S_G for \rho to be

P_{S_G, \rho}(t) = \sum \langle \rho, (S_G)_i \rangle_G t^i.

Since S_G is finite-dimensional, P_{S_G, \rho}(t) is a polynomial of t. One can define similarly the Molien series P_{S, \rho}(t) for an arbitrary graded G-module M with finite dimensional graded pieces. If M is the polynomial algebra S in two variables and if G is a subgroup of SU(2), then the Molien series P_{S, \rho}(t) of S is a rational function of t by [Springer87] and it is well understood as is the connection with the Dynkin diagram corresponding to G (cf. [Springer87] and [McKay99]). In these cases the Molien series P_{S_G, \rho}(t) of S_G is easily derived from the formula for P_{S, \rho}(t).

The first purpose of this paper is to give an explicit formula for P_{S_G, \rho} when G is one of the exceptional finite subgroups of SL(3, C) of type from (E) to (L) in the notation of [YY93]. Using the Koszul complex with G-action, we derive a certain system of equations analogous to the SU(2) case [McKay99] satisfied by the Molien series P_{S, \rho}. The equations are obtained just by taking alternating sums of componentwise generating functions of G-modules in the Koszul complex. They are given explicitly in terms of irreducible decompositions of tensor products with...
the natural representation $\rho_{nat}$ and its second exterior product $\wedge^2 \rho_{nat}$. This will be discussed in Section 2. The consequence of this section enables us to compute $P_{S,\rho}$ explicitly later. However the calculation of $P_{S,\rho}$ in the exceptional cases (E)-(L) is much harder, which will be discussed in Sections 4 and 5. This study of the Molien series $P_{S,\rho}$ was in fact motivated by the study of the $G$-orbit Hilbert scheme explained below, in particular by the study of $\pi^{-1}(0)$.

For a positive integer $N$, $\text{Hilb}^N(\mathbb{C}^n)$ is the universal scheme which parametrizes all zero-dimensional subschemes of $\mathbb{C}^3$ of length $N$. For a finite subgroup $G$ of $\text{GL}(n, \mathbb{C})$, we choose $N = |G|$, the order of $G$. Then the group $G$ acts in the natural manner on $\text{Hilb}^{|G|}(\mathbb{C}^n)$. The $G$-orbit Hilbert scheme $\text{Hilb}^{|G|}(\mathbb{C}^n)$ is by definition the unique irreducible component of the $G$-invariant part of $\text{Hilb}^{|G|}(\mathbb{C}^n)$ dominating $\mathbb{C}^n/G$, the $G$-invariant part of the corresponding Chow scheme of $|G|$ points. In other words, $\text{Hilb}^{|G|}(\mathbb{C}^n)$ is the universal subscheme of the Hilbert scheme $\text{Hilb}^{|G|}(\mathbb{C}^n)$ which parametrizes all smoothable scheme-theoretic $G$-orbits of length $|G|$. The $G$-orbit Hilbert scheme $\text{Hilb}^{|G|}(\mathbb{C}^n)$ is a fairly natural algebro-geometric object which incorporates all representation-theoretic information about $G$ as a subgroup of $\text{GL}(n, \mathbb{C})$. It has already been studied in detail in the SU(2) case [IN99] and in the case where $G$ is a noncommutative simple subgroup $A_5$ or PSL(2, 7) of $\text{SL}(3, \mathbb{C})$ [GNS00]. The scheme $\text{Hilb}^N(\mathbb{C}^n)$ is known to be very singular if $n \geq 3$. However for a finite subgroup $G$ of $\text{SL}(3, \mathbb{C})$, $\text{Hilb}^{|G|}(\mathbb{C}^3)$ is known to be nonsingular by [N01] in the abelian case and by [BKR01] in the general case.

The second purpose of the article is to study $\text{Hilb}^{|G|}(\mathbb{C}^3)$, among other things, the fiber $\pi^{-1}(0)$ of the Hilbert-Chow morphism $\pi : \text{Hilb}^{|G|}(\mathbb{C}^3) \to \mathbb{C}^3/G$ when $G$ is a finite subgroup of $\text{SO}(3)$. This will be discussed in Section 3.

It is well known that there is a surjective homomorphism from SU(2) onto SO(3) having ±1 as its kernel, by which non-abelian subgroups of SU(2) and SO(3) correspond bijectively. For a subgroup $G$ of SO(3) we define the representation graph $R(G)$ of $G$ by using the irreducible decompositions of tensor products with $\rho_{nat}$ in the same manner as in the SU(2) case. First we observe that $\pi^{-1}(0)$ is a union of finitely many smooth rational curves. So we define in the same way as in the SU(2) case the planar dual graph $\overline{R}(G)$ of $\pi^{-1}(0)$ by associating a vertex to each rational curve in $\pi^{-1}(0)$, and by associating an edge connecting a pair of the vertices to each intersection point of the corresponding curves. Then it turns out that the planar dual graph $\overline{R}(G)$ is identified with a particular subgraph of $R(G)$. In other words, every irreducible rational curve in $\pi^{-1}(0)$ is labeled by one of the nontrivial irreducible representations of $G$ and vice versa, whose intersections are described purely in terms of irreducible decompositions of tensor products with $\rho_{nat}$ in a manner similar to the SU(2) case. Thus we have a complete description of $\pi^{-1}(0)$ in the SO(3) case. However in almost all cases other than (A), (H) and (I) in the notation of [YY93] the precise structure of $\pi^{-1}(0)$ is yet to be determined.

This paper is organized as follows. In Section 1, we explain basic lemmas necessary for computing $P_{S,\rho}$. In Section 2, we first recall the Koszul complex over $S$ and show that any alternating sum of componentwise generating functions of the $G$-modules in the Koszul complex is equal to zero, which yields a Springer-McKay type identity of $P_{S,\rho}$. In Section 3, we describe $\pi^{-1}(0)$ completely when $G$ is a subgroup of SO(3).
In Sections 4 and 5 we give tables of $P_{S,G,\rho}$ for every finite subgroup $G$ of $\text{SL}(3,\mathbb{C})$ of type from (E) to (L) and every non-trivial representation $\rho$ of $G$.

We would like to thank J. McKay and A. King for helpful discussions.

1. The coinvariant algebra for a finite group $G$ of $\text{SL}(3,\mathbb{C})$

1.1. The Molien series. Let $V$ be an $n$-dimensional complex vector space, $V^\vee$ the dual of $V$ and $G$ a finite subgroup of $\text{GL}(V)$. We denote by $\rho$ the matrix representation of $G$ afforded by the natural inclusion of $G$ in $\text{GL}(V)$ and by $\rho^\vee$ its contragredient representation. As usual we call $\rho$ the natural representation of $G$. We use the same notation as in [GNS00]; in particular we denote by $S = S(V^\vee)$, $m = S_+$, $S^G$ and $S^G_+$ respectively the symmetric algebra of $V^\vee$ over $\mathbb{C}$, the maximal ideal of $S$ of the origin, the invariant algebra of $G$, and the maximal ideal of $S^G$ of the origin. Let $n$ be the ideal of $S$ generated by $S^G_+$ and $S_G := S/n$ the coinvariant algebra of $G$. Since $n$ is a graded ideal of $S$, $S_G$ is a graded algebra, too.

By the Noether normalization lemma, we can take a minimal system of homogeneous parameters $f_1, f_2, \ldots, f_n$ of $S^G$ so that $S^G$ is a finite module over $\mathbb{C}[f_1, \ldots, f_n]$. Extending them we choose a minimal system of homogeneous generators $f_1, f_2, \ldots, f_r$ of $S^G$ and fix them once for all. The ideal $n$ of $S$ is generated by $f_1, f_2, \ldots, f_r$.

Let $\hat{G} = \{\rho_0 = 1, \rho_1, \ldots, \rho_s\}$ be the set of representatives of equivalence classes of all irreducible representations of $G$ and $\chi_i$ the character of $\rho_i$ for $0 \leq i \leq s$. For an arbitrary graded $CG$-module $M = \oplus_{i \geq 0} M_i$ with $\dim M_i < \infty$, we define the Molien series of $M$ for $\rho_j$ by

$$P_{M,\rho_j}(t) = \sum_{i \geq 0} \langle M_i, \rho_j \rangle_G t^i,$$

where

$$\langle M_i, \rho_j \rangle_G = \dim \text{Hom}_G(\rho_j, M_i) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_j(g)} \text{Tr}_{M_i}(g).$$

The following is derived easily from the formula in [Bourbaki, Lemme 2, p. 110]

$$P_{S,\rho_j}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi_j(g)}}{\det(1 - \rho^\vee(g)t)}.$$  \hspace{1cm} (1)

Now we recall from [Stanley79, (4.9)].

**Theorem 1.2.** Let $f_1, f_2, \ldots, f_r$ be homogeneous generators of $S^G$ chosen as above, $d_i = \deg f_i$, $(f_1, f_2, \ldots, f_n)$ the ideal of $S$ generated by $f_1, f_2, \ldots, f_n$, and let $R = S/(f_1, f_2, \ldots, f_n)$. Then as $CG$-modules we have

$$S \simeq R \otimes \mathbb{C}[f_1, f_2, \ldots, f_n]$$

and $R \simeq (CG)^e$

where $e = |G|^{-1} d_1 d_2 \cdots d_n$.

**Proposition 1.3.** Keeping the notations as above, we have

(i) $$P_{R,\rho_j}(t) = \frac{\prod_{i=1}^n (1 - t^{d_i})}{|G|} \sum_{g \in G} \frac{\overline{\chi_j(g)}}{\det(1 - \rho^\vee(g)t)}.$$
(ii) $P_{R,\rho}(t) - P_{S,\rho}(t)$ is a polynomial with non-negative integer coefficients.

(iii) \[
\sum_{j=0}^{s} (\deg \rho_j) P_{S,\rho_j}(t) = \sum_{j \geq 0} \dim(S_G)_{i} t^{i}.
\]

Proof. (i) It follows from Theorem 1.2 that $P_{S,\rho}(t) = P_{R,\rho}(t)/\prod_{i=1}^{n}(1-t^{d_i})$. From Molien’s formula (1), we infer (i).

(ii) Since we have a canonical surjection from $R$ to $S_G$, $P_{R,\rho}(t) - P_{S,\rho}(t)$ has non-negative integer coefficients.

(iii) Let $S_G = \bigoplus_{j=0}^{s}(S_G)_{\rho_j}$ be the decomposition into homogeneous components, namely $\rho_j$-factors $(S_G)_{\rho_j}$ of $S_G$. Since $\dim(S_G)_{\rho_j} = (\deg \rho_j)(S_G,\rho_j)_G$, the above equation is clear from the definition of $P_{S,\rho_j}(t)$. \qed

We note that if there exists a complex reflection group $\tilde{G}$ of $GL(V)$ containing $G$ with $[\tilde{G} : G] = 2$, then it is easier to calculate $P_{S,\rho_j}(t)$ by using the following

Theorem 1.4. ([Bourbaki] or [GNS00, 1.6]) Assume that there exists a complex reflection subgroup $\tilde{G}$ of $GL(V)$ containing $G$ with $[\tilde{G} : G] = 2$.

(i) There exist $n$ homogeneous $\tilde{G}$-invariants $f_1, f_2, \ldots, f_n$ such that as $\mathbb{C}\tilde{G}$-modules $S^{\tilde{G}} = \mathbb{C}[f_1, f_2, \ldots, f_n]$ and $S_G^{\tilde{G}} = S/(f_1, f_2, \ldots, f_n) \simeq \mathbb{C} \tilde{G}$.

(ii) Let $f_{n+1} = \text{Jac}(f_1, f_2, \ldots, f_n)$. Then we have

$S^{\tilde{G}} = \mathbb{C}[f_1, f_2, \ldots, f_n, f_{n+1}]$ and $S_G^{\tilde{G}} \simeq S_G \oplus \mathbb{C} f_{n+1}$.

Moreover

$$(S_G)_{k} \simeq \begin{cases} (S_G)_{k}, & \text{if } k < d_{n+1}, \\ \mathbb{C} f_{n+1}, & \text{if } k = d_{n+1}, \\ 0, & \text{if } k > d_{n+1}, \end{cases}$$

where $d_{n+1} = \deg f_{n+1} = \sum_{i=1}^{n}(d_i - 1)$.

