Mean iterations derived from transformation formulas for the hypergeometric function

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Abstract

From Goursat’s transformation formulas for the hypergeometric function $F(\alpha, \beta, \gamma; z)$, we derive several double sequences given by mean iterations and express their common limits by the hypergeometric function. Our results are analogies of the fact that the arithmetic-geometric mean of 1 and $x$ $(0, 1)$ can be expressed as the reciprocal of $F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2)$.

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\footnote{Keywords: hypergeometric function, mean iteration.}
1 Introduction

Let \( m_1 \) and \( m_2 \) be the arithmetic mean and the geometric mean:

\[
m_1(x, y) = \frac{x + y}{2}, \quad m_2(x, y) = \sqrt{xy}.
\]

For \( 0 < b < a \), we give a double sequence \( \{a_n\} \) and \( \{b_n\} \) by the iteration of two means \( m_1 \) and \( m_2 \) with initial \((a, b)\):

\[
(a_0, b_0) = (a, b), \quad (a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)).
\]

This double sequence converges and has a common limit, which is called the arithmetic-geometric mean of \( a \) and \( b \), or the compound \( m_1 \circ m_2(a, b) \). It is known that the arithmetic-geometric mean can be expressed by the hypergeometric function, that is

\[
m_1 \circ m_2(a, b) = \frac{a}{F(\frac{1}{2}, \frac{1}{2}, 1; 1 - (\frac{b}{a})^2)},
\]

where

\[
F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} z^n.
\]

We remark that the Gauss quadratic transformation formula for the hypergeometric function implies this fact, refer to Section 3 for details.

In this paper, from Goursat’s transformation formulas in \( [G] \) instead of Gaussian, we induce several double sequences given by mean iterations and express their common limits by the hypergeometric function \( F(\alpha, \beta, \gamma; z) \). We list pairs of means and their common limits induced from quadratic transformations in Theorem 2 and those from cubic ones in Theorem 3. It turns out that the parameters \((\alpha, \beta, \gamma)\) of the hypergeometric function in Theorem 2 satisfy

\[
\left\{ \frac{1}{|1 - \gamma|}, \frac{1}{|\gamma - \alpha - \beta|}, \frac{1}{|\alpha - \beta|} \right\} = \{2, 2, \infty\}, \{2, 4, 4\},
\]

and that those in Theorem 3 satisfy

\[
\left\{ \frac{1}{|1 - \gamma|}, \frac{1}{|\gamma - \alpha - \beta|}, \frac{1}{|\alpha - \beta|} \right\} = \{2, 3, 6\}.
\]
B.C. Carlson considers in [C] the twelve double sequences given by the iteration of means \( m_i \) and \( m_j \) (1 \( \leq i, j \leq 4, i \neq j \)), where

\[
m_3(x, y) = \sqrt{\frac{x + y}{2}}, \quad m_4(x, y) = \sqrt{x + \frac{y}{2}}.
\]

They converge and their common limits \( m_i \triangle m_j(a, b) \) admit integral representations of Euler type. Theorem 2 can be obtain by these results together with some functional equations for the hypergeometric function in Lemma 2.

J.M. and P.B. Borwein study in [BB1] two double sequences given by the iteration of \( m_5 \) and \( m_6 \) and by that of \( m_7 \) and \( m_8 \), where

\[
m_5(x, y) = \frac{x + 2y}{3}, \quad m_6(x, y) = \frac{3}{y} \sqrt{\frac{x^2 + xy + y^2}{3}},
\]
\[
m_7(x, y) = \frac{x + 3y}{4}, \quad m_8(x, y) = \sqrt{\frac{y + x}{2}}.
\]

They converge and their common limits \( m_5 \triangle m_6(a, b) \) and \( m_7 \triangle m_8(a, b) \) can be expressed as

\[
a \frac{F\left(\frac{1}{3}, \frac{2}{3}, 1; 1 - \left(\frac{b}{a}\right)^3\right)}{F\left(\frac{1}{3}, \frac{2}{3}, 1; 1 - \left(\frac{b}{a}\right)^2\right)}.
\]

respectively. We remark that Theorem 3 is independent of the results in [BB1] and [C].

The above expressions of common limits of double sequences in [BB1] are extended to those of multiple sequences by the hypergeometric function \( F_D \) of multi variables, refer to [KS], [KM] and [MO]. Similar extensions of some results in Theorem 2 are studied in [M].

A list of transformation formulas for the generalized hypergeometric function \( _3F_2\left(\alpha_0, \alpha_1, \alpha_2; \beta_1, \beta_2; z\right) \) is given in [K]. We attempt to find double sequences whose common limits can be expressed by \( _3F_2 \). It turns out that we can not get proper expressions of common limits by \( _3F_2 \) because of the reduction and the Clausen formula for \( _3F_2 \), refer to Section 6.

**Acknowledgment.** The authors express their gratitude to Professor M. Kato for informing them of his results in [K].
2 Mean iterations

In this section, we formalize the notion of mean iterations, for which we refer to Section 8 in [BB2].

Let $\mathbb{R}^*_+$ be the multiplicative group of positive real numbers. A *mean* is a continuous function $m : \mathbb{R}^*_+ \times \mathbb{R}^*_+ \to \mathbb{R}^*_+$ satisfying

$$
\min(x, y) \leq m(x, y) \leq \max(x, y),
$$
$$
m(tx, ty) = tm(x, y),
$$

for any $x, y, t \in \mathbb{R}^*_+$. A mean $m(x, y)$ is strict if

$$
m(x, y) = x \text{ or } m(x, y) = y \quad \Rightarrow \quad x = y.
$$

For two means $m_1$ and $m_2$ and two positive real numbers $a$ and $b$, we define a double sequence $\{a_n\}$ and $\{b_n\}$ with initial $(a_0, b_0) = (a, b)$ by

$$
(a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)).
$$

This double sequence is called the $(m_1, m_2)$-sequence with initial $(a, b)$. If $(m_1, m_2)$-sequence with initial $(a, b)$ converges and has a common limit, this value is called the compound of $m_1$ and $m_2$ with initial $(a, b)$ and denoted $m_1 \circ m_2(a, b)$.

If $a \geq b$ and two means $m_1$ and $m_2$ satisfy

$$
m_1(x, y) \geq m_2(x, y), \quad \text{for any } (x, y) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+,
$$

or

$$
(x - y)(m_1(x, y) - m_2(x, y)) \geq 0, \quad \text{for any } (x, y) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+,
$$

then the $(m_1, m_2)$-sequence with initial $(a, b)$ satisfies

$$
b_0 \leq b_1 \leq b_2 \leq \cdots \leq b_n \leq a_n \leq \cdots \leq a_2 \leq a_1 \leq a_0.
$$

If $a \geq b$ and two means $m_1$ and $m_2$ satisfy

$$
(x - y)(m_1(x, y) - m_2(x, y)) \leq 0, \quad \text{for any } (x, y) \in \mathbb{R}^*_+ \times \mathbb{R}^*_+,
$$

then the $(m_1, m_2)$-sequence with initial $(a, b)$ satisfies

$$
b_0 \leq a_1 \leq a_2 \leq \cdots \leq b_{2n} \leq a_{2n+1} \leq b_{2n+1} \leq a_{2n} \leq \cdots \leq a_2 \leq b_1 \leq a_0.
$$
Lemma 1 Suppose that two means $m_1$ and $m_2$ satisfy (1) or (2) or (3). If either $m_1$ or $m_2$ is strict, then the $(m_1,m_2)$-sequence with initial $(a,b)$ converges and has a common limit, and the compound $m_1 \circ m_2$ becomes a mean. Moreover, the convergence is uniform on any compact subset of $\mathbb{R}_+ \times \mathbb{R}_+^*$.

