ARITHMETIC-GEOMETRIC MEANS FOR HYPERELLIPTIC CURVES AND CALABI-YAU VARIETIES

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Abstract. In this paper, we define a generalized arithmetic-geometric mean \( \mu_g \) among \( 2^g \) terms motivated by \( 2\tau \)-formulas of theta constants. By using Thomae’s formula, we give two expressions of \( \mu_g \) when initial terms satisfy some conditions. One is given in terms of period integrals of a hyperelliptic curve \( C \) of genus \( g \). The other is by a period integral of a certain Calabi-Yau \( g \)-fold given as a double cover of the \( g \)-dimensional projective space \( \mathbb{P}^g \).

1. Introduction

Let \( \{a_{n,0}\}_n \) and \( \{a_{n,1}\}_n \) be positive real sequences defined by the recurrence relations

\[
(1.1) \quad a_{n+1,0} = \frac{a_{n,0} + a_{n,1}}{2}, \quad a_{n+1,1} = \sqrt{a_{n,0} a_{n,1}},
\]

and initial terms \( a_{0,0} = a_0, a_{0,1} = a_1 \) with \( 0 < a_1 < a_0 \). One can easily show that \( \{a_{n,0}\}_n \) and \( \{a_{n,1}\}_n \) have a common limit, which is called the arithmetic-geometric mean of \( a_0 \) and \( a_1 \), and is denoted by \( \mu_1(a_0, a_1) \). By the homogeneity of the arithmetic and geometric means, we have \( \mu_1(ca_0, ca_1) = c\mu_1(a_0, a_1) \) for any positive real number \( c \).

On the other hand, two Jacobi’s theta constants \( \theta_0 \) and \( \theta_1 \) satisfy the following \( 2\tau \)-formulas:

\[
(1.2) \quad \theta_0(2\tau)^2 = \frac{\theta_0(\tau)^2 + \theta_1(\tau)^2}{2}, \quad \theta_1(2\tau)^2 = \theta_0(\tau)\theta_1(\tau),
\]

where

\[
\theta_i(\tau) = \sum_{n \in \mathbb{Z}} \exp(\pi \sqrt{-1}(n^2\tau + in)), \quad i = 0, 1,
\]

and \( \tau \) belongs to the upper half space \( \mathbb{H} \). If we find an element \( \tau \in \mathbb{H} \) such that \( \theta_1(\tau)^2/\theta_0(\tau)^2 = a_1/a_0 \) for given initial terms \( a_0 \) and \( a_1 \), then we have

\[
\frac{a_0}{\mu_1(a_0, a_1)} = \frac{\theta_0(\tau)^2}{\mu_1(\theta_0(\tau)^2, \theta_1(\tau)^2)} = \frac{\theta_0(\tau)^2}{\mu_1(\theta_0(2^n\tau)^2, \theta_1(2^n\tau)^2)} = \theta_0(\tau)^2
\]

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by (1.1), (1.2) and \( \lim_{n \to \infty} \theta_i(2^n \tau) = 1 \). Moreover, the Jacobi’s formula between 
\( \theta_0(\tau)^2 \) and an elliptic integral implies that
\[
\frac{a_0}{\mu_1(a_0, a_1)} = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad k = \sqrt{a_0^2 - a_1^2}.
\]

In this paper, we define a generalized arithmetic-geometric mean \( \mu_g \) among 
2\( g \) terms (\( \ldots, a_I, \ldots \)) (\( I \in \mathbb{F}_2^g \)) motivated by the 2\( \tau \)-formulas (2.3) of theta constants obtained by Theorem 2 in [3] p.139. By using Thomae’s formula, we give two expressions of \( \mu_g \) whose initial terms are given as (3.1) for some 2\( g+1 \) real numbers \( p_j \). One is given in terms of period integrals of the hyperelliptic curve \( C \) of genus \( g \) represented by the double cover of the complex projective line \( \mathbb{P}^1 \) branching at \( \infty \) and 2\( g+1 \) points \( p_j \). The other is by a period integral of the Calabi-Yau \( g \)-fold which is the double cover of the \( g \)-dimensional projective space \( \mathbb{P}^g \) branching along the dual hyperplanes of the images of \( \infty \) and \( p_j \) (\( j = 1, \ldots, 2g+1 \)) under the Veronese embedding of \( \mathbb{P}^1 \) into \( \mathbb{P}^g \).