Corollary 1.5. Under the same assumptions in Theorem 1.4

$$P_{S,\rho}(t) = P_{S,\rho}(t) = \prod_{i=1}^{n}(1-t^{d_i}) P_{S,\rho}(t),$$

$$P_{S,\rho}(t) = P_{S,\rho}(t) + t^{n+1} \prod_{i=1}^{n}(1-t^{d_i}) P_{S,\rho}(t) + t^{d_{n+1}}.$$

Proof. Immediate from Theorem 1.4. \qed

Remark 1.6. Let $G$ be a finite subgroup of $SL(3, \mathbb{C})$ of exceptional type (E)-(L). Then homogeneous generators of $S^{\tilde{G}}$ are known explicitly in [YY93]. Moreover, since $(S_G)_{i} \simeq S_i/(n)_i$ and $(n)_i = V^\vee \cdot (n)_{i-1} + \sum_{j \neq i} (n)_j$, we can calculate $(n)_i$ inductively. Thus all the informations of Proposition 1.3 are available, which turns out to be sufficient to determine $P_{S,\rho_j}(t)$ by the case-by-case examination. The results are summarized in Sections 4 and 5.

Either of the groups of type (H), (I) and (L) is a subgroup of some complex reflection group of index two, while the group of type (E), (F) or (J) is a subgroup of some complex reflection group of index 6, 3 or 12 respectively. In these cases
we can apply [Steinberg64] and [Stanley79] to describe \( R := S/(f_1, f_2, f_3) \) in some detail. However no group of type (G) or (K) is a subgroup of a complex reflection group. Nevertheless in any case from (E) to (L) the algebra \( R \) has a remarkable duality as in the cases of complex reflection groups. We will discuss it elsewhere.

2. Koszul complex and Springer-McKay identities of Molien series

We keep the previous notations. We start with the Koszul complex for the symmetric algebra \( S = S(V^\vee) \) (cf. [Lang84, XVI §10]).

**Lemma 2.1.** Let \( \wedge^k V^\vee \) be the \( k \)-th alternating product of \( V^\vee \).

(i) There is a unique homomorphism

\[
d_k : \wedge^k V^\vee \otimes S \longrightarrow \wedge^{k-1} V^\vee \otimes S
\]

such that for \( x_i \in V^\vee \) and \( y \in S \)

\[
d_k((x_1 \wedge x_2 \wedge \cdots \wedge x_k) \otimes y) = \sum_{i=1}^{k} (-1)^{i-1}(x_1 \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k) \otimes (x_i \cdot y).
\]

(ii) There is an exact sequence with \( d_k \) given by (i)

\[
0 \rightarrow \wedge^n V^\vee \otimes S \xrightarrow{d_n} \wedge^{n-1} V^\vee \otimes S \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} V^\vee \otimes S \xrightarrow{d_1} S \xrightarrow{d_0} C \rightarrow 0.
\]

(iii) For each integer \( m \geq 1 \) we have an exact sequence

\[
0 \rightarrow \wedge^n V^\vee \otimes S_{m-n} \rightarrow \wedge^{n-1} V^\vee \otimes S_{m-n+1} \rightarrow \cdots \rightarrow S_m \rightarrow 0,
\]

where \( S_j = 0 \) for \( j < 0 \).

(iv) For each integer \( m \geq 1 \) and for each irreducible representation \( \rho_j \) \((0 \leq j \leq s)\), we have an exact sequence

\[
0 \rightarrow (\wedge^n V^\vee \otimes S_{m-n})_{\rho_j} \rightarrow (\wedge^{n-1} V^\vee \otimes S_{m-n+1})_{\rho_j} \rightarrow \cdots \rightarrow (S_m)_{\rho_j} \rightarrow 0.
\]

**Proof.** For a proof of (i), (ii) and (iii), see [Lang84, (10.13) and (10.14)]. Since \( d_k \) is a \( G \)-homomorphism, we decompose the exact sequence of (iii) into \( \rho_j \)-components, which proves (iv). \( \square \)

We denote by \( \rho^{(k)} \) \( (\rho^\vee)^{(k)} \) the \( CG \)-module \( \wedge^k V \) \( (\wedge^k V^\vee) \). Note that \( \rho^{(0)} = 1 \), \( \rho^{(1)} = \rho \), \( \rho^{(n)} = \text{det} \), and \( \rho^\vee^{(k)} \) is the dual \( \mathbb{C}G \)-module of \( \rho^{(k)} \). Define non-negative integers \( a_{ij}^{(k)} \) by

\[
(2) \quad \rho^{(k)} \otimes \rho_i = \sum_{j=0}^{s} a_{ij}^{(k)} \rho_j, \quad \text{for } 0 \leq i \leq s \text{ and } 0 \leq k \leq n.
\]

**Theorem 2.2.** The Molien series \( P_{S, \rho_j}(t) \) satisfy the following equations:

\[
\sum_{k=0}^{n} \sum_{j=0}^{s} (-1)^{k} a_{ij}^{(k)} t^k P_{S, \rho_j}(t) = \delta_{i,0} \quad \text{for } i = 0, 1, \ldots, s.
\]
Proof. We see
\[
\dim(\wedge^k V \otimes S_{m-k})_{\rho_i} = \deg(\rho_i) \dim \text{Hom}_G(\rho_i, \rho^\vee \otimes S_{m-k}) = \deg(\rho_i) \dim \text{Hom}_G(\rho^k \otimes \rho_i, S_{m-k}) = \deg(\rho_i) \sum_{j=0}^{s} a_{ij}^{(k)} \dim \text{Hom}_G(\rho_j, S_{m-k}).
\]

Thus we obtain
\[
\sum_{m \geq 0} (\dim(\wedge^k V \otimes S_{m-k})_{\rho_i}) t^m = \deg(\rho_i) \sum_{j=0}^{s} a_{ij}^{(k)} t^k P_{S,\rho_j}(t).
\]

Hence our theorem follows from Lemma 2.1 (ii) and (iv).

Remark 2.3. This proposition can be proved directly by using (1).

Corollary 2.4. Keep the same notation in Theorem 2.2.

(i) If \( G \) is a subgroup of \( \text{SL}(V) \), then
\[
\sum_{k=1}^{n-1} \sum_{j=0}^{s} (-1)^k a_{ij}^{(k)} t^k P_{S,\rho_j}(t) = (-1 - (-1)^n t^n) P_{S,\rho_i}(t) + \delta_{i,0}.
\]

(ii) If \( G \) is a subgroup of \( \text{SL}(2, \mathbb{C}) \), then
\[
\sum_{j=0}^{s} a_{ij}^{(1)} P_{S,\rho_j}(t) = (t + t^{-1}) P_{S,\rho_i}(t) - t^{-1} \delta_{i,0},
\]

(iii) If \( G \) is a subgroup of \( \text{SL}(3, \mathbb{C}) \) and if \( \rho^\vee = \rho \), then
\[
\sum_{j=0}^{s} a_{ij}^{(1)} P_{S,\rho_j}(t) = (t + 1 + t^{-1}) P_{S,\rho_i}(t) + (t^2 - t)^{-1} \delta_{i,0}.
\]

(The assumption in (iii) is satisfied if \( G \subset \text{SO}(3) \).)

Proof. If \( G \) is a finite subgroup of \( \text{SL}(V) \), then \( \rho^{(0)} \) and \( \rho^{(n)} \) are trivial. So (i) follows at once from Theorem 2.2. If \( \dim V = 2 \), we obtain (ii) by dividing both sides of (i) by \(-t\). Under the assumption of (iii), we have \( \rho^{(1)} = \rho^{(2)} = \rho \). Dividing both sides of (i) by \((t^2 - t)\), we obtain (iii).

Put \( F_j(t) = P_{S,\rho_j}(t) \prod_{i=1}^{n} (1 - t^{d_i}) \) for \( 0 \leq j \leq s \). By Theorem 1.4
\[
F_j(t) = \begin{cases} 
1 + t^{d_{n+1}} & \text{if } j = 0 \\
P_{S_G,\rho_j}(t) & \text{if } j \neq 0.
\end{cases}
\]

The next corollary is immediate from Corollary 2.4.

Corollary 2.5. Keep the notation as above. Let \( 0 \leq i \leq s \). Then

(i) If \( G \) is a finite subgroup of \( \text{SL}(2, \mathbb{C}) \), then
\[
\sum_{j=0}^{s} a_{ij}^{(1)} F_j(t) = (t + t^{-1}) F_i(t) - \frac{(1 - t^{d_1})(1 - t^{d_2})}{t} \delta_{i,0}.
\]
(ii) If $G$ is a finite subgroup of $\text{SO}(3)$, then
\[ \sum_{j=0}^{s} a^{(1)}_{ij} F_j(t) = (t + 1 + t^{-1}) F_i(t) + \frac{(1 - t^{d_1})(1 - t^{d_2})(1 - t^{d_3})}{(t^2 - t)} \delta_{i,0}. \]

**Remark 2.6.** The system of equations in Corollary 2.4 (ii) were given in [Springer87] and [McKay99] by using corresponding Coxeter-Dynkin diagrams, or McKay’s semi-affine graphs. Corollary 2.4 (i) claims, roughly speaking, that one can calculate all the Molien series once one knows $a^{(k)}_{ij}$, in particular only $a^{(1)}_{ij}$ when $G \subset \text{SL}(2, \mathbb{C})$ or $G \subset \text{SO}(3)$. In this sense the representation graph (or rather the indices $a^{(1)}_{ij}$) of a subgroup $G$ of $\text{SO}(3)$ plays the same role in calculating Molien series as the Coxeter-Dynkin diagram for a finite subgroup of $\text{SL}(2, \mathbb{C})$.

2.7. **Complex reflection groups.** If $G$ is a finite subgroup of $\text{SL}(2, \mathbb{C})$ or $\text{SO}(3)$, there exists a complex reflection group $\tilde{G}$ containing $G$ with $[\tilde{G} : G] = 2$. We list all such pairs $G$ and $\tilde{G}$ in Table 1 and Table 2. We use the notation in [Cohen76]; the group $G_i$ is the complex reflection group with Shephard-Todd number $i$. The symbol $W(A)$ stands for the Weyl group of type $A$. The integer $d_i$ in the tables is the degree of $f_i$ defined in Theorem 1.4.

<table>
<thead>
<tr>
<th>$G$ in $\text{SL}(2, \mathbb{C})$</th>
<th>order</th>
<th>$\tilde{G}$</th>
<th>$d_1, d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cyclic</td>
<td>$l$</td>
<td>$W(I_2^{(l)})$</td>
<td>2, $l$</td>
</tr>
<tr>
<td>binary dihedral</td>
<td>$4l$</td>
<td>$G(2l, l, 2)$</td>
<td>4, $2l$</td>
</tr>
<tr>
<td>binary tetrahedral</td>
<td>24</td>
<td>$G_{12}$</td>
<td>6, 8</td>
</tr>
<tr>
<td>binary octahedral</td>
<td>48</td>
<td>$G_{13}$</td>
<td>8, 12</td>
</tr>
<tr>
<td>binary icosahedral</td>
<td>120</td>
<td>$G_{22}$</td>
<td>12, 20</td>
</tr>
</tbody>
</table>

**Table 1. Subgroups of $\text{SL}(2, \mathbb{C})$**

<table>
<thead>
<tr>
<th>$G$ in $\text{SO}(3)$</th>
<th>order</th>
<th>$\tilde{G}$</th>
<th>$d_1, d_2, d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>cyclic</td>
<td>$l$</td>
<td>$W(I_2^{(l)})$</td>
<td>1, 2, $l$</td>
</tr>
<tr>
<td>dihedral</td>
<td>$2l$</td>
<td>$W(I_2^{(l)} \times A_1)$</td>
<td>2, 2, $l$</td>
</tr>
<tr>
<td>tetrahedral ($\simeq A_4$)</td>
<td>12</td>
<td>$W(A_3)$</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>octahedral ($\simeq S_4$)</td>
<td>24</td>
<td>$W(B_3)$</td>
<td>2, 4, 6</td>
</tr>
<tr>
<td>icosahedral ($\simeq A_5$)</td>
<td>60</td>
<td>$W(H_3)$</td>
<td>2, 6, 10</td>
</tr>
</tbody>
</table>

**Table 2. Subgroups of $\text{SO}(3)$**
3. Geometric McKay correspondence for subgroups of SO(3)

Let $\pi : \text{Hilb}^G(C^3) \to C^3/G$ be the Hilbert-Chow morphism for $G \subset \text{SO}(3)$.