Proof. For a proof of the cases (1) and (2), refer to [BB2]. Suppose that (3) is satisfied. We may assume that $a \geq b$. Let $\{a_n\}$ and $\{b_n\}$ be the $(m_1,m_2)$-sequence with initial $(a,b)$. Then both of four sequences $\{a_{2n}\}$, $\{a_{2n+1}\}$, $\{b_{2n}\}$, and $\{b_{2n+1}\}$ are monotonous and bounded. Thus they converge; we set

$$
\lim_{n \to \infty} a_{2n} = \alpha_0, \quad \lim_{n \to \infty} a_{2n+1} = \alpha_1, \quad \lim_{n \to \infty} b_{2n} = \beta_0, \quad \lim_{n \to \infty} b_{2n+1} = \beta_1.
$$

Let $n \to \infty$ for the inequalities

$$
a_{2n-1} \leq b_{2n} \leq a_{2n+1} \leq b_{2n+1} \leq a_{2n} \leq b_{2n-1},
$$

we have

$$
\alpha_1 \leq \beta_0 \leq \alpha_1 \leq \beta_1 \leq \alpha_0 \leq \beta_1,
$$

i.e., $\beta_0 = \alpha_1$, $\beta_1 = \alpha_0$, $\alpha_1 \leq \alpha_0$.

Let $n \to \infty$ for the equalities

$$
a_{2n+1} = m_1(a_{2n},b_{2n}), \quad b_{2n+1} = m_2(a_{2n},b_{2n}),
$$

we have

$$
\alpha_1 = m_1(\alpha_0,\beta_0) = m_1(\alpha_0,\alpha_1), \quad \alpha_0 = \beta_1 = m_2(\alpha_0,\beta_0) = m_2(\alpha_0,\alpha_1).
$$

Since either $m_1$ or $m_2$ is strict, $\alpha_0$ should be equal to $\alpha_1$.

Let us show that $\mu = m_1 \circ m_2$ is a mean. In order to show that $\mu$ is continuous, take any $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, any $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that

$$
|a_n(a,b) - \mu(a,b)| < \varepsilon
$$

for any $n > N$. We fix a natural number $n$ satisfying $2n > N$. We can regard $a_{2n}$ and $a_{2n+1}$ as continuous functions of initial terms $(a,b)$. Thus there exists $\delta > 0$ such that

$$
|x - a| < \delta, \quad |y - b| < \delta, \quad \Rightarrow \quad |a_{2n}(x,y) - a_{2n}(a,b)| < \varepsilon,
$$

$$
|a_{2n+1}(x,y) - a_{2n+1}(a,b)| < \varepsilon.
$$
If \(|x - a| < \delta, |y - b| < \delta\) and \(x \geq y\) then we have
\[
\mu(x, y) \leq a_{2n}(x, y) < a_{2n}(a, b) + \varepsilon < \mu(a, b) + 2\varepsilon,
\]
i.e.,
\[
|\mu(x, y) - \mu(a, b)| < 2\varepsilon.
\]
If \(|x - a| < \delta, |y - b| < \delta\) and \(x \leq y\) then we have
\[
\mu(x, y) \geq a_{2n+1}(x, y) > a_{2n+1}(a, b) - \varepsilon > \mu(a, b) - 2\varepsilon;
\]
i.e.,
\[
|\mu(x, y) - \mu(a, b)| < 2\varepsilon.
\]
Hence \(\mu\) is continuous at \((a, b)\). It is clear that
\[
\min(x, y) \leq \mu(x, y) \leq \max(x, y),
\]
\[
\mu(tx, ty) = t\mu(x, y),
\]
for any \(x, y, t \in \mathbb{R}_+^*\).

Let \(K\) be any compact subset of \(\mathbb{R}_+^* \times \mathbb{R}_+^*\), and \(K_+\) and \(K_-\) be closed subsets of \(K\) given as \(\{(x, y) \in K \mid \pm(x - y) \geq 0\}\), respectively. Since \(\mu\) is continuous on \(\mathbb{R}_+^* \times \mathbb{R}_+^*\) and the sequences \(\{a_{2n+1}\}\) and \(\{a_{2n}\}\) are monotonous on \(K_+\) (resp. \(K_-\)), they uniformly converge to \(\mu\) on the compact subset \(K_+\) (resp. \(K_-\)) by Dini’s theorem. Thus \(\{a_n\}\) uniformly converges to \(\mu\) on the compact subset \(K\).

The key observation about \(m_1 \circ m_2\) is the following fact in [BB2].

**Fact 1 (Invariant principle)** Suppose that the compound \(m_1 \circ m_2\) of two means \(m_1\) and \(m_2\) exists. Then \(m_1 \circ m_2\) is the unique mean \(\mu\) satisfying
\[
\mu(m_1(a, b), m_2(a, b)) = \mu(a, b)
\]
for any \(a, b \in \mathbb{R}_+^*\).
3 The hypergeometric function and mean iterations

The hypergeometric function $F(\alpha, \beta, \gamma; z)$ with parameters $\alpha, \beta, \gamma$ is defined as

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (1)_n} z^n,$$

where the variable $z$ is in $\{z \in \mathbb{C} \mid |z| < 1\}$, $\gamma \neq 0, -1, -2, \ldots$, and $(\alpha)_n = \alpha (\alpha + 1) \cdots (\alpha + n - 1) = \Gamma(\alpha + n) / \Gamma(\alpha)$. This function admits an integral representation of Euler type

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha} (1-t)^{\gamma-\alpha} (1-zt)^{\beta} \frac{dt}{t(1-t)},$$

and satisfies the hypergeometric differential equation

$$z(1-z) \frac{d^2F}{dz^2} + \left[ \gamma - (\alpha + \beta + 1)z \right] \frac{dF}{dz} - \alpha \beta F = 0.$$

**Theorem 1** Suppose that the compound $m_1 \diamond m_2$ of two means $m_1$ and $m_2$ exists. If $m_1$ and $m_2$ satisfy $m_2(a, b)^p < 2m_1(a, b)^p$ and

$$\frac{m_1(a, b)}{F(\alpha, \beta, \gamma; 1 - \left(\frac{m_2(a, b)}{m_1(a, b)}\right)^p)^q} = \frac{a}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b}{a}\right)^p)^q}$$

for some $\alpha, \beta, \gamma, p, q \in \mathbb{R}$ and for any $a, b \in \mathbb{R}_+$ with $b^p < 2a^p$, then we have

$$m_1 \diamond m_2(a, b) = \frac{a}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b}{a}\right)^p)^q}.$$  

