Some transformation formulas for Lauricella’s hypergeometric functions $F_D$

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Abstract

In this paper, we give some functional equations with a parameter $c$ for Lauricella’s hypergeometric functions; they can be regarded as multivariable versions of the Gauss quadratic transformation formula for the hypergeometric function. These functional equations for $c = 1$ are utilized for the study of arithmetic-geometric means of several terms.

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1 Introduction

It is known that the hypergeometric function $F(\alpha, \beta, \gamma; z)$ satisfies the Gauss quadratic transformation formula:

$$(1 + z)^{2\alpha} F(\alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}; z^2) = F(\alpha, \beta, 2\beta; \frac{4z}{(1 + z)^2}).$$

When $\alpha = \beta = \frac{1}{2}$, this equality reduces to

$$\frac{1 + z}{2} F(\frac{1}{2}, \frac{1}{2}, 1; 1 - z^2) = F(\frac{1}{2}, \frac{1}{2}, 1; 1 - (\frac{2\sqrt{z}}{1 + z})^2),$$

which implies that the reciprocal of the arithmetic-geometric mean of 1 and $x \in (0, 1)$ coincides with $F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2)$, refer to [HKM].
In this paper, we give some functional equations with a parameter $c$ for Lauricella’s hypergeometric functions $F_D$; they can be regarded as multivariable versions of the Gauss quadratic transformation formula. Our functional equations for $c = 1$ are given in [KS1],[KS2] and [KM], and they imply the expressions of arithmetic-geometric means of several terms by Lauricella’s hypergeometric functions $F_D$. We also show that each of our functional equations admits no other parameters when we specify the transformations of variables of $F_D$ and an admissible factor to the product of power functions associated with singular locus of $F_D$.

By considering restrictions of variables for our theorems, we obtain three transformation formulas for the hypergeometric function $F(\alpha, \beta, \gamma; z)$. We remark that they are not listed in [E] and [G], and that one of them was recently found in [BBG].

For proofs of our theorems, we utilize yang [O1], which is a package of computer algebra system Risa/Asir for the ring of differential-difference operators.

2 Lauricella’s hypergeometric function $F_D$

Lauricella’s hypergeometric function $F_D$ of $m$ variables $z_1, \ldots, z_m$ with parameters $\alpha, \beta, \gamma$ is defined as

$$F_D(\alpha, \beta, \gamma; z) = \sum_{n_1, \ldots, n_m \geq 0} (\alpha, \sum_{j=1}^m n_j) \prod_{j=1}^m (\beta_j, n_j) \prod_{j=1}^m z_j^{n_j},$$

where $z = (z_1, \ldots, z_m)$ satisfies $\mid z_j \mid < 1$ ($j = 1, \ldots, m$), $\beta = (\beta_1, \ldots, \beta_m)$, $\gamma \neq 0, -1, -2, \ldots$ and $(\alpha, n) = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha)$.

This function admits the integral representation of Euler type:

$$F_D(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha}(1 - t)^{\gamma-\alpha} \prod_{j=1}^m (1 - z_j t)^{-\beta_j} \frac{dt}{t(1 - t)}. \quad (1)$$

When $m = 1$, $F_D(\alpha, \beta, \gamma; z)$ coincides with the Gauss hypergeometric function $F(\alpha, \beta, \gamma; z)$, and when $m = 2$, $F_D(\alpha, \beta, \gamma; z)$ is Appell’s hypergeometric function $F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$. 
Fact 1 (Proposition 9.1.4 in [IKSY]) The function $F_D(\alpha, \beta, \gamma; z)$ satisfies the integrable Pfaffian system

$$df = \Omega_f(z)f, \quad \Omega_f(z) = \sum_{1 \leq i < j \leq m+2} A_{ij} d\log(z_i - z_j),$$

where $f = \{f_0, f_1, \ldots, f_m\}, f_0 = F_D(\alpha, \beta, \gamma; z), f_i = z_i \frac{\partial f_0}{\partial z_i}$ $(1 \leq i \leq m)$, $z_{m+1} = 0, z_{m+2} = 1$, and $(m+1) \times (m+1)$-matrices $A_{ij}$ are given as