**Theorem 3.1.** Let $G$ be a finite subgroup of SO(3). For $I \in \text{Hilb}^G(C^3)$ with $I \subset \mathfrak{m}$, we define $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$. For $1 \leq i \leq s$, we define $C_j = \{I \in \text{Hilb}^G(C^3); V(I) \supset \rho_j, I \subset \mathfrak{m}\}$. Then

(i) $C_j \simeq \mathbb{P}^1$ and $\pi^{-1}(0) = \bigcup_{j=1}^{s} C_j$.
(ii) If $I \in C_j$ and $I \notin C_i$ for any $j \neq i$, then $V(I) \simeq \rho_j$ as $G$-modules.
(iii) If only two rational curves $C_i$ and $C_j$ meet at $I \in \pi^{-1}(0)$, then $C_i$ and $C_j$ intersect at $I$ transversally and $V(I) \simeq \rho_i + \rho_j$.
(iv) If $G$ is either cyclic, $A_4$ or $D_{4m+2}$, then there are no three rational curves meeting at a point of $\pi^{-1}(0)$.
(v) If $G = D_{4m}, S_4$ or $A_5$, then there is a unique $I \in \pi^{-1}(0)$ such that $\{I\} = C_i \cap C_j \cap C_k$ for $\rho_i, \rho_j, \rho_k \in G$ all distinct. In this case $V(I) \simeq \rho_i + \rho_j + \rho_k$ and the curves $C_i, C_j, C_k$ meet transversally at $I$ as coordinate axes of $(C^3, 0)$.
(vi) No four rational curves $C_i$ meet at a point of $\pi^{-1}(0)$.

Our proof of Theorem 3.1 is carried out by the case by case examination. When $G$ is abelian, our theorem is proved by the same argument as in the two dimensional case. When $G$ is isomorphic to the alternating group $A_4$ or $A_5$, our theorem has been proved in [GNS00]. So we only need to prove our theorem when $G$ is a dihedral group or $G = S_4$. We will give a proof of it in the subsections 3.4, 3.5 and 3.6.

3.2. Graphs of $G$. Here we define three graphs for a finite subgroup $G$ of SO(3).

First we define the planar dual graph $\bar{R}(G)$ of $\pi^{-1}(0)$ as follows: the set of vertices of $\bar{R}(G)$ is $\{C_j\}_{1 \leq j \leq s}$; $C_i$ and $C_j$ are joined by a single edge if and only if $C_i \cap C_j \neq \emptyset$. We note that in Theorem 3.1 there are three rational curves $C_i, C_j$ and $C_k$ in $\pi^{-1}(0)$ meeting at a point, for which we define a planar triangle in $\bar{R}(G)$ with three vertices $C_i, C_j$ and $C_k$ instead of a two cell. See Table 3.

Next we define the (unoriented) representation graph $R(G)$ of $G$ as follows: the set of vertices is $\hat{G}$; let $a_{i,j}^{(1)}$ be the integer defined in (2); $\rho_i$ and $\rho_j$ are joined by an edge of multiplicity $a_{i,j}^{(1)}$ if $a_{i,j}^{(1)} \neq 0$, where if $i = j$ the edge joining $\rho_i$ with itself is understood as a loop of multiplicity $a_{i,i}^{(1)}$. We note $a_{i,j}^{(1)} = 0$ or 1 for $i \neq j$, while $a_{i,i}^{(1)} = 0, 1, \text{or} 2$. We also note that $a_{i,j}^{(1)} = a_{i,j}^{(2)}$ for any finite subgroup $G$ of SO(3).

Finally we define a subgraph $R_0(G)$ of $R(G)$ as follows: the set of vertices is $\{\rho_j\}_{1 \leq j \leq s}$ and $\rho_i$ and $\rho_j$ are joined by a single edge if and only if $i \neq j$ and $a_{i,j}^{(1)} \neq 0$. In other words, $R_0(G)$ is the subgraph of $R(G)$ obtained from $R(G)$ by removing the vertex $\rho_0$, all the edges starting from $\rho_0$ and all the loops in $R(G)$.

The following theorem is a corollary to the proof of Theorem 3.1 once we calculate the representation graph $R(G)$.

**Theorem 3.3.** $\bar{R}(G)$ is isomorphic to $R_0(G)$ under the map $C_i \mapsto \rho_i$ ($1 \leq i \leq s$). The graphs $\bar{R}(G)$ and $R(G)$ are given in Table 3.
<table>
<thead>
<tr>
<th>$G$</th>
<th>$\mathcal{R}(G) (= R_0(G))$</th>
<th>$R(G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}/m\mathbb{Z}$</td>
<td>$\rho_{m-1}$ $\rho_1$ $\rho_{m-2}$ $\rho_2$ $\rho_{j+1}$ $\rho_j$ $\rho_j$ $\rho_{j-1}$</td>
<td>$\rho_0$ $\rho_{m-1}$ $\rho_1$ $\rho_{m-2}$ $\rho_2$ $\rho_{j+1}$ $\rho_j$ $\rho_{j-1}$</td>
</tr>
<tr>
<td>$D_{2l}$ ($l$: odd)</td>
<td>$2_{l-1}$ $2_{l-3}$ $2_2$ $2_1$ $1_2$</td>
<td>$2_{l-1}$ $2_{l-3}$ $2_2$ $2_1$ $1_2$</td>
</tr>
<tr>
<td>$D_{2l}$ ($l$: even)</td>
<td>$2_{l-1}$ $2_{l-2}$ $2_2$ $2_1$ $1_2$</td>
<td>$2_{l-1}$ $2_{l-2}$ $2_2$ $2_1$ $1_2$</td>
</tr>
<tr>
<td>$A_4$</td>
<td>$1_2$ $3$ $1_3$</td>
<td>$1_2$ $1_1$ $3$ $1_3$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$2$ $3_1$ $3_2$ $1_2$</td>
<td>$2$ $1_1$ $3_1$ $3_2$ $1_2$</td>
</tr>
<tr>
<td>$A_5$</td>
<td>$3_1$ $5$ $4$</td>
<td>$1$ $3_1$ $5$ $4$</td>
</tr>
</tbody>
</table>

Table 3. Graphs of subgroups of $SO(3)$
In the rest of this section we give proofs of Theorem 3.1 in the cases where $G$ is a dihedral group or $G$ is isomorphic to $S_4$.

3.4. **Proof of Theorem 3.1 — the dihedral group of order $2\ell = 4m$**. Let $G$ be the dihedral group of order $2\ell$:

$$G = \langle \sigma = \begin{pmatrix} \varepsilon^{-1} & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle, \quad \text{where} \quad \varepsilon = e^{2\pi i/\ell}.$$  

We define

$$f_1 = x^2, \quad f_2 = xy, \quad f_3 = x^2 + y^\ell, \quad f_4 = z(x^\ell - y^\ell).$$

Then we see $\{f_1, f_2, f_3, f_4\}$ is a system of generators of $S^G$ which satisfies

$$f_4^2 - f_1f_3^2 + 4f_1f_2^\ell = 0.$$  

regardless of the parity of $\ell$.

First in this subsection we consider the case where $\ell$ is even. So we write $\ell = 2m$, $|G| = 4m$. The character table of $G$ is as follows.

<table>
<thead>
<tr>
<th>c.c</th>
<th>-1</th>
<th>1</th>
<th>$\tau$</th>
<th>$\tau \sigma$</th>
<th>$\sigma^i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>age</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>1</td>
<td>$m$</td>
<td>$m$</td>
<td>2</td>
</tr>
<tr>
<td>$1_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$1_2$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$1_3$</td>
<td>1</td>
<td>(-1)$^m$</td>
<td>1</td>
<td>-1</td>
<td>$(-1)^i$</td>
</tr>
<tr>
<td>$1_4$</td>
<td>1</td>
<td>(-1)$^m$</td>
<td>-1</td>
<td>1</td>
<td>$(-1)^i$</td>
</tr>
<tr>
<td>$2_j$</td>
<td>2</td>
<td>$(-1)^{i/2}$</td>
<td>0</td>
<td>0</td>
<td>$\varepsilon^{ij} + \varepsilon^{-ij}$</td>
</tr>
</tbody>
</table>

(1 \leq i, j \leq m - 1)

Table 4. Characters of $G(D_{2\ell})$, $\ell = 2m$: even

The coinvariant algebra $S^G$ splits into irreducible components as in Table 5. Using Table 5 we define ideals in $\text{Hilb}^G(\mathbb{C}^3)$ $[a : b] \in \mathbb{P}^1$ as in [GNS00].

$$I([a : b]_{1_2}) = (az + b(x^{2m} - y^{2m}), xz, yz) + n,$$
$$I([a : b]_{1_4}) = (a(x^m + y^m) + b(x^m - y^m), x^m + y^m, x^{m+1}, y^{m+1}, (x^m + y^m)z) + n,$$
$$I([a : b]_{2_2}) = S[G] \cdot (ax^j z + by^{2m-j}, x^{j+1}z, x^{2m-j+1}) + n. \quad \text{for} \quad i = 1, 2, \ldots, m - 1$$

It is clear that $V(I([a : b]_\rho)) \simeq \rho$ as $G$-modules. We note that the following exhaust all the possible cases of coincidence between $I([a : b]_\rho)$.

$$I([0 : 1]_{1_1}) = I([1 : 0]_{2_1}),$$
$$I([0 : 1]_{2_j}) = I([1 : 0]_{2_{j+1}}), \quad \text{for} \quad j = 1, 2, \ldots, m - 2,$$
$$I([0 : 1]_{2_{m-1}}) = I([0 : 1]_{1_3}) = I([0 : 1]_{1_4}).$$
I follow that we consider the second case where $m = 4$. Proof of Theorem 3.1 — the dihedral group of order $2m$.

We define $I([a : b]_{2j}) = (az + bx^{2m-j} + y^{2m-j}, xz, yz) + n$. $I([a : b]_{2j}) = S[G] \cdot (ax^j z + by^{2m-j} , x^{j+1} z, x^{2m-j+2}) + n, j = 1, 2, \ldots, m$,

where

<table>
<thead>
<tr>
<th>degree</th>
<th>$(S_G)_j$</th>
<th>irred. factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\langle x, y \rangle \oplus \langle z \rangle$</td>
<td>$2_1 + 2_2$</td>
</tr>
<tr>
<td>$2 \leq j \leq m - 1$</td>
<td>$\langle x^j, y^j \rangle \oplus \langle x^{j-1} z, -y^{j-1} z \rangle$</td>
<td>$2_j + 2_{j-1}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$\langle x^m + y^m \rangle \oplus \langle x^m - y^m \rangle$</td>
<td>$1_3 + 1_4 + 2_{m-1}$</td>
</tr>
<tr>
<td>$m + 1$</td>
<td>$\langle y^{m+1}, x^{m+1} \rangle \oplus \langle (x^m - y^m) z \rangle$</td>
<td>$2_{m-1} + 1_3 + 1_4$</td>
</tr>
<tr>
<td>$m + 2 \leq j \leq 2m - 1$</td>
<td>$\langle y^j, x^j \rangle \oplus \langle y^{j-1} z, -x^{j-1} z \rangle$</td>
<td>$2_{2m-1} + 2_{2m-1}+1$</td>
</tr>
<tr>
<td>$2m$</td>
<td>$\langle x^{2m} - y^{2m} \rangle \oplus \langle y^{2m-1} z, -x^{2m-1} z \rangle$</td>
<td>$1_2 + 2_1$</td>
</tr>
</tbody>
</table>

Table 5. The coinvariant algebra of $G(D_{2\ell}), \ell = 2m$: even

Now we prove

$$\pi_1^{-1}(0) = \bigcup_{\rho \in \hat{G}} I([a : b]_{\rho}).$$

It is immediate from the definition and the Diagram $D_{4m}$ (see 3.7) that $I([a : b]_{\rho})$ are contained in $\pi^{-1}(0)$. Conversely let $I$ be an ideal contained in $\pi^{-1}(0)$, that is, $n \subseteq I \subseteq m$ and $S/I \cong C[G]$. By the Diagram $D_{4m}$, it is easy to see that $x^{2m-j} z, y^{2m-j} z \in I$ for all $j = 1, 2, \ldots, m - 1$ and that $x^j + ax^j z + by^{2m-j} \notin I$ for any $a, b \in C$ and $j = 1, 2, \ldots, m - 1$. If $x^j z + by^{2m-j} \in I$ for some $b \neq 0$ and some $j = 1, 2, \ldots, m - 1$, then we have $I([1 : 0]_{2j}) \subseteq I$ which implies $I([1 : 0]_{2j}) = I$.

Now we assume the contrary, that is, $x^j z + by^{2m-j} \notin I$ for any nonzero $b$ and any $j = 1, 2, \ldots, m - 1$. Then by the condition $S/I \cong C[G]$ we have either $x^j z \in I$ or $y^{2m-j} \in I$. If there is $j \geq 2$ such that $x^j z \in I$, $x^{j-1} z \notin I$, then $y^{2m-j+1} \in I$. It follows that $I = I([1 : 0]_{2j})$. If $xz \in I$, then $I = I([a : b]_{1z})$.

