SURVEY ON VII_0 SURFACES

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0. 7 classes of surfaces

The purpose of this talk is to give a brief survey on surfaces of class VII_0. This is the updated version of my survey in Sugaku Expositions in 1989.

As you know, algebraic surfaces were classified by Italian school, later by the Russian school of Shafarevich, and then by Kodaira. As everybody knows now, compact complex surfaces, in other words, compact complex manifold of dimension two, free from (-1)-curves, are classified into the following seven classes:

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I.	Ruled surfaces, namely, $\mathbf{P}^2$ or birational to $\mathbf{P}^1 \times C$, (C a curve)
II.	K3 surfaces
III.	Abelian surfaces
IV.	Kähler elliptic surfaces including Enriques surfaces
V.	Projective surfaces of general type, namely, $P_m = O(m^2)$.
VI.	NonKähler elliptic surfaces with $b_1 \geq 3$ odd
VII.	NonKähler elliptic surfaces with $b_1 = 1$

where the classes from I to V are Kähler, while the other are nonKähler.

We say that a compact complex surface is a VII_0-surface if $b_1 = 1$ and free from (-1)-curves.

1. Hopf surfaces

1.1. **Hopf surfaces.** In the study of VII_0-surfaces, the construction of examples was extremely important because there are few examples when Kodaira first defined the class of surfaces. He knows only Hopf surfaces and some elliptic surfaces other than Hopf surfaces.

A Hopf surface is a typical example of VII_0-surfaces. It is the quotient of $\mathbf{C}^2 \setminus O$(:the origin)/ infinite cyclic group action.

Let us define

$$g : (z_1, z_2) = (\alpha z_1 + a z_2^m, \beta z_2).$$

for $\alpha, \beta \in \mathbf{C}$, $0 < |\alpha| < 1, 0 < |\beta| < 1$. Then let us define G to be the infinite cyclic group G generated by g. Then the quotient of $\mathbf{C}^2 \setminus O$ by G is a compact complex surface with $b_1 = 1$, free from (-1)-curves.

This is called a primary Hopf surface.

It has an elliptic curve $z_2 = 0$.

Theorem 1.1.1. *Let S be a primary Hopf surface. Then the following are equivalent:*

1. *S has a nontrivial(=nonconstant) meromorphic functions,*
2. *$a = 0$ and $\alpha^p = \beta^q$ for some positive integers p and q.*

Theorem 1.1.2. *(Kodaira) Let S be a surface with $b_1 = 1$, $b_2 = 0$ having no meromorphic functions. Then the following are equivalent:*

1. *S has a curve, (which is easily proved to be an elliptic curve,)*
2. *There is a primary Hopf surface S' which is a finite unramified covering of S. (Then we call S a (not necessarily primary) Hopf surface.)*

Theorem 1.1.3. *(Kato, see* [N84]*) Let S be a surface with $b_1 = 1$ having no meromorphic functions. If S has precisely two elliptic curevs, the it is isomorphic to a primary Hopf surface.*

2. Inoue surfaces with $b_2 = 0$

2.1. **Construction.** Let M be a 3×3 integral matrix with $\det M = 1$. Let $\alpha, \beta, \overline{\beta}$ be eigenvalues of M with $\alpha > 0$, $\alpha \neq 1$ and $\operatorname{Im} \beta > 0$. Let $(a_1, a_2, a_3) \in \mathbf{R}^3$ and $(b_1, b_2, b_3) \in \mathbf{C}^3$ be eigenvectors of M corresponding to eigenvalues α, β. Let $\mathbf{H}$ be the upper half plane of dimension one and we define transformations of $\mathbf{H} \times \mathbf{C}$ by

$$\begin{aligned} g_0 &: (w,z) \mapsto (\alpha w, \beta z), \\ g_i &: (w,z) \mapsto (w + a_1, z + b_i) \quad (i = 1,2,3), \\ h_0 &: (w,z) \mapsto (\alpha w, \overline{\beta} z), \\ h_i &: (w,z) \mapsto (w + a_1, z + \overline{b_i}) \quad (i = 1,2,3). \end{aligned}$$

Let G^+_M be the group generated by g_j $(j = 0,1,2,3)$, and G^-_M the group generated by h_j $(j = 0,1,2,3)$. Both the groups are isomorphic as abstract groups, and they have respectively compact quotients of $\mathbf{H} \times \mathbf{C}$. Let $S^+_M = \mathbf{H} \times \mathbf{C}/G^+_M$ and $S^-_M = \mathbf{H} \times \mathbf{C}/G^-_M$.