$$A_{ij} = \begin{pmatrix}
0-th & i-th & j-th \\
0-th & -\beta_j & \beta_i \\
i-th & \beta_j & -\beta_i \\
j-th & & \\
0-th & i-th &
\end{pmatrix} (1 \leq i < j \leq m),$$

$$A_{i,m+1} = \begin{pmatrix}
0-th & i-th &
\end{pmatrix}
\begin{pmatrix}
1 & & \\
-\beta_1 & O & O \\
& \vdots & \ddots & \ddots & \ddots \\
& & -\beta_{i-1} & 1 - \gamma + \sum_{1 \leq k \leq m} \beta_k & \\
& & & -\beta_i & 1 - \gamma + \sum_{1 \leq k \leq m} \beta_k \\
& & & & -\beta_{m+1} & O \\
& & & & & -\beta_m & O
\end{pmatrix} (1 \leq i \leq m),$$

$$A_{i,m+2} = \begin{pmatrix}
0-th & i-th &
\end{pmatrix}
\begin{pmatrix}
O & O \\
-\alpha \beta_i & -\beta_i & \cdots & -\beta_i & \gamma - \alpha - \beta_i - 1 & -\beta_i & \cdots & -\beta_i \\
& O & & & O
\end{pmatrix} (1 \leq i \leq m).$$

(Here we correct $A_{ij}$ and $A_{i,m+1}$. ) The singular locus of this Pfaffian system
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$$L_{ij} \subset \mathbb{C}^m,$$ where

$$L_{ij} = \{ \ell_{ij} = z_i - z_j = 0 \}$$ (2)

and $L_{m+1,m+2}$ is regarded as the hyperplane at infinity.

3 Transformation formulas

**Theorem 1** The hypergeometric function $F_D$ of 2 variables satisfies the transformation formula

$$\left(1 + \frac{z_1 + z_2}{3}\right)^c F_D\left(\frac{c}{3}, \frac{c+1}{6}, \frac{c+1}{6}, \frac{c+1}{2}; 1 - z_1^3, 1 - z_2^3\right)$$

$$= F_D\left(\frac{c}{3}, \frac{c+1}{6}, \frac{c+1}{6}, \frac{c+5}{6}; z_1', z_2', \right),$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$, $z = (z_1, z_2)$ is in a small neighborhood $U$ of $(1, 1)$, and the value of $(1+\omega z_1 + \omega z_2)^c$ at $(z_1, z_2) = (1, 1)$ is 1. This functional equation admits no other parameters when we specify the transformations of variables of the both sides of $F_D$ in (3) and an admissible factor to $\Delta(z) = \prod_{i=1}^k p_i(z)^{e_i}$, where $p_1(z), \ldots, p_k(z)$ are the irreducible factors of the product $\prod_{1 \leq i < j \leq m+2} \xi^* \ell_{ij}$ of the pull-back of $\ell_{ij}$ in (2) for $m = 2$ under the map $\xi$.

**Theorem 2** The hypergeometric function $F_D$ of 2 variables satisfies the transformation formula

$$\left(1 + \frac{\sqrt{z_1 z_2}}{2}\right)^{2c-1} F_D\left(\frac{c}{2}, \frac{2c-1}{4}, \frac{2c-1}{4}; c; 1 - z_1^2, 1 - z_2^2\right)$$

$$= F_D\left(\frac{c}{2}, \frac{2c-1}{4}, \frac{2c-1}{4}, \frac{c+1}{2}; z_1', z_2', \right),$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$, $z = (z_1, z_2)$ is in a small neighborhood $U$ of $(1, 1)$, and the value of $(1+\omega z_1 + \omega z_2)^c$ at $(z_1, z_2) = (1, 1)$ is 1. This functional equation admits no other parameters when we specify the transformations of variables of the both sides of $F_D$ in (3) and an admissible factor to $\Delta(z) = \prod_{i=1}^k p_i(z)^{e_i}$, where $p_1(z), \ldots, p_k(z)$ are the irreducible factors of the product $\prod_{1 \leq i < j \leq m+2} \xi^* \ell_{ij}$ of the pull-back of $\ell_{ij}$ in (2) for $m = 2$ under the map $\xi$. 