It remains to consider the case where there is no $j$ such that $x^j z \in I$. Hence $y^{m+1} \in I$. If $x^m + y^m + b(x^m - y^m) z \in I$ (resp. $x^m - y^m + b(x^m + y^m) z \in I$) for some $b \in C$, then $I = I([1 : 1]_{3}) (\text{resp. } I([1 : 1]_{4}))$. Otherwise $I$ contains $(x^m - y^m) z$ and $(x^m + y^m) z$ and then we have $I = I([0 : 1]_{13})$. Thus we complete the proof of Theorem 3.1 when $G$ is a dihedral group of order $4m$.

3.5. Proof of Theorem 3.1 — the dihedral group of order $4m + 2$. Now we consider the second case where $G$ is a dihedral group of order $2\ell = 4m + 2$. Table 6 is the character table of $G$. The coinvariant algebra $S_G$ splits into irreducible components as in Table 7.

We define

$$I([a : b]_{1z}) = (az + bx^{2m+1} - y^{2m+1}, xz, yz) + n, \quad I([a : b]_{2j}) = S[G] \cdot (ax^j z + by^{2m-j+1}, x^{j+1} z, x^{2m-j+2}) + n, \quad j = 1, 2, \ldots, m,$$

where

$$I([0 : 1]_{1z}) = I([1 : 0]_{2z}), \quad I([0 : 1]_{2j}) = I([1 : 0]_{2j+1}), \quad \text{for } j = 1, 2, \ldots, m - 1.$$
<table>
<thead>
<tr>
<th>degree</th>
<th>(SG)</th>
<th>irred. factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>⟨x, y⟩ + ⟨z⟩</td>
<td>2₁ + 1₂</td>
</tr>
<tr>
<td>j</td>
<td>⟨xᵢ, yᵢ⟩ + ⟨x⁻¹_j z, y⁻¹_j z⟩</td>
<td>2₂ + 2₁ j⁻₁</td>
</tr>
<tr>
<td>m</td>
<td>⟨xᵐ, yᵐ⟩ + ⟨x⁻¹ᵐ z, y⁻¹ᵐ z⟩</td>
<td>2₂ m + 2₂ m⁻₁</td>
</tr>
<tr>
<td>m + 1</td>
<td>⟨yᵐ⁺¹, xᵐ⁺₁⟩ + ⟨xᵐ z, yᵐ z⟩</td>
<td>2₂ m + 2₂ m</td>
</tr>
<tr>
<td>m + 2</td>
<td>⟨yᵐ⁺², xᵐ⁺²⟩ + ⟨x⁻¹ᵐ⁺¹ z, y⁻¹ᵐ⁺¹ z⟩</td>
<td>2₂ m⁻¹ + 2₂ m</td>
</tr>
<tr>
<td>j</td>
<td>⟨yᵢ, xᵢ⟩ + ⟨y⁻¹ᵢ z, x⁻¹ᵢ z⟩</td>
<td>2₂ m⁻ⱼ⁻₁ + 2₂ m⁻ⱼ⁺₂</td>
</tr>
<tr>
<td>2m + 1</td>
<td>⟨x²ᵐ⁺¹ − y²ᵐ⁺¹⟩ + ⟨y²ᵐ z, x²ᵐ z⟩</td>
<td>1₂ + 2₁</td>
</tr>
</tbody>
</table>

Table 7. The coinvariant algebra of \( G(D₂ℓ) \), \( ℓ = 2m + 1 \): odd

We see \( π⁻¹(0) = \bigcup_{ρ \in \hat{G} \backslash \{1\}} I([a : b]_ρ) \) in the same manner as in the case of even \( ℓ \). As before we see that \( x^{2m⁻j} z, y^{2m⁻j} z \in I \) for any \( j = 1, 2, \ldots, m - 1 \) and that \( x¹ + a x² z + b y² m⁻j \notin I \) for any \( a, b \in C \) and \( j = 1, 2, \ldots, m - 1 \). If \( x² z + b y² m⁻j \in I \) for some \( b \neq 0 \) and some \( j = 1, 2, \ldots, m - 1 \), then \( I = I([1 : b]_₂) \). If there is \( j ≥ 2 \) such that \( x² z \in I, x⁻¹ z \notin I \), then \( I = I([1 : 0]₂) \). If \( x z \in I \), then \( I = I([a : b]₁₂) \). If \( xᵐ z \notin I \), then \( yᵐ⁺¹, xᵐ⁺¹ \in I \) so that \( I = I([0 : 1]₂ m) \).

3.6. Proof of Theorem 3.1 — the symmetry group \( G = S₄ \). Let

\[
G = \langle σ = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle.
\]

We define

\[
f₁ = x y z, \ f₂ = x² + y² + z², \ f₃ = x⁴ + y⁴ + z⁴,
f₄ = (x² - y²)(y² - z²)(z² - x²).
\]
Then \( \{f_1^2, f_2, f_3, f_1f_4\} \) is a system of generators of \( S^G \) which satisfies
\[
4(f_1f_4)^2 + 108f_1^6 - 20f_1^4f_2^3 + 36f_1^4f_2f_3 + f_1^2f_2^6 - 4f_1f_2^4f_3 + 5f_1^2f_2^2f_3^2 - 2f_1^2f_3^2 = 0.
\]

The following is the character table of \( G \).

<table>
<thead>
<tr>
<th>c.c.</th>
<th>( \sigma^2 )</th>
<th>( \tau )</th>
<th>( \sigma )</th>
<th>( \sigma\tau\sigma^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>age</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>6</td>
</tr>
<tr>
<td>1_1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1_2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>3_1</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3_2</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Table 8.** Characters of \( S_4 \)

The decomposition of the coinvariant algebra \( S_G \) into irreducible components is given in Table 9 where
\[
g = x^2 + \omega y^2 + \omega^2 z^2, \quad \bar{g} = x^2 + \omega^2 y^2 + \omega z^2, \quad \omega = e^{2\pi \sqrt{-1}/3}.
\]

<table>
<thead>
<tr>
<th>( d )</th>
<th>( (S_G)_d )</th>
<th>irred. factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \langle x, y, z \rangle )</td>
<td>3_1</td>
</tr>
<tr>
<td>2</td>
<td>( \langle g, \bar{g} \rangle \oplus \langle yz, zx, xy \rangle )</td>
<td>2 + 3_2</td>
</tr>
<tr>
<td>3</td>
<td>( \langle f_1 \rangle \oplus \langle x^3, y^3, z^3 \rangle \oplus \langle (y^2 - z^2)x, (z^2 - x^2)y, (x^2 - y^2)z \rangle )</td>
<td>1_2 + 3_1 + 3_2</td>
</tr>
<tr>
<td>4</td>
<td>( \langle g^2, y^3 \rangle \oplus \langle (y^2 - z^2)yz, (z^2 - x^2)zx, (x^2 - y^2)xy \rangle \oplus \langle f_1x, f_1y, f_1z \rangle )</td>
<td>2 + 3_1 + 3_2</td>
</tr>
<tr>
<td>5</td>
<td>( \langle f_1g, -f_1\bar{g} \rangle \oplus \langle f_1yz, f_1zx, f_1xy \rangle \oplus \langle (y^2 - z^2)x^3, (z^2 - x^2)y^3, (x^2 - y^2)z^3 \rangle )</td>
<td>2 + 3_1 + 3_2</td>
</tr>
<tr>
<td>6</td>
<td>( \langle f_4 \rangle \oplus \langle f_1(y^2 - z^2)x, f_1(z^2 - x^2)y, f_1(x^2 - y^2)z \rangle \oplus \langle f_1x^3, f_1y^3, f_1z^3 \rangle )</td>
<td>1_2 + 3_1 + 3_2</td>
</tr>
<tr>
<td>7</td>
<td>( \langle f_1g^2, -f_1\bar{g}^2 \rangle \oplus \langle f_1(y^2 - z^2)yz, f_1(z^2 - x^2)zx, f_1(x^2 - y^2)xy \rangle \oplus \langle f_1x^3, f_1y^3, f_1z^3 \rangle )</td>
<td>2 + 3_2</td>
</tr>
<tr>
<td>8</td>
<td>( \langle f_1(y^2 - z^2)x^3, f_1(z^2 - x^2)y^3, f_1(x^2 - y^2)z^3 \rangle )</td>
<td>3_1</td>
</tr>
</tbody>
</table>

**Table 9.** The coinvariant algebra of \( S_4 \)
We define

\[
\begin{align*}
I([a : b]_{12}) &= S \cdot (af_1 + bf_4, f_1x, f_1y, f_1z) + n, \\
I([a : b]_{23}) &= S[G] \cdot (ag^2 + bf_1g, (y^2 - z^2)x^3, f_1yz) + n, \\
I([a : b]_{31}) &= S[G] \cdot (a(y^2 - z^2)yz + bf_1yz, f_1g, (y^2 - z^2)x^3) + n. \\
I([a : b]_{32}) &= S[G] \cdot (af_1x + b(y^2 - z^2)x^3, f_1g, f_1yz) + n.
\end{align*}
\]

Let \( \overline{S}_d = (S_G)_d \), the degree \( d \) part of \( S_G \). Let \( I \in \text{Hilb}^G(\mathbb{C}^3) \) such that \( n \subset I \subset m \). First we note by using the quiver diagram of \( S_4 \) as before that \( I \) does not contain the elements whose projections to \( \overline{S}_1 \oplus \overline{S}_2 \) (the degree one and two parts of \( S_G \)) are nonzero. We note also that \( I \) contains \( \overline{S}_7 \oplus \overline{S}_8 \).

Assume that \( I \) contains an element \( af_1 + bf_4 \) for \( a \neq 0 \). Then by the quiver diagram of \( S_4 \), we see easily that \( I = I([a : b]_{12}) \).

Now we consider the case \( I \) contains no element \( af_1 + bf_4 \) for \( a \neq 0 \). Since \( S_G/I = C[G] \), \( f_4 \in I \), that is \( \overline{S}_6(12) \subset I \). If \( I \) contains an element \( af_1x + b(y^2 - z^2)x^3 \) for \( a \neq 0 \), then \( I = I([a : b]_{32}) \). If \( I \) contains an element \( a(y^2 - z^2)yz + bf_1yz \) for \( a \neq 0 \), then \( I = I([a : b]_{31}) \).

Now we consider the remaining cases. By the quiver diagram of \( S_4 \), we see \( \overline{S}_5(31) \oplus \overline{S}_5(32) \subset I \) and \( \overline{S}_6 \subset I \). If \( ag^2 + bf_1g \in I \) for \( a \neq 0 \), then \( I = I([a : b]_{23}) \). If \( I \) contains no element \( ag^2 + bf_1g \) for \( a \neq 0 \), then \( f_1g \in \overline{S}_5(2) \subset I \) because \( I \) contains no elements without nonzero projections to \( \overline{S}_1 \oplus \overline{S}_2 \). Hence \( \overline{S}_5 \subset I \), and \( I = I([0 : 1]_{2}) = I([0 : 1]_{31}) = I([0 : 1]_{32}) \).

The following exhaust all the possible cases of coincidence between \( I([a : b]_\rho) \).

\[
\begin{align*}
I([0 : 1]_{12}) &= I([1 : 0]_{31}), \\
I([0 : 1]_{2}) &= I([0 : 1]_{31}) = I([0 : 1]_{32}).
\end{align*}
\]

This completes the proof of Theorem 3.1.

3.7. Quiver diagrams. The following diagrams are drawn in the same manner as in [GNS00]. They express the quiver structure of \( S_G \), that is the decomposition of \( S_1 \cdot ((S_G)_d)_\rho \). The rows are indexed by degrees and the columns by irreducible representations. Each irreducible factor \( \rho_j \) of \( (S_G)_d \) has multiplicity one except when \( G = D_{4m+2} \), \( d = m + 1 \), \( \rho_j = 2m \) and \( (S_G)_{m+1} = (g^{m+1}, g^{m+1}) \oplus \langle x^m z, -y^m z \rangle = 2 \cdot 2m \). Each vertex in the diagram stands for nonzero \( ((S_G)_d)_\rho \) and we join \( ((S_G)_d)_\rho \) and \( ((S_G)_{d+1})_{\rho_k} \) with an edge when nonzero \( ((S_G)_{d+1})_{\rho_k} \) appears in \( S_1 \cdot ((S_G)_d)_\rho \). In the unique exceptional case where \( G = D_{4m+2} \), the diagram shows

\[
S_1 \cdot ((S_G)_m)_{2m-1} = \langle x^m z, -y^m z \rangle, \quad S_1 \cdot ((S_G)_m)_{2m} = (S_G)_{m+1}.
\]
Diagram $D_{2\ell}$

Diagram $S_4$
4. The Molien series $P_{SG, \rho_j}$ — the case (E)

Finite subgroups of $\mathrm{SL}(3, \mathbb{C})$ are classified in [Blichfeldt17]. With the notation in [YY93], there are exactly 4 infinite series labeled by (A), (B), (C), (D), and 8 exceptional cases labeled by (E) through (L). Homogeneous generators of the invariant rings for the exceptional 8 groups, together with explicit descriptions of these groups, are given in [YY93], which we shall follow. Since the character tables of these groups can be obtained by using, for example, GAP, we omitted them; instead we give short descriptions of irreducible characters. In what follows we denote by $I_0$ the trivial character (or representation) of $G$.