In 1876, Borchardt studied in [1] the case of \( g = 2 \): the generalized arithmetic-geometric mean \( \mu_2 \) of \( a = (a_{00}, a_{01}, a_{10}, a_{11}) \) was given by the iteration of four means
\[
\frac{a_{00} + a_{01} + a_{10} + a_{11}}{4}, \quad \frac{\sqrt{a_{00}a_{10}} + \sqrt{a_{01}a_{11}}}{2}, \quad \frac{\sqrt{a_{00}a_{11}} + \sqrt{a_{10}a_{01}}}{2},
\]
and \( \mu_2(a) \) was expressed in terms of period integrals of a hyperelliptic curve of genus 2. Mestre showed in [4] that \( \mu_2(a) \) could be expressed in terms of \( \mu_1 \) and some algebraic functions of \( a \) when
\[
a_{00} > a_{01} = a_{10} > a_{11}, \quad a_{00}a_{11} > a_{01}a_{10}.
\]

2. Comparison to Theta Constants

We define a hyperelliptic curve \( C \) of genus \( g \) by
\[
C : y^2 = (x - p_1) \cdots (x - p_{2g+1}),
\]
where \( p_j \)'s are real numbers satisfying \( p_1 < \cdots < p_{2g+1} \). As in [6] p.76, we choose the cycles \( A_1, \ldots, A_g, B_1, \ldots, B_g \) in the union of the following two sheets (I),(II) in Figure 1. Here \( \mathbb{R}_+ \) is the set of non-negative real numbers, the range of values of \( y \) is written, and the cycles in the sheet II are written in thick lines. Note that the cycles satisfy
\[
A_i \cdot A_j = B_i \cdot B_j = 0, \quad A_i \cdot B_j = \delta_{ij}
\]
for \( 1 \leq i, j \leq g \) under the intersection form.

We define holomorphic forms \( \omega_j \) for \( j = 1, \ldots, g \) as
\[
\omega_j = \frac{x^{j-1}dx}{y}.
\]
We define integrals $T_i^{(j)}$ by

$$T_i^{(j)} = \int_{p_i}^{p_i+1} \frac{x^{j-1}dx}{\sqrt[2g+1]{(x - p_k)\prod_{k=i+1}^{2g+1}(p_k - x)}}$$

for $1 \leq i \leq 2g$ and $1 \leq j \leq g$. Then the integrals $T_i^{(j)}$ are positive real numbers. Using these integrals, we express the period integrals of $C$:

$$\int_{A_i} \omega_j = (-1)^i 2T_{2i-1}^{(j)}, \quad \int_{B_i} \omega_j = 2\sqrt{-1}(\sum_{k=1}^{g} (-1)^{k+1}T_{2k}^{(j)}).$$

We set

(2.1) \quad A = (\int_{A_i} \omega_j)_{ij}, \quad B = (\int_{B_i} \omega_j)_{ij}

and consider the normalized period matrix $\tau$ by A-period:

(2.2) \quad \tau = BA^{-1}.

By Riemann's bilinear relations, $\det(A)$ is a non-zero real number and $\tau$ is a symmetric matrix whose imaginary part is positive definite. Note also that $\tau$ is purely imaginary.

Remark 2.1. Since the Vandermonde matrix $\det(x_i^{j-1})_{1 \leq i, j \leq g}$ is positive on $p_{2i-1} \leq x_i \leq p_{2i}$, $(-1)^{g(g+1)/2} \det(A)$ is positive.
For $I = (i_1, \ldots, i_g) \in \mathbb{F}_2^g$, we define theta constants as

$$\theta_I(\tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi \sqrt{-1} \cdot n \tau \cdot \!n + \pi \sqrt{-1} \cdot n \cdot \!I).$$

**Proposition 2.2.** Let $M$ be a positive definite symmetric $g \times g$ real matrix. Then $\theta_I(\sqrt{-1}M)$ is positive for each $I \in \mathbb{F}_2^g$.