Theorem 2.1.1. *(Inoue 1974) The surfaces* $S^+_M = \mathbf{H} \times \mathbf{C}/G^+_M$ *and* $S^-_M = \mathbf{H} \times \mathbf{C}/G^-_M$ *have the following properties:*

1. $b_1 = 1$, $b_2 = 0$,
2. *they have no curves,*
3. *they are diffeomorpic to each other, but they are not isomorphic.*

Suppose that S be a compact complex surface with $b_1 = 1, b_2 = 0$ having no meromorphic functions except constants. Then we note that $\mathrm{Pic}(S) := H^1(S, O^*_S) = \exp H^1(O_S) = H^1(S, \mathbf{C}^*)$, the set of all flat line bundles on S. Because

$$0 \to H^1(S,\mathbf{Z}) \to H^1(S,O_S)(= H^1(S,\mathbf{C})) \stackrel{\exp}{\to} H^1(S,O^*_S) \to H^2(S,\mathbf{Z}) \,(\text{exact}),$$

where $H^2(S,\mathbf{Z})$ is a torsion group. In other words, a certain power of any holomorphic line bundle on S is a flat line bundle.

Theorem 2.1.2. *(Inoue 1974) Let S be a compact complex surface with $b_1 = 1, b_2 = 0$ having no meromorphic functions except constants. If its tangent bundle is not stable in the sense that $H^0(S, T_S(F)) \neq 0$ for some line bundle F on S. (In other words, T_S has a nontrivial sublinebundle.) Then S is isomorphic to either S^+_M or S^-_M.*

Theorem 2.1.3. *(Li-Yau-Zheng 1994, Teleman 1994) Let S be a compact complex surface with $b_1 = 1, b_2 = 0$ having no meromorphic functions except constants. Then the tangent bundle T_S of S has a nontrivial sublinebundle.*

The second theorem is very important in the sense that it suggests that the stability or instability of the tangent bundle might be effective for classification of the surfaces with b_2 positive.

3. Inoue surfaces with b_2 positive

3.1. **Quadratic fields.** There are three kinds of Inoue surfaces, hyperbolic, half and parabolic. A hyperbolic (respectively, a half or a parabolic) Inoue surface has two cycles of rational curves (respectively, a unique cycle of rational curves, or an elliptic curve and a cycle of rational curves). Now let me recall the construction of hyperbolic Inoue surfaces.

Let K be a real quadratic field, M a complete module in K, namely a free $\mathbf{Z}$-module of rank two in K. As usual we define

$$U(M) = \{x \in K; xM = M, x > 0\},$$
$$U_+(M) = \{x \in K; xM = M, x > 0, x' > 0\},$$

where x' denotes the conjugate of x.

By Dirichlet's theorem the groups $U(M)$ and $U_+(M)$ are infinite cyclic, and $U_+(M)$ is a subgroup of $U(M)$ with $[U(M) : U_+(M)] = 1$ or 2.

For instance let $K = \mathbf{Q}(\omega)$, ω a quadratic irrationality, $M = \mathbf{Z} + \mathbf{Z}\omega$, $U(M) = \{\alpha^n; n \in \mathbf{Z}\}$ and $U_+(M) = \{\alpha_+^n; n \in \mathbf{Z}\}$, where $0 < \alpha < 1$, $0 < \alpha_+ < 1$.

For instance, if $\omega = \frac{(3+\sqrt{6})}{2}$, then $\alpha = \alpha_+ = 5 - 2\sqrt{6}$, $U(M) = U_+(M)$.

If $\omega = \frac{(3+\sqrt{5})}{2}$, then $\alpha = \alpha_+^2 = \frac{(-1+\sqrt{5})}{2}$ and $\alpha_+ = \frac{(3-\sqrt{5})}{2}$, $[U(M) : U_+(M)] = 2$.