In this and the next section we calculate $P_{R, \rho_j}$ and $P_{SG, \rho_j}$ explicitly for (E)-(L). See also [GNS00]. In this section we discuss the case (E) in some detail as a prototype for all the other cases. In what follows in order to save space we will not explain the customary notation.

Let

$$G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \rangle, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$V = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = e^{2\pi i/3}$. Then we have $|G| = 108$, and

$$\hat{G} = \{1, e^{\pi i/3}, e^{2\pi i/3}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}, e^{\pi i}\},$$

where $1_1(V) = \sqrt{-1}$, $1_2 = 1_2^2$, $1_3 = 1_3^2$, $1_1 = \rho$, $1_2 = 1_1^2$, $1_3 = 1_2^2$, $3_1 = \rho$, $3_2 = 1_1^2\rho$, $3_3 = 1_2^2\rho$, $3_4 = 1_3^2\rho$, $3_5 = \rho^3$, $3_6 = 1_1^3\rho$, $3_7 = 1_2^3\rho$, $3_8 = 1_3^3\rho$, $4_1(T) = 1$ and $4_2(T) = -2$.

The decompositions of $\rho_i \otimes \rho$ are given in Appendix.

We also have $S^G = C[f_1, f_2, f_3, f_4, f_5]$ with $\deg f_1 = 6$, $\deg f_2 = 6$, $\deg f_3 = 12$, $\deg f_4 = 12$, and $\deg f_5 = 9$.

Put $R = S/(f_1, f_2, f_3)$. Then we can easily compute $P_{R, \rho_j}(t)$ by applying Proposition 1.3. Thus we see $R$ splits into irreducible representations as in Table 10.

Next we calculate the Molien series $P_{SG}(t)$ by the repeated use of the trivial relation $(n)_i = V^\vee \cdot (n)_{i-1} + (S^G)_i$ for any $i$. In the case (E) we need to compute only for $i \leq 21$. What we do is not more than elementary linear algebra, so we omit the details of the computation. We see

$$P_R(t) = \frac{(1 - t^6)^2(1 - t^{12})}{(1 - t)^3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 + 33t^8 + 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} + 26t^{15} + 21t^{16} + 15t^{17} + 10t^{18} + 6t^{19} + 3t^{20} + t^{21},$$

$$P_{SG}(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 + 33t^8 + 34t^9 + 33t^{10} + 30t^{11} + 24t^{12} + 15t^{13} + 6t^{14}.$$

Then in view of Proposition 1.3 we can compute the Molien series $P_{SG, \rho_j}(t)$. Summarizing the computation we see $S_G$ splits as in Table 11.

\[1\text{Since our results use the results in [YY93], we mention here some of their misprints: page 34, line 1, $\sqrt[3]{5}$ should be $\sqrt[6]{5}$; page 80, line 2, $(15 + 5\sqrt{15})y^3z^3$ should be $(15 + 5\sqrt{15})y^3z^3$.}\]
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Table 10. The decomposition of $R$ of type (E)

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Table 11. The decomposition of $S_G$ of type (E)
In other words,
\[
P_{S_6,1_0}(t) = 1,
\]
\[
P_{S_6,1_1}(t) = t^3 + t^9,
\]
\[
P_{S_6,1_2}(t) = 2t^6,
\]
\[
P_{S_6,1_3}(t) = t^3 + t^9,
\]
\[
P_{S_6,3_1}(t) = 3t^5 + 3t^8 + 2t^{11},
\]
\[
P_{S_6,3_2}(t) = t^2 + t^5 + 3t^8 + 2t^{11},
\]
\[
P_{S_6,3_3}(t) = 2t^5 + 2t^8 + 4t^{11} + 2t^{14},
\]
\[
P_{S_6,3_4}(t) = t^2 + t^5 + 3t^8 + 2t^{11},
\]
\[
P_{S_6,3_5}(t) = t + t^4 + 2t^7 + t^{10} + t^{13},
\]
\[
P_{S_6,3_6}(t) = 2t^4 + 2t^7 + 3t^{10} + t^{13},
\]
\[
P_{S_6,3_7}(t) = 4t^7 + 4t^{10} + 2t^{13},
\]
\[
P_{S_6,3_8}(t) = 2t^4 + 2t^7 + 3t^{10} + t^{13},
\]
\[
P_{S_6,4_1}(t) = t^3 + 3t^6 + 4t^9 + 3t^{12},
\]
\[
P_{S_6,4_2}(t) = t^3 + 3t^6 + 4t^9 + 3t^{12}.
\]

As a consequence we see
\[
P_{S_6,\rho_j}(t) = [(1 - t^9)(1 - t^{12})P_{R,\rho_j}(t)]_+
\]
where \([f(t)]_+ = \sum_{d=0}^{21} \max\{a_d, 0\} t^d\) for \(f(t) = \sum a_d t^d \in \mathbb{Z}[t]\). Note that this formula does not imply a similar formula for \(\rho_{S_6}\).

5. The Molien series \(P_{S_6,\rho_j}\)

In this section we report the results for the other types (F)-(L). For the sake of reader’s convenience we list the decompositions of \(\rho_i \otimes \rho\) in Appendix.

5.1. The group of type (F).

\[
G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & \omega \end{pmatrix}, \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \omega^2 \\ \omega & \omega^2 & \omega \end{pmatrix} \rangle,
\]
where \(\omega = e^{2\pi i/3}\).

\(|G| = 216.\)

\(\hat{G} = \{1_0, 1_1, 1_2, 1_3, 2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, 6_1, 6_2, 8\}\).

where \(1_3 = 1_1 1_2, 3_1 = \rho, 3_2 = 1_1 \rho, 3_3 = 1_2 \rho, 3_4 = 1_3 \rho, 3_5 = \rho^\vee, 3_6 = 1_1 \rho^\vee, 3_7 = 1_2 \rho^\vee, 3_8 = 1_3 \rho^\vee, 6_1 = \rho^2 - \rho^\vee, 6_2 = \rho^{\vee 2} - \rho.\)

\(S^G = \mathbb{C}[f_1, f_2, f_3, f_4],\)

with \(\deg f_1 = 6, \deg f_2 = 9, \deg f_3 = 12, \deg f_4 = 12.\)

Let \(R = S/(f_1, f_2, f_3).\) Then we have
\[
P_{R,1_0}(t) = 1 + t^{12} + t^{24},
\]
\[
P_{R,1_1}(t) = P_{R,1_2} = P_{R,1_3} = t^6 + t^{12} + t^{18},
\]
The group of type \((G)\).

\[
\begin{align*}
P_{R,2}(t) &= t^3 + 2t^9 + 2t^{15} + t^{21}, \\
P_{R,31}(t) &= 2t^5 + 2t^8 + 2t^{11} + t^{17} + t^{20} + t^{23}, \\
P_{R,32}(t) &= P_{R,33} = P_{R,34} = t^5 + t^8 + 3t^{11} + 2t^{14} + 2t^{17}, \\
P_{R,33}(t) &= t + t^4 + t^7 + 2t^{13} + 2t^{16} + 2t^{19}, \\
P_{R,34}(t) &= P_{R,37} = P_{R,38} = 2t^7 + 2t^{10} + 3t^{13} + t^{16} + t^{19}, \\
P_{R,61}(t) &= 2t^4 + 2t^7 + 5t^{10} + 3t^{13} + 4t^{16} + t^{19} + t^{22}, \\
P_{R,62}(t) &= t^2 + t^5 + 4t^8 + 3t^{11} + 5t^{14} + 2t^{17} + 2t^{20}, \\
P_{R,8}(t) &= t^3 + 3t^6 + 5t^9 + 6t^{12} + 5t^{15} + 3t^{18} + t^{21}. 
\end{align*}
\]

We see
\[
P_{SG}(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 44t^9 + 48t^{10} \\
+ 51t^{11} + 51t^{12} + 48t^{13} + 42t^{14} + 34t^{15} + 24t^{16} + 15t^{17} + 8t^{18} + 3t^{19},
\]

Hence we have
\[
\begin{align*}
P_{SG,10}(t) &= 1, \\
P_{SG,11}(t) &= P_{SG,12} = P_{SG,13} = t^6 + t^{12}, \\
P_{SG,2}(t) &= t^3 + 2t^9 + t^{15}, \\
P_{SG,31}(t) &= 2t^5 + 2t^8 + 2t^{11}, \\
P_{SG,32}(t) &= P_{SG,33} = P_{SG,34} = t^5 + t^8 + 3t^{11} + 2t^{14} + t^{17}, \\
P_{SG,33}(t) &= t + t^4 + t^7 + t^{13} + t^{16} + t^{19}, \\
P_{SG,34}(t) &= P_{SG,37} = P_{SG,38} = 2t^7 + 2t^{10} + 3t^{13} + t^{16}, \\
P_{SG,61}(t) &= 2t^4 + 2t^7 + 5t^{10} + 3t^{13} + 2t^{16}, \\
P_{SG,62}(t) &= t^2 + t^5 + 4t^8 + 3t^{11} + 4t^{14} + t^{17}, \\
P_{SG,8}(t) &= t^3 + 3t^6 + 5t^9 + 6t^{12} + 4t^{15} + t^{18}. 
\end{align*}
\]

5.2. The group of type \((G)\).

\[
G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle, U = \left( \varepsilon^2 \ 0 \ 0 \\
0 \ \varepsilon^2 \ 0 \\
0 \ 0 \ \varepsilon^6 \right), \frac{1}{\sqrt{-3}} \left( \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \right),
\]

where where \(\varepsilon = e^{2\pi i/9}, \omega = e^{2\pi i/3}\).

\(|G| = 648\).

\(\hat{G} = \{1, 11, 12, 21, 22, 23, 31, 32, 33, 34, 35, 36, 37, 61, 62, 63, 64, 65, 66, 81, 82, 83, 91, 92\}\),

where \(2_1\) and \(3_7\) are rational valued characters and \(1_1(U) = \omega, 1_2 = 1_1^2, 2_2 = 1_1 2_1, \ 2_3 = 1_2 2_1, 3_1 = \rho, 3_2 = 1_1 3_1, 3_3 = 1_2 3_1, 3_4 = \rho^\gamma, 3_5 = 1_1 3_4, 3_6 = 1_2 3_4, 6_1 = \rho^2 - \rho, \ 6_2 = 1_1 6_1, 6_3 = 1_2 6_1, 6_4 = \rho^{12} - \rho, 6_5 = 1_1 6_4, 6_6 = 1_2 6_4, 8_1 = \rho \rho^\gamma - 1_0, 8_2 = 1_1 8_1, \ 8_3 = 1_2 8_1, 9_1 = 3_7 \rho, 9_2 = 3_7 \rho^\gamma\).

