The Stable Types and Minimal Whitney Stratifications of Discriminants and Euler Obstruction.

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In this work we give relations between the minimal Whitney stratification and the stratification given by stable types for discriminants of finitely determined map germs.
Introduction

The minimal Whitney stratification for any reduced equidimensional complex analytic space $V$ is a consequence of Teissier’s work ([11]), and was defined in a joint work of Lê and Teissier in terms of the invariance of a set of multiplicities.
On the other side, we can obtain the stratification by stable types of the discriminant of any finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. 

This stratification is done as a consequence of the Geometric Criterion for finite determinacy given by Mather-Gaffney ([2]), where it is shown that out of the origin, any point of the critical set is stable.

Therefore, the strata of the stratification by the stable types is obtained by the type of singularity which appears in the discriminant set, out of the origin.
The first work that relates polar multiplicities and the stable types in the discriminant is [3]. We also see such relations in [6], or in [7].

In this work, we study when this stratification of the discriminant, which is Whitney, is minimal. We show that this is not always true, and the answer depends of the pair of dimensions \((n, p)\).
The Minimal Whitney Stratification

Consider \((V, 0)\) the germ of a reduced equidimensional analytic complex space of complex dimension \(d\).

We suppose that \((V, 0)\) is embedded in \((\mathbb{C}^{N+1}, 0)\) and that \(V\) is a sufficiently small representative of the germ.

The procedure to construct the minimal Whitney stratification that we describe below, is given in [8] in terms of the multiplicities of the absolute local polar varieties at each point.
For any germ \((V, 0)\) of a reduced equidimensional analytic complex space of complex dimension \(d\), denote by \(\Gamma_x(V)\) the sequence of multiplicities of generic polar varieties of \(V\) in \(x\).

They define a decreasing sequence of algebraic subvarieties of \(V\):

\[
F_0 \supset F_1 \supset \ldots \supset F_k \supset \ldots
\]

where \(F_0 = V\), \(F_1 = \Sigma(V)\), the singular set of \(V\).

To obtain the other \(F_j\), let us denote \((F_{1,j_1})_{j_1 \in J_1}\) the irreducible components of \(F_1\), then \(F_2\) is the union of the singular set of \(F_1\) with a finite number of closed algebraic sets given by the points \(x \in (F_{1,j})\) such that the sequence \(\Gamma_x(V)\) or \(\Gamma_x(F_{1,j_1})\), \(j_1 \in J_1\), is different of one computed on a generic point.
Let us denote \((F_2,j_2)_{j_2 \in J_2}\), the irreducible components of \(F_2\).

In general, let \(F_k\) be the union of the singular set of \(F_{k-1}\) with a finite number of closed algebraic sets given by the points \(x\) such that the sequence of multiplicities

\[
\Gamma_x(V), \; \Gamma_x(F_{1,j_1}), \; (j_1 \in J_1), \ldots, \; \Gamma_x(F_{k-1,j_{k-1}}), \; (j_{k-1} \in J_{k-1}),
\]

does not coincide with the sequence on a generic point of \(F_{k-1,j_{k-1}}\).
In [11] Teissier proved the following result:

**Theorem 2.1.** Let $V$ be a reduced equidimensional analytic space of dimension $d$ and $Y$ a non-singular analytic subspace of $V$ such that $0 \in Y$, then the following conditions are equivalent:

1. The multiplicity, denoted $m_k(V, y)$, of the local polar varieties $P_k(V, y)$ of the germ $(V, y)$ for $y \in Y$ is locally constant on $Y$ in a neighborhood of the origin, for $0 \leq k \leq d - 1$;

2. The pair $(V_{reg}, Y)$ satisfy the Whitney conditions (a) and (b) at the origin, and $V_{reg}$ denotes the regular part of $V$. 
The next corollary was introduced in [8].

**Corollary 2.2.** Let \( V \) be a sufficiently small representative of an analytic germ as above, the stratification of \( V \) such that the strata are given by \( F_{i,j} \setminus \bigcup_{k>i} F_{k,l} \) satisfies the Whitney conditions and it is called minimal stratification.
The stratification by the stable types

Denote by $O(n, p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$.

A map germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable in a finite set $S$ if, by composition with families of holomorphic diffeomorphisms in source and target, every deformation is $\mathcal{A}$-trivial, where $\mathcal{A}$ denotes the group of germs of holomorphic diffeomorphisms in the source and in the target.

We call stable type, an equivalence class of stable map germs.

A germ is $k$-$\mathcal{A}$-determined if any $g \in O(n, p)$ with the same $k$-jet as $f$, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to $f$. The germ $f$ is said to be finitely $\mathcal{A}$-determined if it is $k$-$\mathcal{A}$-determined for some $k$. 
Mather and Gaffney showed the characterization of finitely determined map germs in terms of stable germs.