**Proof.** By the inversion formula of the theta function in [5] p.195, we have

$$\sqrt{\det(M)} \cdot \theta_I(\sqrt{-1}M) = \sum_{n \in \mathbb{Z}^g} \exp \left( \sqrt{-1} \pi (\frac{I}{2}) (\sqrt{-1}M^{-1}) \cdot n + \frac{I}{2} \right),$$

where $\sqrt{\det(M)}$ takes a positive value. Since each term of the right hand side is positive, the left hand side is positive. \qed

We consider variable $u = (u_I)_{I \in \mathbb{F}_2^g}$ whose coordinates are indexed by $\mathbb{F}_2^g$. The pair $(\theta_I(\tau))$ is denoted by $\theta(\tau)$. For $I \in \mathbb{F}_2^g$, we define quadratic polynomials $F_I(u)$ of $2^g$ variables $u = (u_I)_{I \in \mathbb{F}_2^g}$ by

$$F_I(u) = \frac{1}{2^g} \sum_{P \in \mathbb{F}_2^g} u_I \cdot P u_P.$$

We remark that the coefficients of $2^g F_I(u)$ are in $\mathbb{Z}_{\geq 0}$. By Theorem 2 in [3] p.139, we have $2\tau$-formulas of theta constants

$$(2.3) \quad \theta_I(2\tau)^2 = F_I(\theta(\tau))$$

for $I \in \mathbb{F}_2^g$.

Now prepare some combinatorial notations for the statement of Thomae’s formula. For an index $I \in \mathbb{F}_2^g$, we define a subset $S_I$ of $R = \{1, \ldots, 2g+1, \infty\}$ as follows. Let $\eta_i$ be elements of $M(2, g, \mathbb{F}_2)$ defined as

$$\eta_{2i-1} = \begin{pmatrix} 0 & \ldots & 0 & i \text{-th} & 1 & 0 & \ldots & 0 \\ 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

$$\eta_{2i} = \begin{pmatrix} 0 & \ldots & 0 & i \text{-th} & 1 & 0 & \ldots & 0 \\ 1 & \ldots & 1 & 1 & 0 & \ldots & 0 \end{pmatrix},$$

for $i = 1, \ldots, 2g + 1$. Then a subset $T_I$ of $R - \{2g + 1, \infty\} = \{1, 2, \ldots, 2g\}$ is characterized by the equality

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{j \in T_I} \eta_j.$$

We set

$$S_I = \begin{cases} T_I & \text{if } \#T_I \text{ is even,} \\ T_I \cup \{2g + 1\} & \text{if } \#T_I \text{ is odd.} \end{cases}$$

Let $U$ be the set $\{1, 3, 5, \ldots, 2g + 1\}$ and $R_1 \circ R_2$ be the symmetric difference of sets $R_1$ and $R_2$. 
Proposition 2.3 ([6] p.120, [2]). Let A be the period matrix of C in (2.1). Then we have

\[
(2\pi)^{2g}\theta_I(\tau)^4 \frac{\det(A)^2}{\det(\Lambda)^2} = \prod_{i<j,i,j \in S \circ U} (p_j - p_i) \prod_{i<j,i,j \notin S \circ U} (p_j - p_i).
\]

Here we used the fact that \(\theta_I(\tau)\) is a real number to determine the sign of Thomae’s formula in [6].

3. STATEMENT AND PROOF OF THE MAIN THEOREM

Definition 3.1 (AGM sequences).

1. For an element \(u = (u_I)_I \in \mathbb{R}_{+}^{2g}\), we define the termwise root \(\sqrt{u}\) of \(u\) by \((\sqrt{u_I})_I\).

2. Let \(a = (a_I)_I\) be an element in \((\mathbb{R}_{+})^{2g}\). We define \(a_k = (a_{k,I})_I\) inductively by the relation

\[
a_{0,I} = a_I, \quad a_{k+1,I} = F_I(\sqrt{a_k}).
\]

A proof of the following proposition will be left to readers.

Proposition-Definition 3.2 (Generalized arithmetic-geometric mean). For an element \(a = (a_I)_I \in (\mathbb{R}_{+})^{2g}\), the limits \(\lim_{k \to \infty} a_{k,I}\) exist and are independent of indexes \(I\). This common limit is called the generalized arithmetic-geometric mean of \((a_I)_I\) and denoted by \(\mu_g(a_I)\).

Problem 3.3. Is it possible to express the generalized arithmetic-geometric mean \(\mu_g(a_I)\) of \(a = (a_I)_I \in (\mathbb{R}_{+})^{2g}\) in terms of period integrals of a family of varieties parametrized by \(a\)?

Theorem 3.4. Let \(p_1 < \cdots < p_{2g+1}\) be real numbers. We define \(a_I\) by

\[
a_I = \sqrt{\prod_{i<j,i,j \in S \circ U} (p_j - p_i) \prod_{i<j,i,j \notin S \circ U} (p_j - p_i)}.
\]

Then we have

\[
\mu_g(a_I) = \frac{(2\pi)^{2g}}{|\det(A)|},
\]

where \(A\) is the period matrix of \(C\) in (2.1).