$$\xi : (z_1, z_2) \mapsto (z_1', z_2'),$$

$$z_1' = \frac{\sqrt{(1 - z_1^2)(1 - z_2^2)} - \sqrt{-1}(z_1 - z_2)^2}{(1 + \sqrt{z_1 z_2})^4},$$

$$z_2' = \frac{\sqrt{(1 - z_1^2)(1 - z_2^2)} + \sqrt{-1}(z_1 - z_2)^2}{(1 + \sqrt{z_1 z_2})^4},$$
where \( z = (z_1, z_2) \) is in a small neighborhood \( U \) of \((1,1)\), the values of \( \sqrt{z_1z_2} \) and \((1+\sqrt{z_1z_2})^{c-1}\) at \((z_1, z_2) = (1,1)\) are 1, and the value of \( \sqrt{(1-z_1^2)(1-z_2^2)} \) in the expression of \( z'_1 \) is the same as that of \( z'_1 \). Though \( z'_1 \) and \( z'_2 \) are exchanged by the choice of branches of \( \sqrt{(1-z_1^2)(1-z_2^2)} \), the right hand side of (4) is single-valued by the coincidence of the parameters \( \beta_1 \) and \( \beta_2 \) of \( F_D \). This functional equation admits no other parameters when we specify the transformations of variables of the both sides of (3) and an admissible factor to \( \Delta(z) = \prod_{i=1}^{k} p_i(z)^{c_i} \), where \( p_1(z), \ldots, p_k(z) \) are the irreducible factors of the product \( \prod_{1 \leq i < j \leq m+2} \xi^*\ell_{ij} \) of the pull-back of \( \ell_{ij} \) in (2) for \( m = 2 \) under the map \( \xi \).

**Theorem 3** The hypergeometric function \( F_D \) of 3 variables satisfies the transformation formula

\[
\begin{align*}
\left( \frac{1+z_1+z_2+z_3}{4} \right)^2 F_D \left( \frac{c}{4}, \frac{c+2}{12}, \frac{c+2}{12}, \frac{c+2}{3}, \frac{c+5}{6}; z_1', z_2', z_3' \right) &= F_D \left( \frac{c}{4}, \frac{c+2}{12}, \frac{c+2}{12}, \frac{c+2}{3}, \frac{c+5}{6}; z_1, z_2, z_3 \right), \\
\xi : (z_1, z_2, z_3) \mapsto (z_1', z_2', z_3'), \\
z_1' &= \left( 1 - z_1 - z_2 + z_3 \right)^2, \\
z_2' &= \left( 1 - z_1 + z_2 - z_3 \right)^2, \\
z_3' &= \left( 1 + z_1 + z_2 + z_3 \right)^2,
\end{align*}
\]

where \( z = (z_1, z_2, z_3) \) is in a small neighborhood \( U \) of \((1,1,1)\), and the value of \((1+z_1+z_2+z_3)^{c/2}\) at \((z_1, z_2, z_3) = (1,1,1)\) is 1. This functional equation admits no other parameters when we specify the transformations of variables of the both sides of \( F_D \) in (3) and an admissible factor to \( \Delta(z) = \prod_{i=1}^{k} p_i(z)^{c_i} \), where \( p_1(z), \ldots, p_k(z) \) are the irreducible factors of the product \( \prod_{1 \leq i < j \leq m+2} \xi^*\ell_{ij} \) of the pull-back of \( \ell_{ij} \) in (2) for \( m = 3 \) under the map \( \xi \).

**Remark 1** The transformation formulas (3) and (4) for \( c = 1 \) are utilized for the study of arithmetic-geometric means of three terms in [KS1] and [KS2], respectively. The transformation formula (5) for \( c = 1 \) appears in [KM] as Proposition 1, which is a key to express the common limit of a quadruple sequence by Lauricella’s hypergeometric function \( F_D \) of 3 variables.
4 Proof

In this section, we prove Theorem 1. Since we can show the others similarly, we omit their proofs.