**Proof.** Let $\{a_n\}$ and $\{b_n\}$ be the $(m_1, m_2)$-sequence with initial $(a, b)$. The equality (4) implies that

$$\frac{a_0}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b_0}{a_0}\right)^p)^q} = \frac{a_1}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b_1}{a_1}\right)^p)^q} = \frac{a_2}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b_2}{a_2}\right)^p)^q} = \cdots = \frac{a_n}{F(\alpha, \beta, \gamma; 1 - \left(\frac{b_n}{a_n}\right)^p)^q}.$$
Let \( n \to \infty \), then we have
\[
\frac{a}{F(\alpha, \beta, \gamma; 1 - (\frac{b}{a})^p)} = \frac{\lim_{n \to \infty} a_n}{F(\alpha, \beta, \gamma; 1 - \lim_{n \to \infty} (\frac{b_n}{a_n})^p)} = m_1 \circ m_2(a, b),
\]
since \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = m_1 \circ m_2(a, b) \) and \( F(\alpha, \beta, \gamma; 0) = 1 \). \( \square \)

**Corollary 1** Suppose that the compound \( m_1 \circ m_2 \) of two means \( m_1 \) and \( m_2 \) exists and that it satisfies (5) for \( a, b \in \mathbb{R}_+^* \) such that \( b/a \) is sufficiently near to 1. If
\[
m'_{1}(x, y) = m_1(x^r, y^r)^{(s-t)/r} m_2(x^r, y^r)^{t/r} x^{1-s+t} y^{-t},
m'_{2}(x, y) = m_1(x^r, y^r)^{(s-t-1)/r} m_2(x^r, y^r)^{(t+1)/r} x^{1-s+t} y^{-t},
\]
are means for given \( r(\neq 0), s, t \in \mathbb{R} \), and the compound \( m'_1 \circ m'_2 \) exists for such \( a, b \in \mathbb{R}_+^* \), then we have
\[
m'_1 \circ m'_2(a, b) = \frac{a^{t+1}}{b^t F(\alpha, \beta, \gamma; 1 - (\frac{b}{a})^pr)^{qs/r}}.
\]

**Proof.** By Fact 1, we have the equality (4). Since
\[
\frac{m'_2(a, b)}{m'_1(a, b)} = \frac{m_2(a^r, b^r)^{1/r}}{m_1(a^r, b^r)^{1/r}},
\]
we can easily obtain
\[
\frac{m'_1(a, b)^{t+1}}{m'_2(a, b)^t} = \frac{a^{t+1}}{b^t F(\alpha, \beta, \gamma; 1 - (\frac{b}{a})^pr)^{qs/r}}.
\]
Fact 1 implies this theorem. \( \square \)

**Remark 1** Though \( m'_1(x, y) \) and \( m'_2(x, y) \) do not satisfy the condition
\[
\min(x, y) \leq m'_i(x, y) \leq \max(x, y) \quad (i = 1, 2)
\]
for some \( r, s, t \) in Corollary 1, it occurs that the double sequence \( \{a_n\} \) and \( \{b_n\} \) obtained by \( m'_1(x, y) \) and \( m'_2(x, y) \) has a non-zero common limit expressed by the hypergeometric function.
Corollary 2 Suppose that the compound \( m_1 \circ m_2 \) of two means \( m_1 \) and \( m_2 \) exists and that it satisfies (5) for \( a, b \in \mathbb{R}_+^* \) such that \( b/a \) is sufficiently near to 1. If the compound \( m'_1 \circ m'_2 \) of \( m'_1(x, y) = m_2(y, x) \) and \( m'_2(x, y) = m_1(y, x) \) exists for such \( a, b \in \mathbb{R}_+^* \), then we have

\[
m'_1 \circ m'_2(a, b) = \frac{a}{(b/a)^{pqa-1} F(\gamma - \beta, \alpha, \gamma; 1 - (b/a)^p)^q} \cdot \frac{a}{(b/a)^{pq\beta-1} F(\gamma - \alpha, \beta, \gamma; 1 - (b/a)^p)^q}.
\]

Proof. It is shown in [IKSY], p.38 that

\[
F(\alpha, \beta, \gamma; z) = (1 - z)^{-\alpha} F(\gamma - \beta, \alpha, \gamma; \frac{z}{z - 1})
\]

for \( z \in \mathbb{C} \) satisfying \( |z| < 1 \) and \( \text{Re}(z) < \frac{1}{2} \). By the first equality for \( z = 1 - b^p/a^p \) and for \( z = 1 - m_2(a, b)^p/m_1(a, b)^p \), we rewrite (4) as

\[
m_2(a, b)
\]

\[
\frac{(m_2(a, b)/m_1(a, b))^{1-pqa} F(\gamma - \beta, \alpha, \gamma; 1 - (m_1(a, b)/m_2(a, b))^{p})^q}{b}
\]

Recall that we give \( m'_1 \) and \( m'_2 \) by changing the role of \( x, y \) and that of \( m_1, m_2 \). Fact 1 for \( m'_1 \) and \( m'_2 \) implies

\[
m'_1 \circ m'_2(a, b) = \frac{a}{(b/a)^{pqa-1} F(\gamma - \beta, \alpha, \gamma; 1 - (b/a)^p)^q}.
\]

Similarly we can get the second expression of \( m'_1 \circ m'_2(a, b) \).

Let us explain how to utilize Theorem 1 and Corollary 1. The Gauss quadratic transformation formula is as follows:

\[
(1 + z)^{2\alpha} F(\alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}; z^2) = F(\alpha, \beta, 2\beta; \frac{4z}{(1 + z)^2}),
\]

(6)
where \( z \) is in a small neighbourhood of 0, and the value of \((1 + z)^{2\alpha}\) is 1 at \( z = 0 \). By substituting
\[
\frac{b}{a} = \frac{1 - z}{1 + z}, \quad \alpha = \beta = \frac{1}{2}
\]
into the equality (6), we have
\[
\frac{(a + b)/2}{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left(\frac{2\sqrt{ab}}{a+b}\right)^2\right)} = \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left(\frac{b}{a}\right)^2\right)}.
\]
Let \( m_1 \) be the arithmetic mean and \( m_2 \) the geometric mean. It is easy to show that the double sequence \( \{a_n\} \) and \( \{b_n\} \) defined by \((a_0, b_0) = (a, b)\), and
\[
(a_{n+1}, b_{n+1}) = (m_1(a_n, b_n), m_2(a_n, b_n)) = \left(\frac{a_n + b_n}{2}, \sqrt{a_n b_n}\right)
\]
has a common limit \( \mu(a, b) \), which is called the arithmetic-geometric mean of \( a \) and \( b \). Theorem 1 implies a well-known formula
\[
\mu(a, b) = \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left(\frac{b}{a}\right)^2\right)}
\]
for \( 0 < b \leq a \). By applying Corollary 1 for \((r, s, t) = (2, 1, 0)\) to (7), we have
\[
m'_1 \circ m'_2(a, b) = \frac{a}{\sqrt{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left(\frac{b}{a}\right)^4\right)}}
\]
for \( a \geq b > 0 \) and two means
\[
m'_1(x, y) = \sqrt{\frac{x^2 + y^2}{2}}, \quad m'_2(x, y) = \sqrt{xy}.
\]
By applying Corollary 1 for \((r, s, t) = (1, \frac{1}{2}, 0)\) to (7), we have
\[
m'_1 \circ m'_2(a, b) = \frac{a}{\sqrt{F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left(\frac{b}{a}\right)^2\right)}}
\]
for \( a \geq b > 0 \) and two means
\[
m'_1(x, y) = \sqrt{\frac{x + y}{2}}, \quad m'_2(x, y) = \sqrt{\frac{2xy}{x + y}}.
\]
4 Compounds of means by quadratic transformation formulas

In 1881 Goursat gave a list of transformation formulas of the form

\[ F(\alpha, \beta, \gamma; z) = \varphi(z) F(\alpha', \beta', \gamma'; \psi(z)), \]

in [G], where \( \varphi(z) \) and \( \psi(z) \) are algebraic functions with values 1 and 0 at \( z = 0 \), respectively. In this section, we give a list of the compound means expressed by the hypergeometric function derived from Theorem 1 and quadratic transformation formulas G(25), . . . , G(52) in [G].