\(S^G = \mathbb{C}[f_1, f_2, f_3, f_4]\),
with \( \deg f_1 = 9, \deg f_2 = 12, \deg f_3 = 18, \deg f_4 = 18 \).

\[ R = S/(f_1, f_2, f_3). \]

Then we have

\[
\begin{align*}
P_{R,1e}(t) &= 1 + t^{18} + t^{36}, \\
P_{R,1}(t) &= 2t^{12} + t^{30}, \\
P_{R,12}(t) &= t^6 + 2t^{24}, \\
P_{R,21}(t) &= 3t^{15} + 3t^{21}, \\
P_{R,22}(t) &= 2t^9 + 2t^{15} + t^{27} + t^{33}, \\
P_{R,23}(t) &= t^3 + t^9 + 2t^{21} + 2t^{27}, \\
P_{R,31}(t) &= t^8 + 2t^{11} + 3t^{17} + t^{20} + t^{26} + t^{35}, \\
P_{R,32}(t) &= t^5 + 2t^{11} + t^{14} + 2t^{20} + t^{23} + 2t^{29}, \\
P_{R,33}(t) &= t + t^{10} + t^{16} + 3t^{19} + 2t^{25} + t^{28}, \\
P_{R,34}(t) &= 2t^7 + t^{13} + 2t^{16} + t^{22} + 2t^{25} + t^{31}, \\
P_{R,35}(t) &= t^4 + 3t^{13} + 2t^{19} + t^{22} + t^{28} + t^{31}, \\
P_{R,36}(t) &= t^6 + 2t^{12} + 3t^{18} + 2t^{24} + t^{30}, \\
P_{R,61}(t) &= t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 5t^{22} + t^{25} + t^{28} + t^{31}, \\
P_{R,62}(t) &= t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 3t^{22} + t^{25} + 3t^{28}, \\
P_{R,63}(t) &= t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + 2t^{25} + t^{28} + t^{34}, \\
P_{R,64}(t) &= t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + 2t^{23} + t^{26} + t^{29} + t^{32}, \\
P_{R,65}(t) &= t^2 + t^8 + 2t^{11} + 2t^{14} + 2t^{17} + 5t^{20} + t^{23} + 3t^{26} + t^{29}, \\
P_{R,66}(t) &= 3t^8 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23} + 3t^{26} + t^{32}, \\
P_{R,81}(t) &= 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24} + 3t^{27}, \\
P_{R,82}(t) &= t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 5t^{21} + 2t^{24} + 2t^{27} + 2t^{30}, \\
P_{R,83}(t) &= 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 4t^{24} + t^{27} + t^{30} + t^{33}, \\
P_{R,91}(t) &= t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 5t^{23} + 2t^{26} + 2t^{29}, \\
P_{R,92}(t) &= 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 4t^{25} + t^{28} + t^{31}. \\
\end{align*}
\]

We see

\[
\begin{align*}
P_{S_G}(t) &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + 45t^8 + 54t^9 \\
&\quad + 63t^{10} + 72t^{11} + 80t^{12} + 87t^{13} + 93t^{14} + 98t^{15} + 102t^{16} \\
&\quad + 105t^{17} + 105t^{18} + 102t^{19} + 96t^{20} + 88t^{21} + 78t^{22} \\
&\quad + 66t^{23} + 52t^{24} + 36t^{25} + 21t^{26} + 10t^{27} + 3t^{28}, \quad \text{and} \quad P_{S_0,10}(t) = 1, \\
P_{S_0,11}(t) &= 2t^{12},
\end{align*}
\]
5.3. The group of type \((H)\).

\[
G = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & s & t \\ 2 & t & s \end{pmatrix} \right\rangle,
\]

where \(\varepsilon = e^{2\pi i/5}, \omega = e^{2\pi i/3}, s = \varepsilon^2 + \varepsilon^3\) and \(t = \varepsilon + \varepsilon^5\).

\(|G| = 60\).

\(\hat{G} = \{10, 3_1 = \rho = \rho^\times, 3_2, 4, 5\}\).

Let \(G\) be a group generated by \(G\) and \(-I\) where \(I\) is the identity matrix of degree 3.

Then \(\hat{G}\) is a Coxeter group of type \(H_3\) and there exist three homogeneous invariants \(f_1, f_2, f_3\) with \(\deg f_1 = 2, \deg f_2 = 6, \deg f_3 = 10\) such that \(S^G = C[f_1, f_2, f_3]\) and \(S^\hat{G} = C[f_1, f_2, f_3, f_4]\) where \(f_4 = \text{Jac}(f_1, f_2, f_3)\). Hence we have

\[
P_{SG,10}(t) = t^6 + t^{24},
\]
\[
P_{SG,21}(t) = 3t^{15} + 3t^{21},
\]
\[
P_{SG,22}(t) = 2t^9 + 2t^{15},
\]
\[
P_{SG,23}(t) = t^3 + t^9 + t^{21} + t^{27},
\]
\[
P_{SG,31}(t) = t^8 + 2t^{11} + 3t^{17} + t^{20},
\]
\[
P_{SG,32}(t) = t^5 + 2t^{11} + t^{14} + 2t^{20},
\]
\[
P_{SG,33}(t) = t^5 + t^8 + t^{14} + 2t^{17} + 2t^{23},
\]
\[
P_{SG,34}(t) = t + t^{10} + t^{16} + 2t^{19} + 2t^{25},
\]
\[
P_{SG,35}(t) = 2t^7 + t^{13} + 2t^{16} + t^{22},
\]
\[
P_{SG,36}(t) = t^4 + 3t^{13} + 2t^{19} + t^{28},
\]
\[
P_{SG,37}(t) = t^6 + 2t^{12} + 3t^{18} + t^{24},
\]
\[
P_{SG,61}(t) = t^4 + t^7 + t^{10} + 2t^{13} + 4t^{16} + t^{19} + 4t^{22},
\]
\[
P_{SG,62}(t) = t^4 + 3t^{10} + 2t^{13} + 2t^{16} + 3t^{19} + 2t^{22} + t^{25},
\]
\[
P_{SG,63}(t) = t^7 + 3t^{10} + t^{13} + 5t^{16} + 2t^{19} + 2t^{22} + t^{25},
\]
\[
P_{SG,64}(t) = t^5 + t^8 + t^{11} + 5t^{14} + t^{17} + 4t^{20} + t^{23},
\]
\[
P_{SG,65}(t) = t^2 + t^8 + 2t^{11} + 2t^{14} + 4t^{20} + t^{23} + 2t^{26},
\]
\[
P_{SG,66}(t) = 3t^5 + t^{11} + 3t^{14} + 3t^{17} + 2t^{20} + 2t^{23},
\]
\[
P_{SG,81}(t) = 3t^9 + 3t^{12} + 3t^{15} + 6t^{18} + 3t^{21} + 3t^{24},
\]
\[
P_{SG,82}(t) = t^3 + t^6 + t^9 + 4t^{12} + 3t^{15} + 3t^{18} + 4t^{21} + t^{24} + t^{27},
\]
\[
P_{SG,83}(t) = 2t^6 + 2t^9 + 2t^{12} + 5t^{15} + 3t^{18} + 3t^{21} + 2t^{24},
\]
\[
P_{SG,91}(t) = t^5 + t^8 + 4t^{11} + 3t^{14} + 6t^{17} + 3t^{20} + 4t^{23} + t^{26},
\]
\[
P_{SG,92}(t) = 2t^7 + 2t^{10} + 5t^{13} + 3t^{16} + 6t^{19} + 3t^{22} + 2t^{25}.
\]
where $P_{SG,3_2} = t + t^5 + t^6 + t^9 + t^{10} + t^{14}$,
$P_{SG,4} = t^3 + t^4 + t^6 + t^7 + t^8 + t^9 + t^{11} + t^{12}$,
$P_{SG,5} = t^2 + t^4 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{13}$.

5.4. The group of type (I).

\[ G = \langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^4 \end{pmatrix}, \frac{-1}{\sqrt{-7}} \begin{pmatrix} \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \\ \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \\ \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 \end{pmatrix} \rangle, \]

where $\varepsilon = e^{2\pi i/7}$.

$|G| = 168$.

$G = \{10, 3_1 = \rho, 3_2 = \rho^\vee, 6, 7, 8\}$.

Let $G$ be a group generated by $G$ and $-I$ where $I$ is the identity matrix of degree 3. Then $G$ is a complex reflection group of type $J_3(4)$ (c.f. [Cohen76]) and there exist three homogeneous invariants $f_1, f_2, f_3$ with deg $f_1 = 4, \text{deg} f_2 = 6, \text{deg} f_3 = 14$ such that $S^G = \mathbb{C}[f_1, f_2, f_3]$ and $S^G = \mathbb{C}[f_1, f_2, f_3, f_4]$ where $f_4 = \text{Jac}(f_1, f_2, f_3)$.

Hence we have

$P_{SG,1_0} = 1$,
$P_{SG,3_1} = t^3 + t^5 + t^{10} + t^{12} + t^{13} + t^{20}$,
$P_{SG,3_2} = t + t^8 + t^9 + t^{11} + t^{16} + t^{18}$,
$P_{SG,6} = t^2 + t^4 + t^6 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{15} + t^{17} + t^{19}$,
$P_{SG,7} = t^3 + t^5 + t^6 + t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + t^{14} + t^{15} + t^{16} + t^{18}$,
$P_{SG,8} = t^4 + t^5 + t^6 + 2t^7 + t^8 + t^9 + t^{10} + t^{11} + t^{12} + t^{13} + 2t^{14} + t^{15} + t^{16} + t^{17}$.

5.5. The group of type (J).

\[ G = \langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, W = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \rangle, \]

where $\varepsilon = e^{2\pi i/5}, \omega = e^{2\pi i/3}, s = \varepsilon^2 + \varepsilon^3$, and $t = \varepsilon + \varepsilon^4$.

$|G| = 180$.

$G = \{10, 1_1, 1_2, 3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 4_1, 4_2, 4_3, 5_1, 5_2, 5_3\}$

where $4_1$ and $5_1$ are rational valued characters and $l_1(W) = \omega, 1_2 = 1_1^2, 3_1 = \rho, 3_2 = \rho^\vee = 1_1 3_1, 3_3 = 1_2 3_1, 3_4(x) = 3_1(x^7), \forall x \in G, 3_5 = 1_1 3_4, 3_6 = 1_2 3_4, 4_2 = 1_1 4_1, 4_3 = 1_2 4_1, 5_2 = 1_1 5_1$ and $5_3 = 1_2 5_1$.

$S^G = \mathbb{C}[f_1, f_2, f_3, f_4]$, with deg $f_1 = 6, \text{deg} f_2 = 6, \text{deg} f_3 = 15, \text{deg} f_4 = 12$.

Put $R = S/(f_1, f_2, f_3)$. Then we have

$P_{R,1_0}(t) = 1 + t^{12} + t^{24}$,
$P_{R,1_1}(t) = t^2 + t^{14} + t^{20}$,
$P_{R,1_2}(t) = t^4 + t^{10} + t^{22}$,
$P_{R,3_1}(t) = 2t^5 + t^8 + 2t^{11} + 2t^{14} + t^{17} + t^{23}$,
P_{R,3}(t) = t + t^7 + 2t^{10} + 2t^{13} + t^{16} + 2t^{19},
\nonumber
P_{R,3}(t) = t^3 + t^6 + 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{21},
\nonumber
P_{R,3}(t) = 2t^5 + t^8 + t^{11} + 2t^{14} + 3t^{17},
\nonumber
P_{R,3}(t) = 3t^7 + 2t^{10} + t^{13} + t^{16} + 2t^{19},
\nonumber
P_{R,3}(t) = t^3 + 2t^9 + 3t^{12} + 2t^{15} + t^{21},
\nonumber
P_{R,4}(t) = t^3 + 2t^6 + 2t^9 + 2t^{12} + 2t^{15} + 2t^{18} + t^{21},
\nonumber
P_{R,4}(t) = t^5 + 3t^8 + 3t^{11} + 3t^{14} + 2t^{17} + t^{20},
\nonumber
P_{R,4}(t) = t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + t^{19},
\nonumber
P_{R,5}(t) = 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + 3t^{18},
\nonumber
P_{R,5}(t) = t^2 + t^5 + 3t^8 + 3t^{11} + 3t^{14} + 2t^{17} + 2t^{20},
\nonumber
P_{R,5}(t) = 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + 3t^{16} + t^{19} + t^{22}.
\nonumber

We see
\nonumber
P_{S_G}(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 26t^6 + 30t^7 + 33t^8
\nonumber
+ 35t^9 + 36t^{10} + 36t^{11} + 35t^{12} + 33t^{13} + 30t^{14} + 25t^{15}
\nonumber
+ 19t^{16} + 12t^{17} + 5t^{18} + 3t^{19} + t^{20}.
\nonumber

Hence we have
\nonumber
P_{S_G,1}(t) = 1,
\nonumber
P_{S_G,1}(t) = t^2 + t^{20},
\nonumber
P_{S_G,1}(t) = t^4 + t^{10},
\nonumber
P_{S_G,3}(t) = 2t^5 + t^8 + 2t^{11} + 2t^{14},
\nonumber
P_{S_G,3}(t) = t + t^7 + 2t^{10} + t^{13} + t^{16} + t^{19},
\nonumber
P_{S_G,3}(t) = t^3 + t^6 + 2t^9 + t^{12} + t^{15},
\nonumber
P_{S_G,3}(t) = 2t^5 + t^8 + t^{11} + 2t^{14} + t^{17},
\nonumber
P_{S_G,3}(t) = 3t^7 + 2t^{10} + t^{13} + t^{16},
\nonumber
P_{S_G,3}(t) = t^3 + 2t^9 + 3t^{12} + t^{15},
\nonumber
P_{S_G,4}(t) = t^3 + 2t^6 + 2t^9 + 2t^{12} + t^{15},
\nonumber
P_{S_G,4}(t) = t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},
\nonumber
P_{S_G,4}(t) = t^4 + 2t^7 + 2t^{10} + 3t^{13} + 2t^{16},
\nonumber
P_{S_G,5}(t) = 3t^6 + 3t^9 + 3t^{12} + 3t^{15} + t^{18},
\nonumber
P_{S_G,5}(t) = t^2 + t^5 + 3t^8 + 3t^{11} + 2t^{14} + t^{17},
\nonumber
P_{S_G,5}(t) = 2t^4 + 2t^7 + 3t^{10} + 3t^{13} + t^{16}.
5.6. The group of type (K).

\[ G = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
\varepsilon & 0 & 0 \\
0 & \varepsilon^2 & 0 \\
0 & 0 & \varepsilon^4
\end{pmatrix}, \frac{1}{\sqrt{-1}} \begin{pmatrix}
\varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 \\
\varepsilon^2 - \varepsilon^5 & \varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 \\
\varepsilon - \varepsilon^6 & \varepsilon^4 - \varepsilon^3 & \varepsilon^2 - \varepsilon^5
\end{pmatrix}, W = \begin{pmatrix}
\omega & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega
\end{pmatrix}, \]

where \( \varepsilon = e^{2\pi i/7} \) and \( \omega = e^{2\pi i/3} \).

\[ |G| = 504 \]

\[ \tilde{G} = \{ 10, 11, 12, 31, 32, 33, 34, 35, 36, 61, 62, 63, 71, 72, 73, 81, 82, 83 \}, \]

where \( 61, 71 \) and \( 81 \) are rational valued characters and \( 1(W) = \omega, 12 = 1_1^2, 31 = \rho, 32 = 131, 33 = 1231, 34 = \rho^7, 35 = 134, 36 = 1234, 62 = 161, 63 = 1261, 72 = 171, 73 = 171, 82 = 181 \) and \( 83 = 181 \).

\[ S^G = C[f_1, f_2, f_3, f_4], \]

with \( \deg f_1 = 6, \deg f_2 = 12, \deg f_3 = 21, \deg f_4 = 18 \).