3.2. **Hyperbolic Inoue surfaces.** Let V be a subgroup of $U_+(M)$ of finite index. Then by imitating a description of a Hilbert modular surface near one of the cusps, we can construct a new surface with $b_1 = 1$ as follows. For the pair (M, V), we define a group $G(M, V)$ of transformations of $\mathbf{H} \times \mathbf{C}$ as follows:

$$G(M, V) = \left\{ \begin{pmatrix} \varepsilon & m \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2, \mathbf{R}); \varepsilon \in V, m \in M \right\},$$
$$\begin{pmatrix} \varepsilon & m \\ 0 & 1 \end{pmatrix} : (z_1, z_2) \mapsto (\varepsilon z_1 + m, \varepsilon' z_2 + m').$$

The quotient space $Y'(M, V) = \mathbf{H} \times \mathbf{C}/G(M, V)$ is an open complex surface, which is compactified into a compact complex surface $Y(M, V)$ by adding two cusps ∞, ∞'. It is a normal surface, whose minimal resolution we denote by

$$\pi : S(M, V) \to Y(M, V).$$

The surface $S(M, V)$ is called a hyperbolic Inoue surface. However if we define the action

$$\begin{pmatrix} \varepsilon & m \\ 0 & 1 \end{pmatrix} : (z_1, z_2) \mapsto (\varepsilon' z_1 + m', \varepsilon z_2 + m),$$

then this produces another hyperbolic Inoue surface, the transposition ${}^tS(M, V)$ of $S(M, V)$ in the sense of Zaffran. It is quite recent through the discussion with Fujiki that I got aware of this fact. We note that ${}^tS(M, V) \simeq S(M', V')$, which is a source of the duality in Section 7. See also [N86].

This is notable because we have similar pairs S_M^+ and S_M^- in the $b_2 = 0$ cases.

Theorem 3.2.1. *(Inoue 1977) Let V be a subgroup of $U_+(M)$ of finite index. Then the surface $S(M, V)$ has the following properties:*

1. *$b_1 = 1$,*
2. *the inverse images $\pi^{-1}(\infty)$ and $\pi^{-1}(\infty')$ are respectively a cycle of rational curves*
3. *$b_2 = n + m$, where n and m are the numbers of (possibly singular) rational curves of $\pi^{-1}(\infty) \cup \pi^{-1}(\infty')$.*

Theorem 3.2.2. *(Nakamura 1984) Any VII_0 surface with two cycles of rational curves is isomorphic to a hyperbolic Inoue surface.*

3.3. **Half Inoue surfaces.** Assume $[U(M) : U_+(M)] = 2$. Then we choose an infinite cyclic subgroup V of $U(M)$ of odd index. Then $V^2 := \{\alpha^2; \alpha \in V\}$ is a subgroup of $U_+(M)$ and we have a hyperbolic Inoue surface $S(M, V^2)$, on which the group V/V^2 acts on $S(M, V^2)$ as a fixed-point free involution ι. We call the quotient $S(M, V^2)/V/V^2$ a half Inoue surface.

Theorem 3.3.1. *(Inoue 1977) The quotient $\hat{S}(M, V) = S(M, V^2)/\{1, \iota\}$ is a VII$_0$ surface, which has a unique cycle C of rational curves with*

$$C^2 = -b_2(\hat{S}(M, V)) = -b_2(C).$$

Theorem 3.3.2. *(Nakamura 1984) Let S be a VII$_0$ surface with C a cycle of rational curves. Assume one of the following:*

1. $C^2 = -b_2(S)$, *(which implies $b_2(S) > 0$,)*
2. $b_2(C) = b_2(S) > 0$,
3. $[H_1(S, \mathbf{Z}), H_1(C, \mathbf{Z})] \geq 2$.

Then S is isomorphic to a half Inoue surface.

3.4. **Parabolic Inoue surfaces.** See Section 4 below.