It turns out that parameters \((\alpha, \beta, \gamma)\) of the hypergeometric function satisfy

\[ \left\{ \frac{1}{1 - \gamma}, \frac{1}{\gamma - \alpha - \beta}, \frac{1}{\alpha - \beta} \right\} = \{2, 2, \infty\}, \text{ or } \{2, 4, 4\}, \text{ or } \{\infty, \infty, \infty\}, \]

for our consideration. We classify our results by these data. For the case \( \{\infty, \infty, \infty\} \), we have the classical arithmetic-geometric mean explained in the previous section.

**Theorem 2** We have the following table.

<table>
<thead>
<tr>
<th>No.</th>
<th>( m_1(a, b) )</th>
<th>( m_2(a, b) )</th>
<th>type</th>
<th>( m_1 \circ m_2(a, b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q(1)</td>
<td>( \sqrt{ab} )</td>
<td>( \sqrt{\frac{a+b}{2}+\sqrt{ab}} )</td>
<td>(M)</td>
<td>( {a/F} \left(1, 1, \frac{3}{2}; 1-\frac{b}{a}\right) )</td>
</tr>
<tr>
<td>Q(2)</td>
<td>( \frac{a}{\sqrt{\frac{a+b}{2}}} )</td>
<td>( \sqrt{ab} )</td>
<td>(M)</td>
<td>( {a/F} \left(1, \frac{1}{2}, \frac{3}{2}; 1-\frac{b}{a}\right) )</td>
</tr>
<tr>
<td>Q(3)</td>
<td>( \frac{a+b}{2} )</td>
<td>( \frac{a+b}{2} )</td>
<td>(M)</td>
<td>( {a/F} \left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1-\left(\frac{b}{a}\right)^2\right) )</td>
</tr>
<tr>
<td>Q(4)</td>
<td>( \frac{2ab}{a+b} \sqrt{a+b} )</td>
<td>( \frac{a+b}{2} \sqrt{a+b} )</td>
<td>(M)</td>
<td>( {a/F} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1-\left(\frac{b}{a}\right)^2\right)^2 = \sqrt{ab} )</td>
</tr>
<tr>
<td>Q(5)</td>
<td>( \frac{2ab}{a+b} )</td>
<td>( \frac{a+b}{2} )</td>
<td>(M)</td>
<td>( {a/F} \left(\frac{1}{2}, 1, 1, 1-\left(\frac{b}{a}\right)^2\right) = \sqrt{ab} )</td>
</tr>
</tbody>
</table>
\[
\{2, 4, 4\}
\]

<table>
<thead>
<tr>
<th>No.</th>
<th>(m_1(a, b))</th>
<th>(m_2(a, b))</th>
<th>type</th>
<th>(m_1 \circ m_2(a, b))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q(6)</td>
<td>(\sqrt{b(\sqrt{a} + \sqrt{b})} \over 2)</td>
<td>(\sqrt{ab})</td>
<td>(A)</td>
<td>(a/F\left(1, \frac{3}{4}; \frac{5}{4}; 1 - \frac{b}{a}\right))</td>
</tr>
<tr>
<td>Q(7)</td>
<td>(\sqrt{ab})</td>
<td>(\sqrt{b(\sqrt{a} + \sqrt{b})} \over 2)</td>
<td>(A)</td>
<td>(a/F\left(1, \frac{1}{2}; \frac{5}{4}; 1 - \frac{b}{a}\right))</td>
</tr>
<tr>
<td>Q(8)</td>
<td>(\sqrt{\frac{b(a+b)}{2}})</td>
<td>(\frac{a+b}{2})</td>
<td>(A)</td>
<td>(a/F\left(\frac{1}{4}; \frac{3}{4}; \frac{5}{4}; 1 - \left(\frac{b}{a}\right)^2\right)^2)</td>
</tr>
<tr>
<td>Q(9)</td>
<td>(\frac{a+b}{2})</td>
<td>(\sqrt{\frac{a(a+b)}{2}})</td>
<td>(A)</td>
<td>(a/F\left(\frac{1}{4}; \frac{1}{2}; \frac{5}{4}; 1 - \left(\frac{b}{a}\right)^2\right)^2)</td>
</tr>
<tr>
<td>Q(10)</td>
<td>(\sqrt{\frac{2ab}{a+b}} ab^2)</td>
<td>(\sqrt{\frac{a(a+b)}{2}} a^2 b)</td>
<td>(A)</td>
<td>(a/F\left(\frac{1}{4}; \frac{3}{4}; \frac{3}{4}; 1 - \left(\frac{b}{a}\right)^2\right) = \sqrt{ab})</td>
</tr>
<tr>
<td>Q(11)</td>
<td>(\sqrt{\frac{2ab}{a+b}} ab^2)</td>
<td>(\sqrt{\frac{2ab}{a+b}} a^2 b)</td>
<td>(A)</td>
<td>(a/F\left(\frac{1}{4}; \frac{3}{4}; \frac{3}{4}; 1 - \left(\frac{b}{a}\right)^2\right)^2 = \sqrt{ab})</td>
</tr>
</tbody>
</table>

Here \(b/a\) is sufficiently near to 1, the type (M) means the \((m_1, m_2)\)-sequence is monotonous, i.e., they satisfy

\[
b_n \leq b_{n+1} \leq a_{n+1} \leq a_n \quad \text{or} \quad b_n \geq b_{n+1} \geq a_{n+1} \geq a_n;
\]

the type (A) means the \((m_1, m_2)\)-sequence is alternative, i.e., they satisfy

\[
b_0 \leq a_1 \leq b_2 \leq \cdots \leq b_{2n} \leq a_{2n+1} \leq b_{2n+1} \leq a_{2n} \leq \cdots \leq a_2 \leq b_1 \leq a_0.
\]

**Proof.** We show Q(3). The quadratic transformation formula G(41) in [G] is

\[
F(\alpha, 1 - \alpha, \gamma; z) = (1 - z)^{\gamma - 1} F\left(\frac{\gamma - \alpha}{2}, \frac{\gamma + \alpha - 1}{2}, \gamma; 4z(1 - z)\right)
\]

\[
= (1 - z)^{\gamma - 1}(1 - 2z) F\left(\frac{\gamma + \alpha}{2}, \frac{\gamma + 1 - \alpha}{2}, \gamma; 4z(1 - z)\right).
\]

Substitute

\[
\alpha = \frac{1}{2}, \quad \gamma = \frac{3}{2}, \quad \frac{b}{a} = 1 - 2z,
\]

into the first row of this formula, then we have

\[
m_1(a, b) \quad \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1 - \left(\frac{m_2(a,b)}{m_1(a,b)}\right)^2\right)} = \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1 - \left(\frac{b}{a}\right)^2\right)}.
\]
where
\[ m_1(a, b) = \sqrt{\frac{a(a+b)}{2}}, \quad m_2(a, b) = \frac{a+b}{2}. \]

We can easily show that the \((m_1, m_2)\)-sequence converges and has a common limit by Lemma 1. Theorem 1 implies that
\[ m_1 \circ m_2(a, b) = \frac{a}{F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1 - \left(\frac{b}{a}\right)^2\right)}. \]