Let $\Omega_f(z)$ and $\Omega_g(z)$ be the connection 1-forms in Fact 1 for

\[
f_0(z_1, z_2) = F_D(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2),
\]

and

\[
g_0(z_1, z_2) = F_D(\alpha', \beta'_1, \beta'_2, \gamma'; z_1, z_2),
\]

respectively. It is easy to see that the vector-valued functions

\[
f(z) = \left( f_0, \frac{\partial f_0}{\partial z_1}, \frac{\partial f_0}{\partial z_2} \right) \quad \text{and} \quad g(z) = \left( g_0, \frac{\partial g_0}{\partial z_1}, \frac{\partial g_0}{\partial z_2} \right)
\]

satisfy the Pfaffian systems

\[
df = \Omega_f(z)f, \quad dg = \Omega_g(z)g,
\]

respectively, where

\[
\Omega_f(z) = P\Omega_f(z)P^{-1} + dPP^{-1}, \quad \Omega_g(z) = P\Omega_g(z)P^{-1} + dPP^{-1},
\]

\[
P = \text{diag}(1, z_1, z_2) = \begin{pmatrix}
1 & 1 & 1 \\
z_1 & z_1 & z_1 \\
z_2 & z_2 & z_2
\end{pmatrix}.
\]

Consider the vector-valued function

\[
G(x) = \left( G_0, \frac{\partial G_0}{\partial x_1}, \frac{\partial G_0}{\partial x_2} \right)
\]

for the pull-back $G_0(x_1, x_2)$ of $g_0(z_1, z_2)$ under the map

\[
\xi : (x_1, x_2) \mapsto (z_1, z_2) = \left( \frac{1 + \omega x_1 + \omega^2 x_2}{1 + x_1 + x_2}, \left( \frac{1 + \omega^2 x_1 + \omega x_2}{1 + x_1 + x_2} \right)^2 \right).
\]
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It satisfies

$$G(1, 1) = \ell(1, 0, 0)$$

and the Pfaffian system $dG = \Omega_G(x)G$, where

$$\Omega_G(x) = J_2 \Omega_g(x) J_2^{-1} + dJ_2 J_2^{-1}, \quad J_2 = \begin{pmatrix} 1 & \ell \end{pmatrix},$$

$\Omega_g(x)$ is the pull-back of $\Omega_g(z)$ under the map $\xi$, and $J$ is the Jacobi matrix of the map $\xi$. The singular locus of $\Omega_G(x)$ consists of 12 lines $p_i(x) = 0$ ($i = 1, \ldots, 12$), where

$$p_1(x) = 1 + x_1 + x_2, \quad p_2(x) = x_1 - \omega, \quad p_3(x) = x_1 - \omega^2,
$$

$$p_4(x) = x_2 - \omega, \quad p_5(x) = x_2 - \omega^2, \quad p_6(x) = x_1 - \omega x_2,$$

$$p_7(x) = x_1 - \omega x_2, \quad p_8(x) = x_1 - x_2, \quad p_9(x) = x_1 - 1,$$

$$p_{10}(x) = x_2 - 1, \quad p_{11}(x) = 1 + \omega x_1 + \omega^2 x_2, \quad p_{12}(x) = 1 + \omega^2 x_1 + \omega x_2.$$

Put $a_i = p_i(1, 1)$; note that $a_i \neq 0$ for $i = 1, \ldots, 7$, $a_i = 0$ for $i = 8, \ldots, 12$. Since $f_0(1, 1) = g_0(1, 1) = 1$, we have $\Delta(1, 1) = 1$. Thus $\Delta(x)$ should be