Proof. By the initial condition, we have

\[
a_{0,I} = \frac{(2\pi)^{2g}\theta_I(\tau)^2}{|\det(A)|}.
\]

We show that

\[
a_{n,I} = \frac{(2\pi)^{2g}\theta_I(2^n\tau)^2}{|\det(A)|},
\]
by induction on \( n \). Since \( \theta_I(2^n \tau) \) is a positive real number by Proposition 2.2 for each \( I \), we have
\[
 a_{n+1,I} = F(\sqrt{a_n}) \\
= \frac{(2\pi)^g \cdot F(\theta_I(2^n \tau))}{|\det(A)|} \quad \text{(by the induction hypothesis)} \\
= \frac{(2\pi)^g \cdot \theta_I(2^{n+1} \tau)^2}{|\det(A)|} \quad \text{(by the formula (2.3))}
\]
Therefore we have
\[
 \lim_{n \to \infty} a_{n,I} = \frac{(2\pi)^g}{|\det(A)|}.
\]

4. Period of Calabi-Yau variety of certain type

We study a relation between the generalized arithmetic-geometric mean of the last section and a period of a Gorenstein Calabi-Yau variety of a certain type.

**Definition 4.1** (Calabi-Yau varieties). A variety \( X \) only with Gorenstein singularities is called a Calabi-Yau variety if the dualizing sheaf of \( X \) is trivial and \( X \) has a global crepant resolution.

Let \( P = P^g \) be the \( g \) dimensional projective space and \( H_1 \cdots H_{2g+2} \) be hyperplanes of \( P \). There is a unique line bundle \( \mathcal{L} \) on \( P \) and a unique isomorphism \( \varphi : \mathcal{L} \otimes^2 \mathcal{O}_X(\sum_{i=1}^{2g+2} H_i) \) up to a non-zero constant. Using the isomorphism \( \varphi \), we define a double covering \( X = \text{Spec}(\mathcal{O}_X \oplus \mathcal{L}) \), where the multiplication on \( \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_X \) is given by the isomorphism \( \varphi \).

By the following Proposition 4.2, \( X \) becomes a Calabi-Yau variety, since it admits a global crepant resolution.

**Proposition 4.2.**

1. If \( \bigcup_{i=1}^{2g+2} H_i \) is normal crossing, then the variety \( X \) has only Gorenstein singularities. Also it admits a global crepant resolution.
2. Under the above hypotheses, the dualizing sheaf is isomorphic to the structure sheaf.

**Proof.** (1) Locally on \( P \), the variety \( X \) is defined by the equation \( \eta^2 = \xi_1 \cdots \xi_g \), where \( \xi_1, \ldots, \xi_g \) are local coordinates. Therefore this variety \( U \) is an affine toric variety defined by \( \text{Spec}(\sigma \cap M^*) \), where
\[
 M^* = \mathbb{Z}^g + \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \mathbb{Z} \subset \mathbb{Q}^g, \quad \sigma = (R_+)^g.
\]
Let \( \sigma \) be the dual simplex of \( \sigma \) and \( M \) be the dual lattice of \( M \). Since \( \sigma \) is generated by elements contained primitive hyperplanes, \( X \) is Gorenstein. We can construct a global crepant resolution as follows. We make a refinement of the simplex \( \sigma \) into a regular fan \( \bigcup_{w \in \rho_g} \sigma_w \) indexed by the set \( \rho_g \) of “unfair tournament” of \( \{1, \ldots, g\} \). A sequence \( w = (w_1, \ldots, w_{g-1}) \) is an element of the set \( \rho_g \) if it satisfies the following properties:
(i) $w_1$ is equal to 1 or 2 and
(ii) $w_i$ is equal to $w_{i-1}$ or $i+1$ for $2 \leq i \leq g-1$.

For an element $w$ of $\rho_g$, we define $\sigma_w$ as a cone generated by

$$B_w = \{ u_1 = e_1 + e_2, u_2 = e_{w_1} + e_3, u_3 = e_{w_2} + e_4, \ldots, u_{g-1} = e_{w_{g-2}} + e_g, u_g = 2e_{w_{g-1}} \},$$

where $e_i$ is the standard basis of $\mathbb{Z}^g \supset M$. Since the set $B_w$ is a free base of $M$, the fan $\cup_{w \in \rho_g} \sigma_w$ is regular and it defines a smooth toric variety $\tilde{X}$. The coordinates associated to $\mathbb{Z}^g \subset M^*$ are written as $\xi_1, \ldots, \xi_g$. ($\eta$ corresponds to $\frac{1}{2}(1, \ldots, 1).$) Let $z_1, \ldots, z_g$ be the coordinates associated to the dual base $B_w$ of $M$. Then we have