4. Enoki surfaces and the others

4.1. **Enoki surfaces.** Let X be a $\mathbf{P}^1$-bundle over an elliptic curve with an infinity section C_∞ (but possibly with no zero section) with $C_\infty^2 = -n$. Then the complement of C_∞ in X can be it uniquely compactified into a VII$_0$ surface S with $b_2(S) = n$ by replacing C_∞ by a cycle of n-rational curves. This is called *an Enoki surface*. If X has also the zero section, then S has an elliptic curve too. In the second case we call the surface **a parabolic Inoue surface**. Thus the Enoki surfaces have a cycle of rational curves and at most an elliptic curve.

Theorem 4.1.1. *(Enoki 1981) Suppose that a VII$_0$ surface S has C a cycle rational curve with $C^2 = 0$. Then S is isomorphic to an Enoki surface, in other words, the complement of C in S is an affine line bundle (in general with no zero section) over an elliptic curve.*

Theorem 4.1.2. *(Nakamura 1984) Any VII$_0$ surface with an elliptic curve E and C a cycle of rational curves is isomorphic to a parabolic Inoue surface.*

Corollary 4.1.3. *(Nakamura 1984) Let S be a VII$_0$ surface with an elliptic curve E. Then S is either an elliptic VII$_0$ surface, or a Hopf surface or a parabolic Inoue surface.*

4.2. **Surfaces with a cycle each.** I quote one more general fact which appears to be of some importance:

Theorem 4.2.1. *(Nakamura 1990) Suppose that a VII$_0$ surface S has a cycle of rational curves. Then there is a smooth family $\pi : X \to D$ of VII$_0$ surfaces over a disc such that $X_0 = S$ and X_t is a blown-up primary Hopf surface for any $t \neq 0$.*

In particular, S is homeomorphic to the connected sum of $S^1 \times S^3$ and some copies of $(-\mathbf{P}^2)$ (namely, $\mathbf{P}^2$ with reversed orientation).

Proof. Can show $H^2(S, T_S(-logC)) = 0$, which implies that C is deformed into a smooth elliptic curve by deforming S. Then any general fiber of the family smoothing C turns out to be a blown-up primary Hopf surface by using Theorem 4.1.3 □

Corollary 4.2.2. *Suppose that a VII$_0$ surface S has $b_2(S)$ (> 0) rational curves. Then there is a smooth family $\pi : X \to D$ over a disc such that $X_0 = S$ and X_t is a blown-up primary Hopf surface for any $t \neq 0$.*

5. Surfaces with a GSS each

5.1. **Flat deformations.** Let us consider a one-point blow-up Y of $\mathbf{P}^2$. The surface Y has disjoint two rational curves C_0 and C_1, with $C_0^2 = 1$ and $C_1^2 = -1$. The curve C_0 is (the total transform of) a line of $\mathbf{P}^2$, and the curve C_1 is the total transform of the center of blow-up.

We identify the curves to get a singular surface Z with a double curve. This surface can be deformed into a nonsingular surface. The flat family deforming Z into a smooth one was constructed by Kodaira (Amer. J. Math. 1968).

Theorem 5.1.1. *(Kodaira 1968, Oda 1978) There is a proper flat family $\pi : X \to D$ over one dimensional disc D such that*

1. *$X_0 = Z$, X_t $(t \neq 0)$ is nonsigular,*
2. *X_t $(t \neq 0)$ is a primary Hopf surface.*

5.2. **Kato's surfaces with GSS.** The construction due to Kodaira can be generalized to produce quite a lot of new VII$_0$ surfaces. This was done by Oda in some of the toric cases, and later by Nakamura in the general cases.

Theorem 5.2.1. *(Nakamura 1983) Suppose that Y is a rational surface having disjoint two rational curves C_0 and C_1, with $C_0^2 = 1$ and $C_1^2 = -1$. Let Z be a singular surface with a double curve obtained by identifying the curves C_i of Y. Then there is a proper flat family $\pi : X \to D$ over one dimensional disc D such that*

1. *$X_0 = Z$, X_t $(t \neq 0)$ is nonsigular,*
2. *X_t $(t \neq 0)$ is a VII$_0$ surface, which turns out to be a surface with GSS.*

Conversely any surface S with a GSS (which was discovered by Kato) is obtained this way.