We list used formulas in [G] and substitutions to prove this proposition:

| Q(1) | \[\alpha = \beta = 1, b/a = (1-2z)^2,\] |
| G(38) : \[F(\alpha, \beta, \frac{a+b+1}{2}; z) = (1-2z)\frac{a+b+1}{2}; z = 4z(1-z)\] |
| Q(2) | \[\alpha = 1/2, \gamma = 3/2, \quad b/a = 1 - z,\] |
| G(35) : \[F(\alpha, \alpha + 1, \gamma; z) = \left(\frac{1+\sqrt{1-2z}}{2}\right)^{-2\alpha}F(2\alpha, 2\alpha + 1 - \gamma, \gamma; \frac{1+\sqrt{1-2z}}{1+\sqrt{1-2z}})\] |
| Q(3) | \[\alpha = 1/2, \gamma = 3/2, \quad b/a = 1 - 2z,\] |
| G(41) : \[F(\alpha, 1 - \alpha, \gamma; z) = (1-z)^{\gamma-1}F\left(\frac{\gamma-a}{2}, \frac{\gamma+a-1}{2}, \gamma; 4z(1-z)\right)\] |
| Q(4) | \[\alpha = 1/2, \gamma = 1/2, \quad b/a = 1 - 2z,\] |
| G(41) : \[F(\alpha, 1 - \alpha, \gamma; z) = \left(\frac{1-2z}{(1-z)^{\gamma-1}}\right)F\left(\frac{\gamma-a}{2}, \frac{\gamma+a-1}{2}, \gamma; 4z(1-z)\right)\] |
| Q(5) | \[\alpha = 1, \beta = 1/2, \quad b/a = 1 - z,\] |
| G(44) : \[F(\alpha, \beta, 2\beta; z) = \frac{1-z}{(1-z)^{\alpha+1/2}}F\left(\beta, \frac{1-\alpha}{2}, 1+\beta, \frac{1}{2}, \frac{z^2}{(1-z)^{\gamma-1}}\right)\] |
| Q(6) | \[\alpha = 1, \beta = 3/4, \quad b/a = (1+z)^2/(1-z)^2,\] |
| G(49) : \[F(\alpha, \beta, \alpha - \beta + 1; z) = \frac{1+z}{(1-z)^{\alpha+1/2}}F\left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1 - \beta, \alpha - \beta + 1, \frac{-4z}{(1-z)^2}\right)\] |
| Q(7) | \[\alpha = 1, \beta = 1/2, \quad b/a = 1/(1-2z),\] |
| G(39) : \[F(\alpha, \beta, \alpha + 1; z) = (1-2z)^{-\alpha}F\left(\frac{a+1}{2}, \frac{a+1}{2}, \frac{a+1}{2}; \frac{4z(1-z)}{(2-1)\gamma}\right)\] |
| Q(8) | \[\alpha = 3/4, \gamma = 5/4, \quad b/a = 1/(1-2z),\] |
| G(42) : \[F(\alpha, \alpha - \beta + 1; z) = \frac{1-2z}{(1-2z)^{\gamma-1}}F\left(\frac{\gamma-a}{2}, \frac{\gamma+a-1}{2}, \gamma; \frac{-4z(1-z)}{(1-2z)^2}\right)\] |
| Q(9) | \[\alpha = 1/2, \beta = 1/4, \quad b/a = (1+z)/(1-z),\] |
| G(48) : \[F(\alpha, \beta, \alpha - \beta + 1; z) = (1-z)^{-\alpha}F\left(\frac{\alpha+1-2\beta}{2}, 2\alpha + 1 - \beta, \alpha - \beta + 1, \frac{-4z}{(1-z)^2}\right)\] |
| Q(10) | \[\alpha = 1/4, \gamma = 3/4, \quad b/a = 1/(1-2z),\] |
| G(42) : \[F(\alpha, 1 - \alpha, \gamma; z) = \frac{(1-z)^{\gamma-1}}{(1-2z)^{\gamma-1}}F\left(\frac{\gamma-a}{2}, \frac{\gamma+a-1}{2}, \gamma; \frac{-4z(1-z)}{(1-2z)^2}\right)\] |
| Q(11) | \[\alpha = 1/2, \beta = 3/4, \quad b/a = (1+z)/(1-z),\] |
| G(49) : \[F(\alpha, \beta, \alpha - \beta + 1; z) = \frac{1+z}{(1-z)^{\alpha+1/2}}F\left(\frac{\alpha+1}{2}, \frac{\alpha+1}{2}, 1 - \beta, \alpha - \beta + 1, \frac{-4z}{(1-z)^2}\right)\] |

Here we remark that the formulas (G38), (G41) and (G42) consist of some equalities.
Lemma 2 We have
\[
F \left( \frac{\gamma - \alpha}{2}, \frac{\gamma + \alpha - 1}{2}, \gamma; 1-t^2 \right) = tF \left( \frac{\gamma + \alpha}{2}, \frac{\gamma + 1 - \alpha}{2}, \gamma; 1-t^2 \right),
\]
\[
F \left( \frac{\alpha}{2}, \frac{\alpha + 1}{2}, \beta + 1; 1-t^2 \right) = t^{2(\beta - \alpha)} F \left( \beta - \frac{\alpha}{2}, \beta + \frac{1 - \alpha}{2}, \beta + 1; 1-t^2 \right),
\]
for \( t \) in a small neighbourhood of 1. Especially,
\[
F \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1-t^2 \right) = tF \left( 1, 1, \frac{3}{2}; 1-t^2 \right),
\]
\[
F \left( \frac{1}{2}, \frac{1}{2}, \frac{5}{4}; 1-t^2 \right) = tF \left( \frac{3}{4}, 1, \frac{5}{4}; 1-t^2 \right),
\]
\[
F \left( \frac{1}{4}, \frac{5}{4}, \frac{1}{4}; 1-t^4 \right) = tF \left( \frac{1}{2}, 1, \frac{5}{4}; 1-t^4 \right).
\]

Proof. By substituting \( t = 1 - 2z \) into the formula (G41), and \( t = \frac{2\sqrt{1-z}}{2-2z} \) into (G45), we obtain the first and second equalities in this lemma, respectively. In order to get the rest, put \((\alpha, \gamma) = (1/2, 3/2)\) and \((\alpha, \gamma) = (1/4, 5/4)\) in the first equality, and \((\alpha, \beta) = (1/2, 3/4), \sqrt{t} = t'\) in the second. \( \square \)

Remark 2 Carlson studied in [C] compound means of two means taken from the following four means:
\[
m_1(x, y) = \frac{x + y}{2}, \quad m_2(x, y) = \sqrt{xy},
\]
\[
m_3(x, y) = \sqrt{\frac{x + y}{2}}, \quad m_4(x, y) = \sqrt{\frac{x + y}{2}y}.
\]
Refer also to §8.5 in [BB2] for these results. Note that the compound mean \( m_1 \circ m_2 \) is the classical arithmetic-geometric mean. It is shown that the compound means \( m_3 \circ m_4(a, b) \) and \( m_4 \circ m_3(a, b) \) are expressed as
\[
\sqrt{\frac{a^2 - b^2}{2\log(a/b)}},
\]
which is called Carlson’s log expression. The other compound means \( m_i \circ m_j \) can be expressed by the hypergeometric function by Theorem 2, Corollary 1 and Lemma 2.
For example, $Q(3)$ in Theorem 2 coincides with the expression of $m_3 \odot m_1$ shown in [BB2] and [C]. The compound mean $m_2 \odot m_4$ is expressed as

$$a \sqrt{\frac{\alpha}{\pi}} F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - \left(\frac{b}{a}\right)^2\right)^{1/2}$$

by Exercises 1 of §8.5 in [BB2]. This result is obtained by $Q(1)$ in Theorem 2, Corollary 1 for $(r, s, t) = (2, 1, 0)$ and Lemma 2.