**Proposition 3.1.** Suppose $f \in \mathcal{O}(n, p)$. Then $f$ is finitely determined iff for each representative $\tilde{f}$ of $f$, there exist neighborhoods of the origin $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^p$ such that $\tilde{f}^{-1}(0) \cap U \cap \Sigma(\tilde{f}) = 0$ and for each $y \in V$, $y \neq 0$, the germ $\tilde{f}_y : (\mathbb{C}^n, S_y) \to (\mathbb{C}^p, y)$ is stable, where $S_y = \tilde{f}^{-1}(y) \cap U \cap \Sigma(\tilde{f})$ and $\Sigma(\tilde{f})$ denotes the critical set of $\tilde{f}$. 
Remarks. 1) The critical set $\Sigma(f)$ of any map germ $f$ is defined as the set of points $x \in \mathbb{C}^n$ such that the differential of $f$ at $x$ is not a submersion.

2) When $n < p$, we have $\Sigma(f) = \mathbb{C}^n$.

3) We shall use the same symbol $\Sigma$ to denote different objects, the singular set of any complex analytic space $V$ is denoted by $\Sigma(V)$ and also the critical set $\Sigma(f)$ of a map germ $f$. 
The stable types which appear in the critical set of any finitely determined germ $f$ are given in any versal unfolding of $f$.

We say that $f$ has discrete stable type if there exist a versal unfolding of $f$ in which only a finite number of stable types occur.

If the pairs $(n, p)$ are in Mather’s “nice dimensions” ([10]), or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n, p)$ has discrete stable type.
The main point now is to describe for each pair \((n, p)\) all stable types which appear in any finitely determined map germ.

In general, for any pair of dimensions \((n, p)\) the description of the stable types is done in terms of sub-schemes of multiple points sets of a germ \(f\).

This description leads to the Thom-Boardman stratification by the stable types.
First we decompose the critical set $\Sigma(f)$ by the mono-germs and then we obtain a first stratification in the discriminant by the images of the strata of the critical set.

Now, to obtain the final stratification in the discriminant $\Delta(f) = f(\Sigma(f))$ we need to consider the images of these strata and refine this stratification by the strata composed by the stable multi-germs.
For any Boardman symbol $i = (i_1, \ldots, i_r)$, we denote by $\Sigma^i(f)$ the set of points in $\Sigma(f)$ of type $i$. Then in $\Sigma(f)$ we first consider the stratification done by the smooth parts of the sets $\Sigma^i(f)$ for all Boardman symbol $i$ which $\Sigma^i(f) \neq \emptyset$.

We shall denote by $r$-stable type any stable class of singularities which is $r$-dimensional in the discriminant of $f$. We can have $0 \leq r \leq d - 1$ if the discriminant is $d$-dimensional.
Theorem 4.1. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ. Let us consider $V$ as the discriminant of $f$, then the minimal Whitney stratification of $V$ coincides with the stratification of $V$ given by stable types.
Theorem 4.2. Let $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{2n-1}, 0)$ be a finitely determined map germ. Let us consider $V$ as the discriminant of $f$, then the minimal Whitney stratification of $V$ is coincident with the stratification of $V$ given by the stable types.
Map germs from \((\mathbb{C}^{n+p}, 0)\) to \((\mathbb{C}^p, 0)\)

To show how the stratifications are obtained for these dimensions, we shall follow Arnold’s notation and say that a singularity is of type \(A_k\) if it is the image of a point of type \(\Sigma^i(f)\), with \(i = 1, \ldots, 1\) with the number 1 appearing \(k\) times. The stable multiple points sets in the discriminant appear as normal crossings of these germs.

For each \(\ell\) with \(1 \leq \ell \leq p - 2\) and a partition \(I = \{i_1, \ldots, i_s\}\) of \((p - \ell)\) we denote by \(A_I\) the \(\ell\)-stable singularity of type \(A_{i_1, \ldots, i_s}\).

For instance if we fix \(p = 3\) and \(\ell = 1\), we can have the \(A_2\) and the \(A_{1,1}\) stable singularities of dimension 1 in the discriminant.
For $p = 4$, the possible one dimensional singularities are $A_{1,1,1}$, $A_{2,1}$ or $A_{3}$. The possible two dimensional singularities are $A_{1,1}$ and $A_{2}$.

In the general case of co-rank 1 map germs from $\mathbb{C}^{n+p}$ to $\mathbb{C}^{p}$, the discriminant is $(p - 1)$-dimensional, then if we consider the $(p - 2)$-dimensional strata, can appear the stratum formed by the $(p - 2)$-multiple points or points $A_{1,\ldots,1}$ with 1 appearing $(p - 2)$ times, we can have also the $(p - 3)$-multiple points which are of of type $A_{2,1,\ldots,1}$ with 1 appearing $(p - 4)$ times, and so on, until we get the mono germs which are of type $A_{p-2}$.

We remark that when $p \geq 4$ there are other stable corank one singularities different from the $A_{k}$ that can appear in the discriminant, for example we cite the umbilic points which appear as a corank one stable map germ from $\mathbb{C}^{5}$ to $\mathbb{C}^{4}$, see [1]. We do not need to consider such singularities here.
Now we show that in these dimensions the Whitney stratification given by the stable types can be different from the minimal Whitney stratification.