The Enoki surfaces in Section 4 are obtained in the way of Theorem 5.2.1. For instance, when we choose a successive blow up of $\mathbf{P}^2$ having C_0 with $C_0^2 = 1$, and a chain of rational curves $C_1, C_2, \cdots, C_n$ with $C_1^2 = C_2 = \cdots = C_{n-1}^2 = -2$, $C_n^2 = -1$, then we have from one of the family of the above theorem a VII$_0$ surface with an elliptic curve and a cycle of rational curves. by identifying the rational curves C_0 and C_n carefully. The surface thus constructed is a parabolic Inoue surface.

If we twist the family a little by choosing generic identification of two rational curves C_0 and C_n, we will still have a family from Theorem 5.2.1 with general fiber an Enoki surface. It is a VII$_0$ surface S with a cycle C of rational curves such that $S - C$ is an affine line bundle over an elliptic curve.

Theorem 5.2.2. *(Kato,Nakamura,Dloussky)*

1. *Let S be a surface with a GSS. Then it has b_2 rational curves, which is in general positive, and can be arbitrarily large.*

2. *Enoki surfaces, hyperbolic, half and parabolic Inoue surfaces have a GSS each.*

Theorem 5.2.3. *(Dloussky-Oeljeklaus-Toma 2003) Suppose that a VII_0 surface S has $b_2(S)$ rational curves. Then S has a GSS, in other words, it is isomorphic to a general fiber of the family in Theorem 5.2.1.*

	curves	classification
1	at least 3 elliptic	elliptic VII_0 surfaces
2	exactly 2 elliptic	Hopf surfaces
3	elliptic + no cycle	Hopf surfaces
4	elliptic + a cycle	parabolic Inoue surfaces
5	2 cycles	hyperbolic Inoue surfaces
6	a cycle C with $C^2 = 0$	Enoki surfaces
7	a cycle C with $C^2 < 0$	
	(7-1) $b_2(S) = b_2(C)$	half Inoue surfaces
	(7-2) $b_2(S)$ curves	surfaces with GSS (D-O-T)
8	Otherwise	Teleman in progress

6. Surfaces with $b_2 = 1, 2$

6.1. **Surfaces with $b_2 = 1$.** Let me give some examples of VII_0 surfaces with small b_2. If $b_2 = 1$, then the examples are given as follows. First an Enoki surface with $n = 1$ enjoys the property. It is a VII_0 surface with a rational curve C with a node with $C^2 = 0$. This has a nontrivial two-dimensional moduli as Enoki surfaces , to be more precise, one modulus parametrizing the affine bundle structures, and one modulus of elliptic curves. It can have an elliptic curve E in addition, which is then a parabolic Inoue surface. If we choose $\omega = \frac{(3+\sqrt{5})}{2}$, and $M = \mathbf{Z} + \mathbf{Z}\omega$, then as we saw in Section 3,

$$U(M) = \{\alpha^n; n \in \mathbf{Z}\}, U_+(M) = \{\alpha_+^n; n \in \mathbf{Z}\}, [U(M) : U_+(M)] = 2$$

where $\alpha = \frac{(-1+\sqrt{5})}{2}$ and $\alpha_+ = \frac{(3-\sqrt{5})}{2}$. Let $S = S(M, U(M))$. Then S is a half Inoue surface with $b_2(S) = 1$ having a unique rational curve with a node. We see $C^2 = -b_2(S) = -1$. Thus there are only

1. Enoki surfaces with $n = 1$,
2. a parabolic Inoue surface with $b_2 = 1$,
3. a half Inoue surface $\hat{S}(M, U(M))$ where $M = \mathbf{Z} + \mathbf{Z}\frac{(3+\sqrt{5})}{2}$.

In other words, let C be a cycle of rational curves on S with $b_1 = b_2 = 1$. Then C is a rational curve with a node and $C^2 = 0$, or $C^2 = -1$. Moreover we can choose a homology basis e of S with $e^2 = -1$ so that

1. $K_S = e$ in any case,
2. either $C = 0$,
3. or $C = -e$.

where e comes from a (-1) curve on S_t in Theorem 4.2.1, whose reminiscent remains in the formula $K_S = e$.