Put \( R = S/(f_1, f_2, f_3) \). Then we have

\[ P_{R,10}(t) = 1 + t^{18} + t^{36}, \]
\[ P_{R,11}(t) = t^8 + t^{14} + t^{12}, \]
\[ P_{R,12}(t) = t^4 + t^{22} + t^{28}, \]
\[ P_{R,31}(t) = t^5 + t^{11} + t^{14} + 2t^{17} + 2t^{20} + t^{23} + t^{35}, \]
\[ P_{R,32}(t) = t^7 + t^{10} + 2t^{13} + t^{16} + t^{19} + t^{25} + t^{28} + t^{31}, \]
\[ P_{R,33}(t) = t^3 + t^9 + t^{12} + t^{18} + 2t^{21} + t^{24} + 2t^{27}, \]
\[ P_{R,34}(t) = t + t^{13} + 2t^{16} + 2t^{19} + t^{22} + t^{25} + t^{31}, \]
\[ P_{R,35}(t) = 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{24} + t^{27} + t^{33}, \]
\[ P_{R,36}(t) = t^5 + t^8 + t^{11} + t^{17} + t^{20} + 2t^{23} + t^{26} + t^{29}, \]
\[ P_{R,61}(t) = 2t^6 + t^9 + 3t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 3t^{24} + t^{27} + 2t^{30}, \]
\[ P_{R,62}(t) = t^2 + 2t^8 + t^{11} + 2t^{14} + 3t^{17} + 3t^{20} + 2t^{23} + 2t^{26} + t^{32}, \]
\[ P_{R,63}(t) = t^4 + 3t^{10} + 2t^{13} + 3t^{16} + 3t^{19} + 2t^{22} + t^{25} + 2t^{28} + t^{34}, \]
\[ P_{R,71}(t) = t^3 + t^6 + 2t^9 + 2t^{12} + 3t^{15} + 3t^{18} + 3t^{21} + 2t^{24} + 2t^{27} + t^{30} + 3t^{33}, \]
\[ P_{R,72}(t) = t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29}, \]
\[ P_{R,73}(t) = 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + t^{28} + t^{31}, \]
\[ P_{R,81}(t) = t^6 + 2t^9 + 3t^{12} + 4t^{15} + 4t^{18} + 4t^{21} + 3t^{24} + 2t^{27} + t^{30}, \]
\[ P_{R,82}(t) = t^5 + 2t^8 + 3t^{11} + 4t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 2t^{26} + 2t^{29} + t^{32}, \]
\[ P_{R,83}(t) = t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 4t^{22} + 3t^{25} + 2t^{28} + t^{31}. \]

We see

\[ P_{SG}(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 27t^6 + 33t^7 + 39t^8 + 45t^9 + 5t^{10} + 57t^{11} + 62t^{12} + 66t^{13} + 69t^{14} + 71t^{15} + 72t^{16} + 72t^{17} + 71t^{18} + 69t^{19} + 66t^{20} + 61t^{21} + 54t^{22} + 45t^{23} + 35t^{24} + 24t^{25} + 13t^{26} + 3t^{27} + t^{28}. \]
Hence we have

\[ P_{S_G,1_0}(t) = 1, \]
\[ P_{S_G,1_1}(t) = t^8 + t^{14}, \]
\[ P_{S_G,1_2}(t) = t^4 + t^{28}, \]
\[ P_{S_G,3_1}(t) = t^5 + t^{11} + t^{14} + 2t^{17} + 2t^{20}, \]
\[ P_{S_G,3_2}(t) = t^7 + t^{10} + 2t^{13} + t^{16} + t^{19}, \]
\[ P_{S_G,3_3}(t) = t^3 + t^9 + t^{12} + t^{18} + t^{21} + t^{24} + t^{27}, \]
\[ P_{S_G,3_4}(t) = t + t^{13} + 2t^{16} + t^{19} + t^{22} + t^{25}, \]
\[ P_{S_G,3_5}(t) = 2t^9 + t^{12} + 2t^{15} + t^{18} + t^{24}, \]
\[ P_{S_G,3_6}(t) = t^5 + t^8 + t^{11} + t^{17} + t^{20} + t^{23}, \]
\[ P_{S_G,6_1}(t) = 2t^6 + t^9 + 3t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + t^{24}, \]
\[ P_{S_G,6_2}(t) = t^2 + 2t^8 + t^{11} + 2t^{14} + 3t^{17} + 2t^{20} + 2t^{23} + t^{26}, \]
\[ P_{S_G,6_3}(t) = t^4 + 3t^{10} + 2t^{13} + 3t^{16} + 3t^{19} + t^{22} + t^{25}, \]
\[ P_{S_G,7_1}(t) = t^3 + t^6 + 2t^9 + 2t^{12} + 3t^{15} + 3t^{18} + 2t^{21} + t^{24}, \]
\[ P_{S_G,7_2}(t) = t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 2t^{23} + t^{26}, \]
\[ P_{S_G,7_3}(t) = 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + t^{25}, \]
\[ P_{S_G,8_1}(t) = t^6 + 2t^9 + 3t^{12} + 4t^{15} + 4t^{18} + 4t^{21} + 2t^{24}, \]
\[ P_{S_G,8_2}(t) = t^5 + 2t^8 + 3t^{11} + 4t^{14} + 3t^{17} + 3t^{20} + 2t^{23}, \]
\[ P_{S_G,8_3}(t) = t^4 + 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + t^{25}. \]

5.7. The group of type (L).

\[ G = \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & t & s \\ 2 & t & s \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \lambda_2 & s & t \\ 2 \lambda_2 & t & s \end{pmatrix} \right), \]

where \( \varepsilon = e^{2\pi i/5}, s = \varepsilon^2 + \varepsilon^3, t = \varepsilon + \varepsilon^4, \lambda_1 = -\frac{1+\sqrt{-15}}{4} \) and \( \lambda_2 = -\frac{1+\sqrt{-15}}{4} \).

\( |G| = 1080. \)

\( \tilde{G} = \{ 1, 3_1, 3_2, 3_3, 3_4, 5_1, 5_2, 6_1, 6_2, 8_1, 8_2, 9_1, 9_2, 9_3, 10, 15_1, 15_2 \}, \)

where \( 3_1 = \rho, 3_2 = \rho^y, 3_3(x) = 3_1(x^7) \) for all \( x \in G, 3_4(x) = 3_2(x^7) \) for all \( x \in G, \)

\( 6_1 = \rho^2 - \rho^y, 6_2 = \rho^{y^2} - \rho, 8_1 = 3_1 3_2 - 1_0, 8_2 = 3_3 3_4 - 1_0, 9_1 = 3_1 3_4, 9_2 = 3_1 3_3, 9_3 = 3_3 3_4, 15_1 = 3_1 5_1 \) and \( 15_2 = 3_2 5_1. \)

Let \( \tilde{G} \) be a group generated by \( G \) and \( -I \) where \( I \) is the identity matrix of degree 3.

Then \( \tilde{G} \) is a complex reflection group of type \( J_3(5) \) (c.f. [Cohen76]) and there exist three homogeneous invariants \( f_1, f_2, f_3 \) with \( \deg f_1 = 6, \deg f_2 = 12, \deg f_3 = 30 \) such that \( S^G = C[f_1, f_2, f_3] \) and \( S^G = C[f_1, f_2, f_3, f_4] \) where \( f_4 = \text{Jac}(f_1, f_2, f_3). \)

Hence we have

\[ P_{S_G,1_0} = 1, \]
\[ P_{S_G,3_1} = t^5 + t^{11} + t^{20} + t^{26} + t^{29} + t^{44}, \]
Let $d = 26$ YASUSHI GOMI, IKU NAKAMURA AND KEN-ICHI SHINODA

\[ P_{G, 3_1} = t + t^{16} + t^{19} + t^{25} + t^{34} + t^{40}, \]
\[ P_{G, 3_2} = t^5 + t^{17} + t^{20} + t^{23} + t^{32} + t^{38}, \]
\[ P_{G, 3_3} = t^7 + t^{13} + t^{22} + t^{25} + t^{28} + t^{40}, \]
\[ P_{G, 5_1} = t^6 + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39}, \]
\[ P_{G, 5_2} = t^6 + t^{12} + t^{15} + t^{18} + t^{21} + t^{24} + t^{27} + t^{30} + t^{33} + t^{39}, \]
\[ P_{G, 6_1} = t^4 + 2t^{10} + t^{16} + t^{19} + t^{22} + 2t^{25} + t^{28} + t^{31} + t^{37} + t^{43}, \]
\[ P_{G, 6_2} = t^2 + t^8 + t^{14} + t^{17} + 2t^{20} + t^{23} + t^{26} + t^{29} + 2t^{35} + t^{41}, \]
\[ P_{G, 8_1} = t^6 + t^9 + t^{12} + 2t^{15} + t^{18} + 2t^{21} + 2t^{24} + t^{27} + t^{30} + t^{33} + t^{36} + t^{39}, \]
\[ P_{G, 8_2} = t^9 + 2t^{12} + 2t^{15} + 2t^{18} + t^{21} + t^{24} + 2t^{27} + 2t^{30} + 2t^{33} + t^{36}, \]
\[ P_{G, 9_1} = t^6 + t^9 + t^{12} + t^{15} + 2t^{18} + t^{21} + 2t^{24} + 2t^{27} + t^{30} + 2t^{33} + t^{36} + t^{39}, \]
\[ P_{G, 9_2} = t^4 + t^{10} + 2t^{13} + 2t^{16} + 2t^{19} + 2t^{22} + 2t^{25} + 2t^{28} + 2t^{31} + t^{34} + 2t^{37}, \]
\[ P_{G, 9_3} = 2t^8 + t^{11} + 2t^{14} + t^{17} + t^{20} + 2t^{23} + 2t^{26} + 2t^{29} + 2t^{32} + t^{35} + t^{41}, \]
\[ P_{G, 10} = t^3 + 2t^9 + t^{12} + 2t^{15} + 2t^{18} + 2t^{21} + 2t^{24} + 2t^{27} + 2t^{30} + t^{33} + 2t^{36} + t^{42}, \]
\[ P_{G, 15_1} = t^5 + t^8 + 3t^{11} + 3t^{14} + 3t^{17} + 3t^{20} + 3t^{23} + 3t^{26} + 3t^{29} + 3t^{32} + 2t^{35} + 2t^{38}, \]
\[ P_{G, 15_2} = 2t^7 + 2t^{10} + 3t^{13} + 3t^{16} + 3t^{19} + 3t^{22} + 3t^{25} + 3t^{28} + 3t^{31} + 3t^{34} + t^{37} + t^{40}. \]

5.8. Summary. Here we list the invariants for the subgroups of type (E)-(L) where $d_{\text{max}} = d_1 + d_2 + d_3 - 3$:

| type | $d_1, d_2, d_3$ | $d_4, d_5$ | $d_{\text{max}}$ | $|G|$ | $e$ |
|------|-----------------|------------|------------------|------|----|
| $E$  | 6, 6, 12        | 12, 9      | 21               | 108  | 4  |
| $F$  | 6, 9, 12        | 12         | 24               | 216  | 3  |
| $G$  | 9, 12, 18       | 18         | 36               | 648  | 3  |
| $H$  | 2, 6, 10        | 15         | 15               | 60   | 2  |
| $I$  | 4, 6, 14        | 21         | 21               | 168  | 2  |
| $J$  | 6, 6, 15        | 12         | 24               | 180  | 3  |
| $K$  | 6, 12, 21       | 18         | 36               | 504  | 3  |
| $L$  | 6, 12, 30       | 45         | 45               | 1080 | 2  |

Table 12. Groups (E)-(L)

Summarizing the calculation in the previous subsections we infer

**Theorem 5.9.** Let $G$ be a subgroup of $\text{SL}(3, \mathbb{C})$ of type from (E) to (L). Let $f_i$ be generators of the invariant ring $S_G^G$ and $d_i = \deg f_i$ ($1 \leq i \leq n$), $d_{\text{max}} = d_1 + d_2 + d_3 - 3$ as in Table 12, and $S_G$ the coinvariant algebra. Then for any irreducible
representation \( \rho_j \) of \( G \) the Molien series \( P_{S_G, \rho_j} \) is given by the formula

\[
P_{S_G, \rho_j}(t) = \prod_{i=1}^{n}(1-t^{d_i})P_{R, \rho_j}(t) + \begin{cases} 
\begin{aligned}
t^{18}(\delta_{j,8} + \delta_{j,51}) & \quad \text{if } G = (F) \text{ or } (J), \\
0 & \quad \text{otherwise}.
\end{aligned}
\end{cases}
\]

where \([f(t)]_+ := \sum_{d=0}^{d_{\max}} \max\{a_d, 0\} t^d\) for \(f(t) = \sum a_d t^d \in \mathbb{Z}[t]\).