Carlson’s log expression can be obtained by the following functional equation for the hypergeometric function.

**Lemma 3** For $\alpha, n \in \mathbb{C}$ and $x$ in a small neighbourhood of 1, we have

$$n(1-x)F(n(\alpha-1)+1, 1, 1; 1-x) = (1-x^n)F(\alpha, 1; 1-x^n) = \frac{x^{(1-\alpha)n} - 1}{\alpha - 1},$$

where the value of $x^n$ is 1 at $x = 1$. Especially, if $\alpha = 1$ and $n \in \mathbb{N}$ then it reduces

$$F(1, 1, 2; 1-x) = \left(1 + \frac{x + x^2 + \cdots + x^{n-1}}{n}\right) F(1, 1, 2; 1-x^n) = \frac{\log x}{x-1}.$$  

**Proof.** It is easy to show that the functions $n(1-x)F(n(\alpha-1)+1, 1, 1; 1-x)$ and $(1-x^n)F(\alpha, 1; 1-x^n)$ satisfy the differential equation

$$\frac{d^2 \varphi}{dx^2} = -\frac{n(\alpha-1) + 1}{x} \frac{d \varphi}{dx}$$

with initial conditions $\varphi(1) = 0$ and $\frac{d \varphi}{dx}(1) = -n$. Thus these functions coincide with $(x^{(1-\alpha)n} - 1)/(\alpha - 1)$. Note that this function converges to $-n \log x$ as $\alpha \to 1$. \hfill $\Box$

By Lemma 3 for $\alpha = 1, n = 2$ and $b/a = x$, we have

$$a \sqrt{F\left(1, 1, 2; 1 - \left(\frac{b}{a}\right)^2\right)} = \frac{m_3(a, b)}{\sqrt{F\left(1, 1, 2; 1 - \left(\frac{m_3(a, b)}{m_4(a, b)}\right)^2\right)}}$$

$$= \frac{m_4(a, b)}{\sqrt{F\left(1, 1, 2; 1 - \left(\frac{m_3(a, b)}{m_4(a, b)}\right)^2\right)}}.$$  

Theorem 1 implies Carlson’s log expression.
5 Compounds of means by cubic transformation formulas

We give a list of the compound means expressed by the hypergeometric function derived from cubic transformation formulas (G78), . . . , (G125) in [G], Theorem 1 and Corollary 2.

Theorem 3 We have the following table:

<table>
<thead>
<tr>
<th>No.</th>
<th>$m_1(a, b)$</th>
<th>$m_2(a, b)$</th>
<th>type</th>
<th>$m_1 \circ m_2(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C(1)</td>
<td>$b^\frac{1}{3}X_1$</td>
<td>$b^\frac{1}{3}X_2$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, 1, \frac{2}{3}; 1 - (\frac{b}{a})^2\right)$</td>
</tr>
<tr>
<td>C(2)</td>
<td>$X_1X_2^2$</td>
<td>$X_2^3$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{2}{3}, \frac{7}{6}; 1 - (\frac{b}{a})^2\right)^3$</td>
</tr>
<tr>
<td>C(3)</td>
<td>$X_1X_2^2$</td>
<td>$X_2^3X_3$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; 1 - (\frac{b}{a})^2\right)$</td>
</tr>
<tr>
<td>C(4)</td>
<td>$b^\frac{1}{3}X_1X_2$</td>
<td>$b^\frac{1}{3}X_2X_3$</td>
<td>(A)</td>
<td>$a/F\left(\frac{5}{6}, 1, \frac{2}{3}; 1 - (\frac{b}{a})^2\right)^\frac{1}{2}$</td>
</tr>
<tr>
<td>C(5)</td>
<td>$b^\frac{1}{3}X_1$</td>
<td>$b^\frac{1}{3}X_3$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{1}{2}, \frac{7}{6}; 1 - (\frac{b}{a})^2\right) = \sqrt{ab^2}$</td>
</tr>
<tr>
<td>C(6)</td>
<td>$a^\frac{1}{2}Y_2$</td>
<td>$a^\frac{1}{2}Y_1$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1 - (\frac{b}{a})^2\right)$</td>
</tr>
<tr>
<td>C(7)</td>
<td>$Y_2^2$</td>
<td>$Y_1Y_2^2$</td>
<td>(A)</td>
<td>$a/F\left(\frac{2}{3}, 1, \frac{2}{3}; 1 - (\frac{b}{a})^2\right)^3$</td>
</tr>
<tr>
<td>C(8)</td>
<td>$Y_2^2Y_3$</td>
<td>$Y_1Y_2^2$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; 1 - (\frac{b}{a})^2\right)$</td>
</tr>
<tr>
<td>C(9)</td>
<td>$a^\frac{1}{2}Y_2Y_3$</td>
<td>$a^\frac{1}{2}Y_1Y_2$</td>
<td>(A)</td>
<td>$a/F\left(\frac{2}{3}, 1, \frac{3}{2}; 1 - (\frac{b}{a})^2\right)^\frac{1}{2}$</td>
</tr>
<tr>
<td>C(10)</td>
<td>$a^\frac{1}{2}Y_3$</td>
<td>$a^\frac{1}{2}Y_1$</td>
<td>(A)</td>
<td>$a/F\left(\frac{1}{3}, \frac{1}{2}, \frac{3}{2}; 1 - (\frac{b}{a})^2\right) = \sqrt{a^2b}$</td>
</tr>
</tbody>
</table>

where $b/a$ is sufficiently near to 1,

\[X_1 = \frac{\xi_1^3 + \xi_2^3}{2}, \quad X_2 = \sqrt{\frac{\xi_1^3 + \xi_1^3\xi_2^3 + \xi_2^3}{3}}, \quad X_3 = \sqrt{\xi_1^3 - \xi_1^3\xi_2^3 + \xi_2^3},\]

$(\xi_1, \xi_2)$ is the preimage of $(a, b)$ under the arithmetic and geometric means:

\[\frac{\xi_1 + \xi_2}{2} = a, \quad \sqrt{\xi_1\xi_2} = b, \quad \{\xi_1, \xi_2\} = \{a \pm \sqrt{a^2 - b^2}\},\]

and $-\frac{\pi}{6} < \arg(\xi_i^3) < \frac{\pi}{6}$ $(i = 1, 2), \quad \xi_1^3\xi_2^3 = b^2 \in \mathbb{R}_+^*$ ;

\[Y_1 = \frac{\eta_1^3 + \eta_2^3}{2}, \quad Y_2 = \sqrt{\frac{\eta_1^3 + \eta_1^3\eta_2^3 + \eta_2^3}{3}}, \quad Y_3 = \sqrt{\eta_1^3 - \eta_1^3\eta_2^3 + \eta_2^3},\]
\((\eta_1, \eta_2)\) is the preimage of \((a, b)\) under the geometric and arithmetic means:

\[
\sqrt{\eta_1 \eta_2} = a, \quad \frac{\eta_1 + \eta_2}{2} = b, \quad \{\eta_1, \eta_2\} = \{b \pm \sqrt{b^2 - a^2}\},
\]

and \(-\frac{\pi}{6} < \arg(\eta_i) < \frac{\pi}{6} \ (i = 1, 2), \ \eta_1 \eta_2 = a^2 \in \mathbb{R}_+^*.

