The Gibbs-Thomson relation for anisotropic phase transitions

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1 Introduction and Main result

This is a joint work with Marco Cicalease in Università di Napoli Federico II and Giovanni Pisante in Seconda Università di Napoli.

In this talk we consider the Gibbs-Thomson relation between the coarse grained chemical potential and the non homogeneous and anisotropic mean curvature of a phase interface within the (non homogeneous and anisotropic) gradient theory of phase transitions thus proving a generalization of a conjecture stated by Gurtin and proved by Luckhaus and Modica in the isotropic case.

We consider the following energy functional

\[ E_{\varepsilon}(u_{\varepsilon}) = \int_{\Omega} \varepsilon f(x, Du_{\varepsilon}) + \frac{W(u_{\varepsilon})}{\varepsilon} \, dx, \]  

where \( W(s) : \mathbb{R} \to \mathbb{R} \) is a standard double-well potential which has exactly two zeros at \( \pm 1 \) and \( \varepsilon > 0 \) is a parameter which denote the order of the width of the phase interface. We assume the function \( f(x, p) \in C^2(\Omega \times (\mathbb{R}^n \setminus \{0\})) \), which denote non homogeneous anistropic surface tension, satisfies

\[ f(x, tp) = t^2 f(x, p) \quad \text{for} \quad x \in \Omega, \ p \in \mathbb{R}^n, \ t \in \mathbb{R} \]  

and

\[ c_1|p|^2 \leq f(x, p) \leq c_2|p|^2 \quad \text{for} \quad x \in \Omega, \ p \in \mathbb{R}^n, \]  

for positive constants \( 0 < c_1 \leq c_2 \). Also we assume \( f^2(x, p) \) is strictly convex with respect to \( p \in \mathbb{R}^n \).

We consider the variational problem of this energy under the volume constraint;

\[ \min \left\{ E_{\varepsilon}(u) \mid \int_{\Omega} u \, dx = m \right\} \]  

for \( -|\Omega| \leq m \leq |\Omega| \).
If we choose \( f(x, p) = |p|^2 \), that is, isotropic case, the energy functional is known as Ginzburg-Landau type and Modica proved that the energy functional converges to the minimal surface area with fixed volume in [8], that is,

\[
\int_{\Omega} \varepsilon |Du_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon} \, dx \to 2c_0 \mathcal{H}^{n-1}(\Omega \cap \partial\{u_0 = -1\}) \tag{5}
\]
as \( \varepsilon \to 0 \), where \( c_0 = \int_{-1}^{+1} \sqrt{W(s)} \, ds \).

More generally Bouchitté proved that our energy functional (1) \( \Gamma \)-converges to the generalized perimeter of the limit interface in [6],

\[
\int_{\Omega} \varepsilon f(x, Du_\varepsilon) + \frac{W(u_\varepsilon)}{\varepsilon} \, dx \to 2c_0 \int_{\Omega \cap \partial\{u_0 = -1\}}^f(x, n(x)) \, d\mathcal{H}^{n-1} \tag{6}
\]
as \( \varepsilon \to 0 \), where \( n \) is unit normal of \( \partial\{u_0 = -1\} \).

We note that there is one more interesting model which Braides in [1] and [7] considered the phase transition in a periodic medium, that is, the case of \( f(x, p) \) is \( \delta \)-periodic with respect to \( x \). \( \delta \) represents the length scale of inhomogeneities in the medium. The energy functional \( \Gamma \)-converges to two different types of the generalized perimeter by the order of \( \varepsilon \) and \( \delta \).

Now we focus on the curvature of the limit interface. The minimizer \( u_\varepsilon \) satisfies the following Euler-Lagrange equation

\[
-\varepsilon \sum_{i=1}^{n} D_i (f_p_i(x, Du_\varepsilon)) + \frac{W'(u_\varepsilon)}{\varepsilon} = \lambda \varepsilon \quad \text{in} \quad \Omega, \tag{7}
\]
where \( \lambda_\varepsilon \) is Lagrange multiplier from the volume constraint. Luckhaus and Modica proved that in isotropic case, that is \( f(x, p) = |p|^2 \), the Lagrange multiplier \( \lambda_\varepsilon \) converges to the constant mean curvature of the limit interface in [8]. Moreover in the isotropic case, not only as the minimizer but also as the solution of the PDE, convergence to the curvature in suitable weak sense was proved, which is strongly related to the modified De Giorgi conjecture and recently studied in [10] and [11].

We consider this analogy for non homogenous anisotropic surface tension. Here we prove the Lagrange multiplier \( \lambda_\varepsilon \) converges to the anisotropic curvature for the energy functional (1). For the variational problem (4) the following fact holds.

**Proposition 1.1.** Let \( \{u_{\varepsilon_i}\}_{i=1}^{\infty} \subset C^1(\Omega) \) be a sequence of the minimizers of (4). Assume that \( u_{\varepsilon_i} \) converge to some \( u_0 \in BV(\Omega) \) in \( L^1(\Omega) \) as \( \varepsilon_i \to 0 \). Then we have

(i) \( u_0 = \pm 1 \) for almost every \( x \in \Omega \).