Remark 5.10. Theorem 5.9 implies the following. Suppose \( j \neq 8 \) if \( G \) is type (F) or \( j \neq 51 \) if \( G \) is of type (J). For any fixed irreducible representation \( \rho_j \) multiplication by \( f_{\alpha} (\alpha = 4, 5) \) is a homomorphism \( \phi_{d, \rho_j}^\alpha \) from \( (R_d)_{\rho_j} \) to \( (R_{d+a})_{\rho_j} \). Then \( \phi_{d, \rho_j}^\alpha \) is surjective if \( \dim(R_d)_{\rho_j} \geq \dim(R_{d+a})_{\rho_j} \), while it is injective if \( \dim(R_d)_{\rho_j} \leq \dim(R_{d+a})_{\rho_j} \). In other words, rank \( \phi_{d, \rho_j}^\alpha \) is equal to \( \min\{\dim(R_d)_{\rho_j}, \dim(R_{d+a})_{\rho_j}\} \). Moreover \( f_4 R \cap f_5 R = f_4 f_5 R \simeq f_4 f_5 \mathbb{C} \). In the exceptional case, for instance, of type (J) and \( j = 51 \), the nonzero coefficient of \( t^{19} \) in \( P_{S_G, 51} \) explains nonvanishing of the coefficient of \( t^{18} \) in \( P_{S_G, 51} \). We note that the above theorem does not imply \( P_{S_G}(t) = \prod_{i=4}^{n}(1-t^{d_i})P_{R}(t) \) even in the case other than (F) and (J).

6. Appendix

In this appendix we list the decompositions of irreducible representations tensored with the natural representation \( \rho \).

### 6.1. Type (E).

\[
\begin{align*}
1_0 \otimes \rho &= 3_1, & 1_1 \otimes \rho &= 3_2, \\
1_2 \otimes \rho &= 3_3, & 1_3 \otimes \rho &= 3_4, \\
3_1 \otimes \rho &= 5_3 + 3_6 + 3_8, & 3_2 \otimes \rho &= 3_5 + 3_6 + 3_7, \\
3_3 \otimes \rho &= 3_6 + 3_7 + 3_8, & 3_4 \otimes \rho &= 3_5 + 3_7 + 3_8, \\
3_5 \otimes \rho &= 1_0 + 4_1 + 4_2, & 3_6 \otimes \rho &= 1_1 + 4_1 + 4_2, \\
3_7 \otimes \rho &= 1_2 + 4_1 + 4_2, & 3_8 \otimes \rho &= 1_3 + 4_1 + 4_2, \\
4_1 \otimes \rho &= 3_1 + 3_2 + 3_3 + 3_4, & 4_2 \otimes \rho &= 3_1 + 3_2 + 3_3 + 3_4.
\end{align*}
\]

### 6.2. Type (F).

\[
\begin{align*}
1_0 \otimes \rho &= 3_1, & 1_1 \otimes \rho &= 3_2, \\
1_2 \otimes \rho &= 3_3, & 1_3 \otimes \rho &= 3_4, \\
2 \otimes \rho &= 6_2, & 3_1 \otimes \rho &= 3_5 + 6_1, \\
3_2 \otimes \rho &= 3_6 + 6_1, & 3_3 \otimes \rho &= 3_7 + 6_1, \\
3_4 \otimes \rho &= 3_8 + 6_1, & 3_5 \otimes \rho &= 1_0 + 8, \\
3_6 \otimes \rho &= 1_1 + 8, & 3_7 \otimes \rho &= 1_2 + 8, \\
3_8 \otimes \rho &= 1_3 + 8, & 6_1 \otimes \rho &= 2 + 2 \cdot 6_2, \\
6_2 \otimes \rho &= 3_5 + 3_6 + 3_7 + 3_8 + 6_1, & 8 \otimes \rho &= 3_1 + 3_2 + 3_3 + 3_4 + 2 \cdot 6_2.
\end{align*}
\]
6.3. Type (G).

\[\begin{align*}
1_0 \otimes \rho &= 3_1, & 1_1 \otimes \rho &= 3_2, \\
1_2 \otimes \rho &= 3_3, & 2_1 \otimes \rho &= 6_5, \\
2_2 \otimes \rho &= 6_6, & 2_3 \otimes \rho &= 6_4, \\
3_1 \otimes \rho &= 3_4 + 6_1, & 3_2 \otimes \rho &= 3_5 + 6_2, \\
3_3 \otimes \rho &= 3_6 + 6_3, & 3_4 \otimes \rho &= 1_0 + 8_1, \\
3_5 \otimes \rho &= 1_1 + 8_2, & 3_6 \otimes \rho &= 1_2 + 8_3, \\
3_7 \otimes \rho &= 9_1, & 6_1 \otimes \rho &= 2_2 + 8_1 + 8_3, \\
6_2 \otimes \rho &= 2_3 + 8_1 + 8_2, & 6_3 \otimes \rho &= 2_1 + 8_2 + 8_3, \\
6_4 \otimes \rho &= 3_4 + 6_2 + 9_2, & 6_5 \otimes \rho &= 3_5 + 6_3 + 9_2, \\
6_6 \otimes \rho &= 3_6 + 6_1 + 9_2, & 8_1 \otimes \rho &= 3_1 + 6_4 + 6_6 + 9_1, \\
8_2 \otimes \rho &= 3_2 + 6_4 + 6_5 + 9_1, & 8_3 \otimes \rho &= 3_3 + 6_5 + 6_6 + 9_1, \\
9_1 \otimes \rho &= 6_1 + 6_2 + 6_3 + 9_2, & 9_2 \otimes \rho &= 3_7 + 8_1 + 8_2 + 8_3.
\end{align*}\]

6.4. Type (H).

\[\begin{align*}
1_0 \otimes \rho &= 3_1, & 3_1 \otimes \rho &= 1_0 + 3_1 + 5, \\
3_2 \otimes \rho &= 4 + 5, & 4 \otimes \rho &= 3_2 + 4 + 5, \\
5 \otimes \rho &= 3_1 + 3_2 + 4 + 5.
\end{align*}\]

6.5. Type (I).

\[\begin{align*}
1_0 \otimes \rho &= 3_1, & 3_1 \otimes \rho &= 3_2 + 6, \\
3_2 \otimes \rho &= 1_0 + 8, & 6 \otimes \rho &= 3_2 + 7 + 8, \\
7 \otimes \rho &= 6 + 7 + 8, & 8 \otimes \rho &= 3_1 + 6 + 7 + 8.
\end{align*}\]

6.6. Type (J).

\[\begin{align*}
1_0 \otimes \rho &= 3_1, & 1_1 \otimes \rho &= 3_2, \\
1_2 \otimes \rho &= 3_3, & 3_1 \otimes \rho &= 1_2 + 3_2 + 5_3, \\
3_2 \otimes \rho &= 1_0 + 3_3 + 5_1, & 3_3 \otimes \rho &= 1_1 + 3_1 + 5_2, \\
3_4 \otimes \rho &= 4_3 + 5_3, & 3_5 \otimes \rho &= 4_1 + 5_1, \\
3_6 \otimes \rho &= 4_2 + 5_2, & 4_1 \otimes \rho &= 3_4 + 4_2 + 5_2, \\
4_2 \otimes \rho &= 3_5 + 4_3 + 5_3, & 4_3 \otimes \rho &= 3_6 + 4_1 + 5_1, \\
5_1 \otimes \rho &= 3_1 + 3_4 + 4_2 + 5_2, & 5_2 \otimes \rho &= 3_3 + 5_3 + 4_3 + 5_3, \\
5_3 \otimes \rho &= 3_3 + 3_6 + 4_1 + 5_1.
\end{align*}\]

6.7. Type (K).

\[\begin{align*}
1_0 \otimes \rho &= 3_1, & 1_1 \otimes \rho &= 3_2, \\
1_2 \otimes \rho &= 3_3, & 3_1 \otimes \rho &= 3_4 + 6_3, \\
3_2 \otimes \rho &= 3_5 + 6_1, & 3_3 \otimes \rho &= 3_6 + 6_2, \\
3_4 \otimes \rho &= 1_0 + 8_1, & 3_5 \otimes \rho &= 1_1 + 8_2, \\
3_6 \otimes \rho &= 1_2 + 8_3, & 6_1 \otimes \rho &= 3_6 + 7_2 + 8_2, \\
6_2 \otimes \rho &= 3_4 + 7_3 + 8_3, & 6_3 \otimes \rho &= 3_5 + 7_1 + 8_1, \\
7_1 \otimes \rho &= 6_2 + 7_2 + 8_2, & 7_2 \otimes \rho &= 6_3 + 7_3 + 8_3, \\
7_3 \otimes \rho &= 6_1 + 7_1 + 8_1, & 8_1 \otimes \rho &= 3_1 + 6_2 + 7_2 + 8_2, \\
8_2 \otimes \rho &= 3_2 + 6_3 + 7_3 + 8_3, & 8_3 \otimes \rho &= 3_3 + 6_1 + 7_1 + 8_1.
\end{align*}\]
6.8. Type (L).

\[
\begin{align*}
1_0 \otimes \rho &= 3_1, & 3_1 \otimes \rho &= 3_2 + 6_1, \\
3_2 \otimes \rho &= 1_0 + 8_1, & 3_3 \otimes \rho &= 3_2, \\
3_4 \otimes \rho &= 9_1, & 5_1 \otimes \rho &= 15_1, \\
5_2 \otimes \rho &= 15_1, & 6_1 \otimes \rho &= 8_1 + 10, \\
6_2 \otimes \rho &= 3_2 + 15_2, & 8_1 \otimes \rho &= 3_1 + 6_2 + 15_1, \\
8_2 \otimes \rho &= 9_3 + 15_1, & 9_1 \otimes \rho &= 3_3 + 9_3 + 15_1, \\
9_2 \otimes \rho &= 8_2 + 9_1 + 10, & 9_3 \otimes \rho &= 3_4 + 9_2 + 15_2, \\
10 \otimes \rho &= 6_2 + 9_3 + 15_1, & 15_1 \otimes \rho &= 6_1 + 9_2 + 2 \cdot 15_1, \\
15_2 \otimes \rho &= 5_1 + 5_2 + 8_1 + 8_2 + 9_1 + 10.
\end{align*}
\]

6.9. Addendum. In [GNS00, p.52, p.53] there are a few errors in notation and formulation, though harmless for the consequences in the subsequent sections of [GNS00]. As the arguments in this article are entirely independent from [GNS00] we would like to correct the errors in [GNS00] in the paper [GNS3] much closer to [GNS00].

We acknowledge Professor Li Chiang for pointing out the following errors in [GNS00] (different from the above) to us. The fourth line of [GNS00, p.57] must be replaced by

\[
f^3 + \bar{f}^3 = \prod_{i=0}^{2} (f + \omega^i \bar{f}) = 27f_3^2 - 9f_2f_4 + 2f_2^3.
\]

The fifth column of $S_d[\rho]$ of [GNS00, p.57, Table 2.2] must be replaced by

\[
\{\bar{f}^2\} + \{f^2\} + \{yzf, \omega^2zf, \omega xyf\}.
\]

References


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