**Remark 3** Parameters \((\alpha, \beta, \gamma)\) of the hypergeometric function in Theorem 3 satisfy

\[
\left\{ \begin{array}{c}
\frac{1}{|1 - \gamma|}, \quad \frac{1}{|\gamma - \alpha - \beta|}, \quad \frac{1}{|\alpha - \beta|}
\end{array} \right\} = \{2, 3, 6\}.
\]

We prepare two lemmas.

**Lemma 4** If \(b < a\) then

\[
b < b^{\frac{3}{2}} X_1 < b^{\frac{3}{2}} X_2 < b^{\frac{3}{2}} X_3 < X_2^2 X_3 < a,
\]

\[
b < Y_2^2 Y_3 < a^{\frac{3}{2}} Y_3 < a^{\frac{3}{2}} Y_2 < a^{\frac{3}{2}} Y_1 < a;
\]

if \(a < b\) then

\[
a < X_2^2 X_3 < b^{\frac{3}{2}} X_3 < b^{\frac{3}{2}} X_2 < b^{\frac{3}{2}} X_1 < b,
\]

\[
a < a^{\frac{3}{2}} Y_1 < a^{\frac{3}{2}} Y_2 < a^{\frac{3}{2}} Y_3 < Y_2^2 Y_3 < b.
\]

**Proof.** Suppose that \(b < a\). Since \(\xi_1, \xi_2\) are real and \(a = \frac{(\xi_1^3)^3 + (\xi_2^3)^3}{2}\), it is easy to show that

\[
b < b^{\frac{3}{2}} X_1 < b^{\frac{3}{2}} X_2 < b^{\frac{3}{2}} X_3 < X_2^2 X_3
\]

and

\[
a^2 - X_1^2 X_3^2 = \frac{1}{36}(5\xi_1^3 + 11\xi_1^3 \xi_2^3 + 5\xi_2^3)(\xi_1^3 - \xi_1^3 \xi_2^3 + \xi_2^3)(\xi_1^3 - \xi_2^3)^2 > 0.
\]

In order to show the other inequalities, we assume that \(a = 1\) by the homogeneity. Note that \(\eta_i\) do not belong to \(\mathbb{R}\) and that

\[
|\eta| = 1, \quad \text{Re}(\eta) = b, \quad -\frac{\pi}{2} < \arg(\eta) < \frac{\pi}{2}.
\]

If we take branches of \(\eta_i^\frac{1}{3}\) so that \(-\frac{\pi}{6} < \arg(\eta_i^\frac{1}{3}) < \frac{\pi}{6}\), then we have

\[
\eta_1^\frac{1}{3} \eta_2^\frac{1}{3} = 1, \quad \frac{\sqrt{3}}{2} < \frac{\eta_1^\frac{1}{3} + \eta_2^\frac{1}{3}}{2} = Y_1 < 1.
\]
Since
\[ b = 4Y_1^3 - 3Y_1, \quad Y_2 = \sqrt{\frac{4Y_1^2 - 1}{3}}, \quad Y_3 = \sqrt{4Y_1^2 - 3}, \]
we have
\[
\begin{align*}
Y_1^2 - Y_2^2 &= \frac{1}{3}(1 - Y_1^2) > 0, \\
Y_2^2 - Y_3^2 &= \frac{8}{3}(1 - Y_1^2) > 0, \\
Y_3 - Y_2^2Y_3 &= Y_3(1 - Y_2^2) > Y_3(Y_1^2 - Y_2^2) > 0, \\
Y_2^4Y_3^2 - b^2 &= \frac{1}{9}(1 - Y_1^2)(1 + 20Y_1^2)(4Y_1^2 - 3) > 0,
\end{align*}
\]
for \( \frac{\sqrt{3}}{2} < Y_1 < 1. \) We can similarly show the inequalities when \( a < b. \) \( \square \)

**Lemma 5** For any \( a, b \in \mathbb{R}^*_+, \) we have
\[
\frac{2\sqrt{2}}{3} < X_2 < \frac{2\sqrt{3}}{3}, \quad 0 < \frac{X_3}{X_1} < 2, \quad \frac{\sqrt{3}}{2} < \frac{Y_1}{Y_2} < \frac{3\sqrt{2}}{4}, \quad \frac{1}{2} < \frac{Y_1}{Y_3} < \infty.
\]

**Proof.** By the homogeneity of \( X_i, \) we normalize \( b = 1. \) Note that
\[
\frac{\sqrt{3}}{2} < X_1 < \infty,
\]
and that
\[
\frac{X_2}{X_1} = \sqrt{\frac{4X_1^2 - 1}{3X_1^2}}, \quad \frac{X_3}{X_1} = \sqrt{\frac{4X_1^2 - 3}{X_1^2}}
\]
are monotonous as functions of \( X_1. \) Consider their limits as \( X_1 \to \frac{\sqrt{3}}{2} \) and as \( X_1 \to \infty. \) Normalize \( a = 1 \) to show the inequalities for \( Y_1/Y_i. \) \( \square \)

**Proof of Theorem 3.** Lemmas 1 and 4 imply that the \((m_1, m_2)\)-sequence alternatively converges for \((m_1, m_2)\) in Theorem 3 and for any \( a, b \in \mathbb{R}^*_+. \) We show C(1). Substitute \( \alpha = 1/6 \) into the formula G(112):
\[
\begin{align*}
F\left(\alpha, \alpha + \frac{1}{2}, 2\alpha + \frac{5}{6}; z\right) &= (1 - z)^{\frac{1}{2}} F\left(\alpha + \frac{1}{3}, \alpha + \frac{5}{6}, 2\alpha + \frac{5}{6}; z\right) \\
&= (1 - 9\theta)^{2\alpha} F\left(3\alpha, 3\alpha + \frac{1}{2}, 2\alpha + \frac{5}{6}; t\right),
\end{align*}
\]
where $27t(1 - t)^2 + (1 - 9t)^2z = 0$. We have

$$F\left(\frac{1}{2}, 1, \frac{7}{6}; t\right) = \frac{3t + 1}{1 - 9t} F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \frac{(3t + 1)^3}{(1 - 9t)^2}\right).$$

Put $u = (3t + 1)/(1 - 9t)$, then $t = (u - 1)/(3(1 + 3u))$ and

$$F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \frac{4(1 + 2u)}{3(1 + 3u)}\right) = u F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \frac{4u^3}{1 + 3u}\right).$$

Solve the equation

$$\left(\frac{b}{a}\right)^2 = \frac{4u^3}{1 + 3u}$$

with variable $u$ for given $a, b > 0$. Then we have

$$u = \frac{1}{a} b^2 X_1, \quad \frac{1}{X_1} = \frac{X_1^2}{a},$$

$$\frac{4}{1 + 3u} \frac{1 + 2u}{3} = \left(\frac{b}{a}\right)^2 \frac{1}{u^2} \frac{1}{3} + \frac{2}{3} = \frac{1}{X_1^2} \frac{X_1^2 + 2b^2}{3} = \frac{X_2^2}{X_1^2},$$

and

$$m_1(a, b) \bigg/F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \left(\frac{m_2(a, b)}{m_1(a, b)}\right)^2\right) = \frac{a}{F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \left(\frac{a}{b}\right)^2\right)}$$

for

$$m_1(a, b) = b^2 X_1, \quad m_2(a, b) = b^2 X_2.$$

Theorem 1 implies

$$m_1 \odot m_2(a, b) = \frac{a}{F\left(\frac{1}{2}, 1, \frac{7}{6}; 1 - \left(\frac{b}{a}\right)^2\right)}.$$
In order to get (C2), . . . , (C5), we use the following.