(ii) \( E_0 = \{x \in \Omega | u_0 = -1\} \) is a solution of the variational problem;

\[
\min\{P_{\Omega}^J(E) | E \subset \Omega, |E| = \frac{1}{2}(|\Omega| - m)\} \tag{8}
\]

(iii)

\[
\lim_{i \to \infty} E_{\varepsilon_i}(u_{\varepsilon_i}) = 2c_0 P_{\Omega}^J(E_0), \tag{9}
\]

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where $P_{\Omega}^{f}$ is a generalized perimeter defined by (17).

Now we show the convergence of the Lagrange multiplier. We remark that we assume that the regularity of the limit interface because even in isotropic case, it is known that in higher dimension ($n \geq 8$) minimal surface may have singularities.

**Theorem 1.2. (Main Result)** Let $\{u_{\varepsilon}\} \subset C^{1}(\Omega)$ be a family of minimizers for the variational problem (4) and let $\{\lambda_{\varepsilon}\}$ be a family of Lagrange multiplier in (7). Then there exists subsequence $\{\varepsilon_{i}\}_{i=1}^{\infty}$ satisfying $u_{\varepsilon_{i}}$ converges to $u_{0} \in BV(\Omega; \{-1, 1\})$ in $L^{1}(\Omega)$ and

$$\lim_{i \to \infty} \lambda_{\varepsilon_{i}} = c_{0}\lambda_{0}$$

where $\lambda_{0}$ is a constant anisotropic curvature of the limit interface $\partial E_{0} \cap \Omega$ with $E_{0} = \{x \in \Omega | u_{0} = -1\}$ in the sense of (24).

2 Definitions and lemmata

We use the framework of Finsler metric to define the anisotropic curvature. Here, we state the perimeter and the first variation under the anisotropy, refer to [2], [3], [4] and [5].

Let $\phi$ be a non-negative function $\phi \in C^{2}(\Omega \times (\mathbb{R}^{n}\{0\}))$ satisfying

$$\phi(x, tp) = |t|\phi(x, p) \quad \text{for} \quad x \in \Omega, \ p \in \mathbb{R}^{n}, \ t \in \mathbb{R},$$

and

$$\lambda|p| \leq \phi(x, p) \leq \Lambda|p| \quad \text{for} \quad x \in \Omega, \ p \in \mathbb{R}^{n},$$

for positive constants $\lambda$ and $\Lambda$. The dual function $\phi^{\circ}$ of $\phi$ is defined by

$$\phi^{\circ}(x, p^{*}) = \sup\{p^{*} \cdot p \ | \ p \in B_{\phi}(x)\},$$

where $B_{\phi}(x) = \{p \in \mathbb{R}^{n} \ | \ \phi(x, p) \leq 1\}$. Here we only treat $\phi(x, p)$ and $\phi^{\circ}(x, p)$ which are strictly convex with respect to $p \in \mathbb{R}^{n}$. We apply the following terminology to

$$\phi^{\circ}(x, p) = \sqrt{f(x, p)}.$$  

(\phi-total variation) We define the generalized total variation by

$$\int_{\Omega} |D\alpha_{\phi}| = \sup \left\{ \int_{\Omega} u \ \text{div} g \ dx \ | \ g \in C^{1}_{0}(\Omega; \mathbb{R}^{n}) \text{ and } g \in B_{\phi^{\circ}}(x) \right\}$$

for $u \in BV(\Omega)$. For a measurable set $E$ in $\mathbb{R}^{n}$, we define the perimeter $P_{\Omega}^{\phi}(E)$ of $E$ as

$$P_{\Omega}^{\phi}(E) = \int_{\Omega} |D\chi_{E}|_{\phi^{\circ}}.$$
Under the assumption of the convexity and (11) of $\phi$, for smooth set $E$ the definition (16) is equivalent to the following:

$$P_\Omega^\phi(E) = \int_{\Omega \cap \partial E} \phi^\circ(x, n) \, d\mathcal{H}^{n-1},$$
(17)

where $n$ is outer normal of $\partial E$.

(\textit{$\phi$-normal vector and Cahn-Hoffman vector}) Let $n_\phi^*$ and $n_\phi$ be

$$n_\phi^* = \frac{n}{\phi^\circ(x, n)} \quad \text{and} \quad n_\phi = \nabla_p \phi^\circ(x, n_\phi^*),$$
(18)

which $n_\phi^*$ is the normal vector with respect to anisotropy $\phi$ and $n_\phi$ is Cahn-Hoffman vector. For these vectors, the property

$$n_\phi^* \cdot n_\phi = 1$$
(19)

holds. We define the signed distance function $d_\phi$ associated to $\phi$

$$d_\phi(x) = \begin{cases} 
- \inf \{ \delta_\phi(x, y) \mid y \in \mathbb{R}^n \setminus E \} & \text{if } x \in E, \\
\inf \{ \delta_\phi(x, y) \mid y \in E \} & \text{if } x \in \mathbb{R}^n \setminus E,
\end{cases}$$
(20)

where

$$\delta_\phi(x, y) = \inf \{ \int_0^1 \phi(\gamma, \gamma) \, dt \mid \gamma \in W^{1,1}([0, 1]; \Omega), \quad \gamma(0) = x, \quad \gamma(1) = y \}.$$ 