This is the case Teleman is going to discuss. He proved

Theorem 6.1.1. *(Teleman 2005) Any VII_0 surface with $b_2 = 1$ has a rational curve (with a node).*

This completes the classification of VII_0 surfaces with $b_2 = 1$.

6.2. **Surfaces with $b_2 = 2$.** In this case, there are Enoki surfaces with $b_2 = 2$ and parabolic Inoue surfaces with $b_2 = 2$. Moreover, a hyperbolic Inoue surface $S(M, U_+(M))$ has $b_2 = 2$ where $M = \mathbf{Z} + \mathbf{Z}\frac{(3+\sqrt{5})}{2}$. Besides these surfaces there are VII_0 surfaces, each having a pair of curves C_1, C_2 with

$$C_1^2 = -1, C_2^2 = -2, C_1C_2 = 1$$

where C_1 is a rational curve with a node and C_2 a smooth rational curve.

In other words, we can choose a homology basis e_1, e_2 of S with $e_i^2 = -1, e_ie_j = 0 (i \neq j)$such that

1. $K_S = e_1 + e_2$ in any case,
2. either $C_1 = -e_1 + e_2, C_2 = e_2 - e_1,\underline{C = C_1 + C_2 = 0}$,
3. or $\underline{C_1 = -e_1}$, $\underline{C_2 = -e_2}$,
4. or $\underline{C_1 = -e_1}$, $C_2 = e_1 - e_2$

This will appear in the discussion of Teleman for $b_2 = 2$. He proved

Theorem 6.2.1. *(Teleman 2007) Any VII_0 surface with $b_2 = 2$ has a cycle of rational curves. Hence in particular, it can be deformed into a blown-up primary Hopf surface.*

7. Duality

7.1. **Cusp singularity.** Let us recall first that the hypersurface

$$T_{p,q,r} : x^p + y^q + z^r - xyz = 0$$

has an isolated singularity at the origin. Its minimal resolution has a cycle of rational curves as the exceptional set. It is as a germ the same as one of the cusps of a Hilbert modular surface.

7.2. **The pair $T_{3,4,4}$ and $T_{2,5,6}$.** Let me take a pair of hypersurfaces

$$T_{3,4,4},\ T_{2,5,6}.$$

We resolve the singularities to get cycles of rational curves

$$C_1 + C_2,\ D_1 + D_2 + D_3$$

with

$$C_1^2 = -3, C_2^2 = -4,\ D_1^2 = -2, D_2^2 = -3, D_3^2 = -3.$$

We note that $C = C_1 + C_2$ ad $D_1 + D_2 + D_3$ are cycles of rational curves.

Blowing up at one of the intersection point of C_1 and C_2 we get a cycle of three rational curves

$$C_0' + C_1' + C_2'$$

with

$$(C_0')^2 = -1, (C_1')^2 = -4, (C_2')^2 = -5.$$

We have a pair of triples

$$(1, 4, 5),\ (2, 3, 3).$$

Let us add $(1,1,1)$ to each to get

$$(2,5,6),\ (3,4,4),$$

which turn out to be the same as the triples interchanged.

This is the strange duality of the cusp singularities I discovered in the course of the study of hyperbolic Inoue surfaces. If we choose $\omega = \frac{(3+\sqrt{6})}{2}$, $M = \mathbf{Z} + \mathbf{Z}\omega$, and $V = U_+(M)$, then we have a hyperbolic Inoue surface $S = S(M, U_+(M))$. It has two cycles of rational curves, which are just C and D. They show the duality as above, and moreover, for instance,

$$C^2 = -3 = -b_2(D), D^2 = -2 = -b_2(C)$$

and so on.

7.3. **K3 surfaces.** Let Z be the above hyperbolic Inoue surface $S = S(M, U_+(M))$ with C and D contracted to one point each. This singular surface can be deformed into a smooth surface, which is a K3 surface. See also [Loojenga, Ann. Math.]

7.4. **Further details.** See the references of the references below for more details of all the topics of this survey.

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