<table>
<thead>
<tr>
<th>C(2)</th>
<th>α = 1/6, u = 2(1 + 3z)/(1 − 9z), (b/a)^2 = u^3/(3u + 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>G(119)</td>
<td>(1 − z)^2 F(1/3 − α, 5/6 − α, 2α + 5; z) = (1 − 9z)^2 F(α, α + 1/2, 2α + 5, −27z(1 − z)^2/(1 − 9z)^2)</td>
</tr>
<tr>
<td>C(3)</td>
<td>α = 0, t = 1 − 4/(1 + 3u), (b/a)^2 = 4u^3/(1 + 3u),</td>
</tr>
<tr>
<td>G(87)</td>
<td>F(α + 1/2, 2, α, 3/2, z) = \frac{\phi(1 − 3u)^{2a + 1}}{2a + 1} F(3α + 1, α + 2, 3, 2, t)</td>
</tr>
<tr>
<td>(t − 9)^2 t + 27(1 − t)^2 z = 0</td>
<td></td>
</tr>
<tr>
<td>C(4)</td>
<td>α = 1/6, t = 1 − 4/(1 + 3u), (b/a)^2 = 4u^3/(1 + 3u),</td>
</tr>
<tr>
<td>G(87)</td>
<td>(1 − z)^2 F(1 − α, α + 5/6, 3/2, z) = \frac{\phi(1 − 3u)^{2a + 1}}{2a + 1} F(3α + 1, α + 2, 3, 2, t)</td>
</tr>
<tr>
<td>(t − 9)^2 t + 27(1 − t)^2 z = 0</td>
<td></td>
</tr>
<tr>
<td>C(5)</td>
<td>α = 1/6, t = 1 − 4/(1 + 3u), (b/a)^2 = 4u^3/(1 + 3u),</td>
</tr>
<tr>
<td>G(86)</td>
<td>(1 − z)^2 F(1/2 − α, α + 1/3, 1/2, z) = (1 − t)^2 F(3α, α + 1, 1/2, t)</td>
</tr>
<tr>
<td>(t − 9)^2 t + 27(1 − t)^2 z = 0</td>
<td></td>
</tr>
</tbody>
</table>

We remark that formulas G(86), G(87) and G(119) consist of some equalities. The equality C(5 + k) is obtained by Corollary 2 for C(k) (k = 1, . . . , 5).

6 Compounds of means by transformation formulas for \(3F_2\)

The generalized hypergeometric function \(3F_2\) is defined as

\[
3F_2 \left( \frac{\alpha_0, \alpha_1, \alpha_2}{\beta_1, \beta_2}; z \right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n (\alpha_1)_n (\alpha_2)_n}{(\beta_1)_n (\beta_2)_n} z^n ,
\]

where \(\beta_1, \beta_2 \neq 0, -1, -2, \ldots, \) and \(|z| < 1\). Note that this function reduces to the hypergeometric function \(F(\alpha_0, \alpha_1, \beta_1; z)\) when \(\alpha_2 = \beta_2\). In this section, we attempt to find pairs of means whose compounds can be expressed by \(3F_2\) by using transformation formulas for \(3F_2\) in [K].

**Proposition 1** We have functional equations of the form

\[
3F_2 \left( \frac{\alpha_0, \alpha_1, \alpha_2}{\beta_1, \beta_2}; z \right) = \varphi(z) 3F_2 \left( \frac{\alpha_0, \alpha_1, \alpha_2}{\beta_1, \beta_2}; \psi(z) \right), \quad (8)
\]
where \(\{\alpha_0, \alpha_1, \alpha_2\}\), \(\{\beta_1, \beta_2\}\), \(\varphi(z)\) and \(\psi(z)\) are given as

\[
\begin{array}{|c|c|c|c|}
\hline
\text{No.} & \{\alpha_0, \alpha_1, \alpha_2\} & \{\beta_1, \beta_2\} & \varphi(z) \\psi(z) \\
\hline
\text{K(1)} & \{\frac{1}{2}, \frac{3}{4}, 1\} & \{\frac{5}{6}, \frac{7}{4}\} & \frac{1}{1-z} & 1 - \left(\frac{1+z}{1-z}\right)^2 \\
\text{K(2)} & \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\} & \{\frac{3}{4}, \frac{5}{4}\} & \frac{1}{\sqrt{1-z}} & 1 - \left(\frac{1+z}{1-z}\right)^2 \\
\text{K(3)} & \{\frac{2}{3}, \frac{5}{3}, 1\} & \{\frac{7}{6}, \frac{1}{3}\} & \frac{1}{1-4z} & 1 - \frac{(1-z)(1+8z)^2}{(1-4z)^3} \\
\text{K(4)} & \{\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\} & \{\frac{5}{6}, \frac{7}{6}\} & \frac{1}{\sqrt{1-4z}} & 1 - \frac{(1-z)(1+8z)^2}{(1-4z)^3} \\
\hline
\end{array}
\]

**Proof.** We can easily show these functional equations by the formulas (2.1) and (2.2) in [K].

**Remark 4** Non-trivial functional equations of the form (8) can not directly obtained any more by the formulas (2.1), . . . , (2.5) in [K].

Note that each \(3F2\) in the functional equations K(2) and K(4) has a common parameter in the sets \(\{\alpha_0, \alpha_1, \alpha_2\}\) and \(\{\beta_1, \beta_2\}\). Thus these functional equations reduce to

\[
3F2\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; z\right) = \frac{1}{\sqrt{1-z}} F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \left(\frac{1+z}{1-z}\right)^2\right), \tag{9}
\]

\[
3F2\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; z\right) = \frac{1}{\sqrt{1-4z}} F\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{(1-z)(1+8z)^2}{(1-4z)^3}\right), \tag{10}
\]

which appear when we study Q(9) and C(6), respectively.

By the Clausen formula

\[
3F2\left(\begin{array}{c}
2\alpha, 2\beta, \alpha + \beta \\
2\alpha + 2\beta, \alpha + \beta + 1/2
\end{array}; z\right) = F(\alpha, \beta, \alpha + \beta + 1/2; z)^2,
\]

we have

\[
3F2\left(\begin{array}{c}
1/2, 3/4, 1 \\
5/4, 3/2
\end{array}; z\right) = F(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; z)^2,
\]

\[
3F2\left(\begin{array}{c}
1/3, 2/3, 1 \\
7/6, 4/3
\end{array}; z\right) = F(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; z)^2.
\]

Thus the functional equations K(1) and K(3) reduce to (9) and (10), respectively.

Hence we conclude that proper expressions of compounds of means by \(3F2\) can not directly obtained by transformation formulas for \(3F2\) in [K].
References


Mean iterations and hypergeometric function

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