$$\Delta(x) = \prod_{i=1}^{7} \left( \frac{p_i(x)}{a_i} \right)^{c_i}.$$

Consider the vector-valued function

$$F(x) = \ell(F_0, \partial F_0 / \partial x_1, \partial F_0 / \partial x_2)$$

for

$$F_0(x_1, x_2) = \Delta(x)f_0(1 - x_1^3, 1 - x_2^3).$$

It satisfies

$$F(1, 1) = \ell(1, \sum_{i=1}^{7} \frac{c_i}{a_i} \frac{\partial p_i}{\partial x_1} - \frac{3\alpha \beta_1}{\gamma}, \sum_{i=1}^{7} \frac{c_i}{a_i} \frac{\partial p_i}{\partial x_2} - \frac{3\alpha \beta_2}{\gamma})$$

and the Pfaffian system $dF = \Omega_F(x)F$, where

$$\Omega_F(x) = Q[J_1 \Omega_f(x) J_1^{-1} + dJ_1 J_1^{-1}]Q^{-1} + dQQ^{-1}, \quad J_1 = \text{diag}(1, -3x_1^2, -3x_2^2),$$
The singular locus of $\Omega$ and $\Omega$ quadratic equations of 15 variables for the identity differential-difference operators. We have a necessary and sufficient condition which is a package of computer algebra system Risa/Asir for the ring of

$$Q = \begin{pmatrix} \Delta(x) & \Delta(x) \\ \frac{\partial \Delta(x)}{\partial x_1} & \frac{\partial \Delta(x)}{\partial x_2} \end{pmatrix} = \Delta(x) \begin{pmatrix} 1 \\ \frac{\partial \log \Delta(x)}{\partial x_1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and $\Omega_f(x)$ is the pull-back of $\Omega_f(z)$ under the map

$$(x_1, x_2) \mapsto (z_1, z_2) = (1 - x_1^3, 1 - x_2^3).$$

The singular locus of $\Omega_F(x)$ consists of 12 lines $x_1 = 0$, $x_2 = 0$ and $p_i(x) = 0$ ($i = 1, \ldots, 10$).

Note that $F_0(x) = G_0(x)$ on $U$ if and only if $F(1, 1) = G(1, 1)$ and $\Omega_F(x) = \Omega_G(x)$. By (6) and (7), we have

$$c_1 + \frac{c_2 + c_6}{1 - \omega} + \frac{c_3 + c_7}{1 - \omega^2} - \frac{3\alpha \beta_1}{\gamma} = \frac{c_1}{3} + \frac{c_4 - c_6 \omega}{1 - \omega} + \frac{c_5 - c_7 \omega^2}{1 - \omega^2} - \frac{3\alpha \beta_2}{\gamma} = 0. \quad (8)$$

We compare the entries of $\Omega_F(x)$ with those of $\Omega_G(x)$ by utilizing yang [O1], which is a package of computer algebra system Risa/Asir for the ring of differential-difference operators. We have a necessary and sufficient condition for the identity $\Omega_F(x) = \Omega_G(x)$ expressed as a system of 802 linear and 399 quadratic equations of 15 variables $^t\!v = (c_1, \ldots, c_7, \alpha, \beta_1, \beta_2, \gamma, \alpha', \beta'_1, \beta'_2, \gamma')$. The 802 linear equations include the followings 14 linear equations:

$$
\begin{align*}
-c_1 - c_4 - c_5 - c_6 - c_7 + 3\beta_2 + \alpha' + \beta'_1 + \beta'_2 - \gamma' &= 0, \\
-2c_4 - 2c_5 + 3\alpha + 3\beta_2 - 3\gamma - 2\alpha' + \beta'_1 + \beta'_2 + 2\gamma' &= 1, \\
-2c_6 - 2c_7 + 3\alpha + 3\beta_1 + 3\beta_2 - 3\gamma - \alpha' - \beta'_1 - \beta'_2 + \gamma' &= 0, \\
-c_6 - c_7 + 3\beta_1 - \alpha' - \beta'_1 - \beta'_2 + \gamma' &= 1, \\
c_6 + c_7 + \alpha' + \omega\beta'_1 - (\omega + 1)\beta'_2 - \gamma' &= -1, \\
3\alpha - 3\beta_1 + 3\beta_2 - 3\beta'_1 - 3\beta'_2 &= -1, \\
\alpha' - 2\beta'_1 - 2\beta'_2 + 2\gamma' &= 1, \\
3\alpha + 3\beta_1 - 3\gamma &= -1, \\
3\alpha + 3\beta_2 - 3\gamma &= -1, \\
(\omega + 1)c_2 - \omega c_3 &= 0, \\
c_2 + c_3 &= 0, \\
(\omega + 1)c_4 - \omega c_5 &= 0, \\
c_4 + c_5 &= 0, \\
\omega c_6 - (\omega + 1)c_7 &= 0.
\end{align*}
$$
Thus we have a 1-dimensional solution space of these 14 linear equations, which can be expressed as