**Theorem 5.1.** Let \( f : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0), \) \( n \geq 0, \) be a finitely determined map germ. Suppose that for some \( \ell \) with \( 1 \leq \ell \leq p - 2, \) there appear in the discriminant \( V \) of \( f, \) different types of \( \ell \)-stable singularities \( A_I \) and \( A_J, \) with \( I \) and \( J \) partitions of \( p - \ell. \) Then the stratification of \( V \) given by the stable types is different from the minimal Whitney stratification of \( V. \)
To prove this result we shall show that these \( \ell \)-stable singularities have equal sequence of polar multiplicities, therefore they are in the same stratum in the minimal stratification and in different strata in the stratification by the stable types.

For any \( n \geq 0 \), the singularities \( A_I \) are suspensions of the singularities \( A_I \) which appear for map germs from \( (\mathbb{C}^p, 0) \) to \( (\mathbb{C}^p, 0) \), since in this case the multiplicities are preserved, we shall describe only these.
First we show it for the map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$ and for the other cases the strata appear in an analogous way.

Here, as the discriminant is 2-dimensional, the possible 0-stable singularities are: swallowtails, the $A_3$ singularities, normal crossings between cuspidal edges and planes, the $A_{2,1} = A_2 \cap A_1$ singularities, and the triple points $A_{1,1,1} = A_1 \cap A_1 \cap A_1$.

The 1-stable singularities which can appear are: the cuspidal edge, or points $A_2$ and the double points curve $A_{1,1} = A_1 \cap A_1$. 
In this case, the double points curve and the cuspidal edge belong
to different strata in the stratification by the stable types, but all
multiplicities considered in the minimal stratification are equal for
all points in these curves, so they are in the same stratum.

To show this we see first that in both sets, the double points curve
and the cuspidal edge, the multiplicity of $\Delta(f)$ in any point of these
sets is 2, hence the polar multiplicity $m_0$ is also 2, since it is equal
to the multiplicity of $\Delta(f)$. Now, as the local polar curve of $\Delta(f)$ at
any point of these sets is empty, the polar multiplicity $m_1$ at these
points is 0.
In the general case of corank 1 map germs from $\mathbb{C}^{n+p}$ to $\mathbb{C}^p$, when we consider the stratification given by the stable types, we have from the hypothesis at least two $\ell$-dimensional strata, the stratum formed by the points $A_I$ and the stratum formed by the points $A_J$.

On the other side, we can show in an analogous way than the case of map germs from $\mathbb{C}^3$ to $\mathbb{C}^3$ that, for all these points the multiplicity $m_0$ is $p - \ell$ and all polar varieties are empty, therefore the corresponding polar multiplicities $m_i$ with $i = 1, \ldots, p - 2$ are equal to 0 and these sets are in the same stratum of the minimal Whitney stratification.
The local Euler obstruction

The local Euler obstruction at a point $p$ of an algebraic variety $V$, denoted by $Eu_V(p)$, was defined by MacPherson in [9].

With the aid of Gonzales-Sprinberg’s purely algebraic interpretation of the local Euler obstruction, Lê and Teissier in [8] showed that the local Euler obstruction is an alternate sum of the multiplicities of the local polar varieties.
Theorem 6.1. [8] Let $X \subset \mathbb{C}^{n+1}$ be an analytic space of dimension $d$ reduced at 0. Then

$$Eu_0(X) = \sum_{i=0}^{d-1} (-1)^{d-i-1} m_i(X, 0),$$

where $Eu_0(X)$ denotes the Euler obstruction of $X$ at 0 and $m_i(X, 0)$ is the polar multiplicity of the polar varieties $P_i(X, 0)$. 
Here we apply the formula above to give an easy way to compute the local Euler obstruction at zero of the discriminant of finitely determined map germs from $\mathbb{C}^{n+p}$ to $\mathbb{C}^p$.

**Corollary 6.2.** Let $f : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$, $n \geq 0$, be a finitely determined map germ. Then the local Euler obstruction at 0 of the discriminant $V$ is equal to the multiplicity of $V$ at this point, or

$$Eu_0(V) = m_0(V, 0).$$
Corollary 6.3. If we suppose that the origin is in the $\ell$-stratum formed by singularities of type $A_1$. Then the local Euler obstruction at 0 of the discriminant $V$ is

$$Eu_0(V) = p - \ell.$$
Now we state a corollary which gives an easy way to find map germs such that both stratifications of the discriminant are coincident.

**Corollary 6.4.** Let \( f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0) \), be a finitely determined map germ with \((n, p)\) in the nice dimensions of Mather or in its border. Let us call \( V \) the discriminant of \( f \) and suppose that the stratification of \( V \) given by stable types has the property that for each \( j = 0, \ldots, \ell \), with \( \ell = \min(n, p) \) the \( j \)-dimensional stratum has at most one type of stable singularity. Then the minimal Whitney stratification of \( V \) is coincident with the stratification by the stable types.
Referências Bibliográficas