$$z_1^{u_1} \cdots z_g^{u_g} = \xi_1^{e_1} \cdots \xi_g^{e_g}.$$

Thus $\xi_1^{\frac{1}{2}} \cdots \xi_g^{\frac{1}{2}} = z_1 \cdots z_g$. Therefore the pull back of the rational differential form $\omega_X$ to the affine toric variety associated to $\sigma_w$ is a non-zero constant multiple of $dz_1 \wedge \cdots \wedge dz_g$, which shows that the map $\tilde{X} \to X$ is a crepant resolution. Since the local crepant resolutions depend only on the choice of order of the components of the branching divisor, they are patched together into a global crepant resolution.

(2) Let $\xi_1, \ldots, \xi_g$ be inhomogeneous coordinates of $\mathbb{P}$ with the infinite hyperplane $H_{g+2}$ and $l_i = l_i(\xi)$ be inhomogeneous linear forms defining the hyperplane $H_i$ for $i = 1, \ldots, 2g+1$. Then defining equation of the double covering $X$ can be written as

$$\eta^2 = \prod_{i=1}^{2g+1} l_i(\xi).$$

As is shown in the proof of (1),

$$(4.1) \quad \omega_X = \frac{1}{\eta} d\xi_1 \wedge \cdots \wedge d\xi_g$$

is a global generator of the dualizing sheaf of $X$. \hfill \Box

For real numbers $p_1 < \cdots < p_{2g+1}$, we define linear forms $l_i$ by

$$l_i = \xi_1 - p_i \xi_2 + p_i^2 \xi_3 + \cdots + (-1)^{g-1} p_i^{g-1} \xi_g + (-1)^g p_i^g$$

and set $H_i = \{ l_i = 0 \}$. By using the Vandermonde matrix, we see that $\cup_{i=1}^{2g+2} H_i$ is a normal crossing divisor.

We define a subset $\Delta$ of $\mathbb{R}^g$ as

$$\Delta = \{(x_1, \ldots, x_g) \mid (-1)^{i-1} l_{2i-1}(x_1, \ldots, x_g) \geq 0 \text{ for } i = 1, \ldots, g+1, \text{ and } (-1)^i l_{2i}(x_1, \ldots, x_g) \geq 0 \text{ for } i = 1, \ldots, g \}.$$
Then $\gamma = \gamma_+ - \gamma_-$ defines a $g$-chain in $X$. We have the following relation between the generalized arithmetic-geometric mean and a period of the Calabi-Yau variety $X$. The following theorem is obtained by Theorem 2 in [7].

**Theorem 4.3.** Let $(a_I)_I$ be an element of $\mathbb{R}^g_+$ defined in (3.1). Under the above notation, we have

$$\mu(a_I) = \frac{2\pi g}{\int_\gamma \omega_X}.$$ 

**Proof.** Let $C_j$ be a copy of the curve $C$ given by $y_j = \prod_{i=1}^{2g-1}(x_j - p_i)$. We define a map $\pi : C_1 \times \cdots \times C_g \to X$ by sending $((x_1, y_1), \ldots, (x_g, y_g))$ to the point whose $\xi_k$-coordinate and $\eta$-coordinate are the ($g+1-k$)-th elementary symmetric function of $x_1, \ldots, x_g$ and $\prod_{i=1}^{g} y_i$, respectively. Then we have

$$\pi^* \omega_X = \sum_{\sigma \in S_g} \text{sgn}(\sigma) \sum_{i=1}^{g} \omega_{\sigma(i)}.$$ 

Since $\pi_*(A_1 \times \cdots \times A_g) = (-1)^{g(g+1)/2}2^{g-1}\gamma$, we have

$$2^{g-1} \int_\gamma \omega_X = | \det(A) |.$$ 

By Theorem 3.4, we have the theorem. \hfill \Box

5. **Genus Two Case**

In this section, we will give a detailed study for the case of $g = 2$. Refer to [1] and [4] for the original results by Borchardt and recent related works by Mestre, respectively. We begin with $(a_{00}, a_{01}, a_{10}, a_{11})$ as initial data for AGM sequences. The recursive relations for $a_{k, I}$ ($I \in \mathbb{F}_2^g, k = 0, 1, \cdots$) are given as $a_{0, I} = a_I$ and $a_{k+1, I} = F_I(\sqrt{a_{k,00}}, \cdots, \sqrt{a_{k,11}})$, where

$$F_{00}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{4}(u_{00}^2 + u_{01}^2 + u_{10}^2 + u_{11}^2),$$

$$F_{01}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{01} + u_{11}u_{10}),$$

$$F_{10}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{10} + u_{11}u_{01}),$$

$$F_{11}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{11} + u_{10}u_{01}).$$