\[
\begin{align*}
\alpha &= \alpha' = \frac{c_1}{3}, \\
\beta_1 &= \beta_2 = \beta'_1 = \beta'_2 = \frac{c_1 + 1}{6}, \\
\gamma &= \frac{c_1 + 1}{2}, \\
\gamma' &= \frac{c_1 + 5}{6}, \\
\gamma &= \frac{c_1 + 5}{6}, \\
c_2 &= c_3 = c_4 = c_5 = c_6 = c_7 = 0,
\end{align*}
\]

where we regard \( c_1 \) as a free parameter in \( \mathbb{C} \).

We can see that the solution (9) satisfies the 802 linear and 399 quadratic equations and (8) by Risa/Asir. Hence \( F_0(x) = G_0(x) \) on \( U \) if and only if the condition (9) holds. Refer to [O2] for our computation by yang and Risa/Asir.

\[\Box\]

5 Restrictions

In this section, we derive some corollaries by considering restrictions of variables for our transformation formulas.

**Corollary 1 (Theorem 2.3 in [BBG])** We have

\[
\left( \frac{1 + 2z}{3} \right)^c F\left( \frac{c}{3}, \frac{c + 1}{3}, \frac{c + 1}{2}; 1 - z^3 \right) = F\left( \frac{c}{3}, \frac{c + 1}{3}, \frac{c + 5}{6}; (1 - z)^3 \right)
\]

for \( z \) sufficiently near to 1, where the value of \( \left( \frac{1 + 2z}{3} \right)^c \) at \( z = 1 \) is 1.

**Proof.** Put \( z = z_1 = z_2 \) for the transformation formula (3) and use the integral representation (1). \[\Box\]

**Corollary 2** We have

\[
\left( \frac{1 + z}{2} \right)^{2c - 1} F\left( \frac{c}{2}, \frac{2c - 1}{4}, c; 1 - z^4 \right) = F\left( \frac{c}{2}, \frac{2c - 1}{4}, \frac{c + 1}{2}; -(1 - z)^2 \right)
\]

for \( z \) sufficiently near to 1, where the value of \( \left( \frac{1 + z}{2} \right)^{2c - 1} \) at \( z = 1 \) is 1.

**Proof.** Put \( z = \sqrt{z_1}, z_2 = 1 \) for the transformation formula (4) and use the integral representation (1). \[\Box\]
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**Corollary 3** We have

$$
\left( \frac{1 + 3z}{4} \right)^2 F\left( \frac{c}{4}; \frac{c+2}{4}; \frac{c+2}{3}; 1 - z^2 \right) = F\left( \frac{c}{4}; \frac{c+2}{4}; \frac{c+5}{6}; \frac{1 - z}{1 + 3z} \right).
$$

for $z$ sufficiently near to 1, where the value of $(\frac{1+3z}{4})^{c/2}$ at $z = 1$ is 1.

**Proof.** Put $z = z_1 = z_2 = z_3$ for the transformation formula (3) and use the integral representation (1). \hfill \Box

**Remark 2** The equalities in Corollaries 1 and 3 for $c = 1$ are used in [BB] to study modified arithmetic-geometric means.

**Remark 3** It is written in [BBG] that Corollary 1 can not be deduced from the cubic transformation formulas in [G]. The authors remark that our corollaries are not listed in [E] and [G], and think that Corollaries 2 and 3 can not be obtained by classical results.

**References**


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