(21)

Thus on the boundary $\partial E$

$$n_\phi^* = Dd_\phi \quad \text{and} \quad n_\phi = \nabla_p \phi^\circ(x, Dd_\phi).$$
(22)

We can consider the extension of $n_\phi^*$ and $n_\phi$ on the neighborhood of $\partial E$ by using this distance $d_\phi$.

(\textit{$\phi$-tangential divergence}) We define the $\phi$-tangential divergence on $\partial E$ by

$$\text{div}_{\partial E}^\phi g = \text{tr}[(\text{Id} - n_\phi \otimes n_\phi^*) D\tilde{g} + \phi^\circ(x, n_\phi^*) \otimes \tilde{g}]$$
(23)

for $g \in C^\infty_0(\partial E; \mathbb{R}^n)$ where $\tilde{g}$ is any smooth extension to $\mathbb{R}^n$. We notice that it holds independent of the extension.

(\textit{$\phi$-anisotropic curvature}) We define the $\phi$-anisotropic curvature $\kappa_\phi$ by

$$\kappa_\phi = -\text{div}_{\partial E}^\phi n_\phi.$$ 
(24)

Since $-n_\phi^* \cdot D_j n_\phi = 0$ by (19), we have

$$\text{div}_{\partial E}^\phi n_\phi = \text{div} n_\phi \quad \text{on } \partial E.$$ 
(25)

(\textit{first variation}) Let $E_t := \{ x + tg(x) | x \in E \}$ for $g \in C^\infty_0(\Omega; \mathbb{R}^n)$ and $t \in \mathbb{R}^n$. The first variation of the perimeter $P_\Omega^\phi$ is calculated (Theorem 5.1 in [5])

$$\frac{d}{dt} P_\Omega^\phi(E_t)|_{t=0} = \int_{\Omega \cap \partial E} g \cdot n_\phi^* \kappa_\phi \phi^\circ(x, n) \, d\mathcal{H}^{n-1}.$$ 
(26)
for \( g \in C^\infty_0(\Omega; \mathbb{R}^n) \). For \( \phi \)-tangential divergence, by the property (25) the following lemma holds.

**Lemma 2.1.** If \( E \) is a \( C^2 \) set in \( \mathbb{R}^n \) and \( g \in C^\infty_0(\Omega; \mathbb{R}^n) \), then we have

\[
\int_{\Omega \cap \partial E} \text{div}_E^\phi g \phi^\circ(x, n) \, dH^{n-1} = \int_{\Omega \cap \partial E} g \cdot n_\phi^* \text{div}_E n_\phi \phi^\circ(x, n) \, dH^{n-1}. \tag{27}
\]

Thus together with (26) and (27) for the first variation it follows that

\[
\frac{d}{dt} P^\phi(\Omega)(E_t) \big|_{t=0} = \int_{\Omega \cap \partial E} \text{div}_E g \phi^\circ(x, n) \, dH^{n-1}. \tag{28}
\]

*(weak convergence)* In order to justify the convergence, we consider the suitable form of Reshetnyak theorem. For this total variation (15), we can generalize the theorem of [8].

**Lemma 2.2.** Let \( \{v_\varepsilon\} \subset C^1(\Omega) \) and \( v_0 \in BV(\Omega) \) satisfying \( v_\varepsilon \) converges to \( v_0 \) in \( L^1(\Omega) \). If we assume

\[
Dv_\varepsilon \rightharpoonup Dv_0 \quad \text{in} \quad \Omega \tag{29}
\]

and

\[
\lim_{\varepsilon \to 0} \int_\Omega |Dv_\varepsilon|_\phi = \int_\Omega |Dv_0|_\phi, \tag{30}
\]

then for the function \( F(x, p) \in C(\Omega \times \mathbb{R}^n) \) satisfying

\[
F(x, tp) = tF(x, p) \quad \text{for} \quad x \in \Omega, \ p \in \mathbb{R}^n, \ t \geq 0 \tag{31}
\]

and

\[
F(x, p) = 0 \quad \text{for} \quad x \notin K, \ p \in \mathbb{R}^n \tag{32}
\]

with \( K \) is a fixed compact subset of \( \Omega \), we have

\[
\lim_{\varepsilon \to 0} \int_\Omega F(x, Dv_\varepsilon) \, dx = \int_\Omega F(x, n_\phi^*) \, d|Dv_0|_\phi, \tag{33}
\]

where \( n_\phi^* = \frac{Dv_0}{|Dv_0|_\phi} \) of \( \partial\{v_0 = -1\} \).

**References**