In the following, we assume that $a_{00} > a_{10} > a_{11} > a_{01}$ and $a_{00}a_{01} > a_{10}a_{11}$. First we define positive real numbers $k_1 > k_2$ and $0 < l_2 < l_1 < 1$ such that

$$(a_{00} + a_{01})^2 - (a_{10} + a_{11})^2 = k_1^2, \quad (a_{00} - a_{01})^2 - (a_{10} - a_{11})^2 = k_2^2.$$
Therefore by Theorem 3.4, we have

\[ a_{00} + a_{01} = \frac{1 + l_1^2}{1 - l_1^2} k_1, \quad a_{10} + a_{11} = \frac{2l_1}{1 - l_1^2} k_1, \]

\[ a_{00} - a_{01} = \frac{1 + l_2^2}{1 - l_2^2} k_2, \quad a_{10} - a_{11} = \frac{2l_2}{1 - l_2^2} k_2, \]

We set

\[ p_1 = 0, \quad p_2 = \frac{1}{(1 - l_2^2)(1 - l_1^2)}, \]

\[ p_3 = \frac{2(l_1 l_2 + 1) a_{00}}{(1 - l_1^2)(1 - l_2^2)(k_1 + k_2)(1 - l_1 l_2)}, \]

\[ p_4 = \frac{2(l_1 l_2 + 1) a_{01}}{(1 - l_1^2)(1 - l_2^2)(k_1 - k_2)(1 - l_1 l_2)}, \]

\[ p_5 = \frac{4 a_{00} a_{01}}{(k_1 - k_2)(k_1 + k_2)(1 - l_2^2)(1 - l_1^2)}. \]

Then we have

\[ (a_{00}^2 : a_{01}^2 : a_{10}^2 : a_{11}^2) = ((p_3 - p_1)(p_5 - p_1)(p_5 - p_3)(p_4 - p_2) : (p_4 - p_1)(p_5 - p_1)(p_5 - p_4)(p_3 - p_2) : (p_3 - p_2)(p_5 - p_2)(p_5 - p_3)(p_4 - p_1) : (p_4 - p_2)(p_5 - p_2)(p_5 - p_4)(p_3 - p_1)). \]

Therefore by Theorem 3.4, we have

\[
\lim_{n \to \infty} a_{n,00} = \frac{4\pi^2 a_{00}}{|\det(A)| \sqrt{(p_3 - p_1)(p_5 - p_1)(p_5 - p_3)(p_4 - p_2)}}
\]

\[
= \frac{8\pi^2}{|\det(A)|} \cdot (1 - l_1^2)^2(1 - l_2^2)^2 \sqrt{\frac{(a_{00} a_{01} - a_{10} a_{11})^3 (1 - l_1 l_2)^3}{a_{00} a_{01} a_{10} a_{11} (l_1^2 - l_2^2)(1 + l_1 l_2)}}.
\]

where \( A \) is the period matrix of \( C \) in (2.1).

Using the result of §4, we have

\[
|\det(A)| = 4 \cdot \int_{\Delta} \frac{d\xi_1 \wedge d\xi_2}{\sqrt{\prod_{i=1}^5 (\xi_1 - p_i \xi_2 + p_i^2)}},
\]

where \( \Delta \) is a domain in \( \mathbb{R}^2 \) defined by \( l_1 \geq 0, -l_2 \geq 0, -l_3 \geq 0, l_4 \geq 0 \) and \( l_5 \geq 0 \). This is a period integral of the covering \( X \) of \( \mathbb{P}^2 \) defined by

\[
\eta^2 = \prod_{i=1}^5 (\xi_1 - p_i \xi_2 + p_i^2).
\]

We notice that the variety \( X \) is the (nodal) Kummer surface of the Jacobian of \( C \).

**Remark 5.1.** When

\[ a_{00} > a_{01} = a_{10} > a_{11}, \quad a_{00} a_{11} > a_{01} a_{10}, \]
μ_2(a) can be expressed in terms of the arithmetic-geometric mean μ_1 and expressions p_2, ..., p_5 by a (see [4]).

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