Examples of solvmanifolds without LCK structures

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1. Introduction

**Definition 1.**
\[ G \]: simply-connected solvable Lie group.
\[ \Gamma \]: lattice, that is, discrete co-compact subgroup of \( G \).
\[ \Rightarrow \Gamma \backslash G \]: solvmanifold.

\((G \): nilpotent Lie group \[ \Rightarrow \Gamma \backslash G \]: nilmanifold)
**Definition 3.**

\((M, g, J)\) : Hermitian manifold.

\(\Omega\) : the fundamental 2-form \((\Omega(X, Y) = g(X, JY))\).

\((M, g, J)\) : locally conformal Kähler (LCK) \(\iff\) def \(\exists \omega\) : closed 1-form such that \(d\Omega = \omega \wedge \Omega\).

(We call \(\omega\) Lee form.)

**Remark 4.**

If \(\omega = df\), then \((M, e^{-f}g, J)\) is Kähler.

**Definition 5.**

\((M, g, J)\) : LCK manifold.

\((M, g, J)\) : Vaisman manifold \(\iff\) def Lee form \(\omega\) is parallel with respect to \(g\).
Definition 6.

\( M \) : manifold,
\( \alpha \) : closed 1-form on \( M \).
\( d_\alpha : A^p(M) \to A^{p+1}(M) \)

\[ d_\alpha \beta := \alpha \wedge \beta + d\beta \quad (d_\alpha^2 = 0). \]

We call \( \beta \) \( \alpha \)-closed (\( \alpha \)-exact), if \( d_\alpha \beta = 0 \) (\( \beta = d_\alpha \gamma \)).

Similarly, we can define the new differential operator on a Lie algebra.

\( (M, g, J) \) : LCK manifold.

\[ \iff d\Omega = \omega \wedge \Omega \quad (\omega \text{ : closed 1-form}). \]
\[ \iff 0 = -\omega \wedge \Omega + d\Omega. \]
\[ \iff 0 = d_{-\omega} \Omega, \text{ that is, } \Omega: \text{ } -\omega\text{-closed.} \]
**Theorem 7.** [León-López-Marrero-Padrón, ’03]

\((M, g)\) : compact Riemannian manifold

\(\alpha\) : parallel 1-from with respect to \(g\)

\[\Rightarrow\] any \(\alpha\)-closed form is \(\alpha\)-exact.

The fundamental 2-form \(\Omega\) of a Vaisman manifold is \(-\omega\)-exact:

\[\Omega = d_{-\omega} \eta = -\omega \wedge \eta + d\eta.\]

**Remark 8.** [S. ’15]

On solvmanifolds, the inverse is hold.
Examples

• Hopf surface (Vaisman ’79) : Vaisman manifold

• Inoue surfaces (Tricerri ’82) : non-Vaisman manifold

nilmanifold $S^1 \times \Gamma \backslash H$

• where, $H$ is a Hisenberg Lie group : Vaisman manifold (Fernandez et al ’86)

• Oeljeklaus-Toma manifold (Oeljeklaus-Toma, ’05) : non-Vaisman manifold
**LCK nilmanifold** $\Gamma \backslash G$

**Theorem 9.** [S. ’07]
$\Gamma \backslash G$ has a LCK structure $(g, J)$ such that $J$ is left-invariant, 
$\Rightarrow G = \mathbb{R} \times H$, where $H$ is a Heisenberg Lie group.

**Theorem 10.** [Bazzoni. ’17]
$\Gamma \backslash G$ has a Vaisman structure $(g, J)$, 
$\Rightarrow G = \mathbb{R} \times H$, where $H$ is a Heisenberg Lie group.
In this talk, we consider the following solvable Lie group:

\[ G_n = \left\{ \left( t, \left( x_i \right), z \right) : t, x_i, y_i, z \in \mathbb{R}, i = 1, \cdots, n \right\}, \]

where a structure group on \( G_n \) is defined by

\[
\left( t, \left( x_i \right), z \right) \cdot \left( t', \left( x_i' \right), z' \right) = \left( t + t', \left( e^{a_i t} x_i' + x_i \right), z' + \frac{1}{2} \sum_{i=1}^{n} (-e^{a_i t} y_i x_i' + e^{-a_i t} x_i y_i') + z \right)
\]

and \( a_i \in \mathbb{Z} - \{0\} \).

\([G_n, G_n] : (2n + 1)\)-dimensional Heisenberg Lie group
Construction of solvmanifold $\Gamma_n \backslash G_n$

$B = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}$ be a unimodular matrix with distinct positive eigenvalues $\lambda, \lambda^{-1}$

$\exists P = \begin{pmatrix} 1 & \lambda \\ 1 & \lambda^{-1} \end{pmatrix}$ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

$\varphi : \mathbb{R}^{2n+2} \to G_n$ diffeomorphism

$\varphi(A) = \left( (\log \lambda)t, 2P \begin{pmatrix} x_i \\ y_i \end{pmatrix}, |P| z \right)$

$\Rightarrow$ We can prove that $\varphi(\mathbb{Z}^{2n+2})$ is a lattice $\Gamma_n$ on $G_n$
Main Theorem.
Let $J_n$ be a left-invariant complex structure on $\Gamma_n \backslash G_n$.

- In the case of $n = 1$, $(\Gamma_1 \backslash G_1, J_1)$ has a LCK structure, but has no Vaisman structures.
- In the case of $n \geq 2$, $(\Gamma_n \backslash G_n, J_n)$ has no LCK structures.

Remark 11.
On a LCK structure, in absence of a complex structure, it is said to be *locally conformal symplectic (LCS)*. The solvmanifold $\Gamma_n \backslash G_n$ has LCS structures.
2. Preliminary

**Definition 12.**

\( G \) : simply-connected solvable Lie group.

\( G \) : completely solvable

\( \iff \) For \( \forall X \in \mathfrak{g}, \text{ad}(X) : \mathfrak{g} \to \mathfrak{g} \) has only real eigenvalues,

where \( \mathfrak{g} \) : Lie algebra of \( G \).

**Theorem 13.** [Hattori ’60]

\( H^*_\text{DR}(\Gamma \backslash G) \cong H^*(\mathfrak{g}) \), where \( \mathfrak{g} \) : Lie algebra of \( G \).

Note that \( G_n \) is completely solvable.
(\(M = \Gamma \backslash G, g, J\)) : LCK solvmanifold with Lee form \(\omega\) such that

1. \(J\) is left-invariant.

2. \(\exists\) left-invariant closed 1-form \(\omega_0\) s.t \(\omega_0 - \omega = df\).

**Theorem 14.** [Belgun '00]
For \(X, Y \in g\),

\[
\langle X, Y \rangle := \int_M e^f g(X, Y) d\mu
\]

\(\implies (g, \langle \ , \ \rangle, J) : LCK\ solvable\ Lie\ algebra\ with\ Lee\ form\ \omega_0\).

\(\Omega_0\) : the fundamental 2-form of \((\langle \ , \ \rangle, J)\)

**Remark 15.** [S. '12]
\((M = \Gamma \backslash G, g, J) : Vaisman\ solvmanifold\ (\Omega = d-\omega \eta)\)

\(\implies \Omega_0 = d-\omega_0 \eta_0\)
Definition 16.
\[ J_1, J_2 : \text{complex structure on } g \]
\[ J_1, J_2 \text{ are equivalent} \iff \exists F \in \text{Aut}(g) \text{ such that } J_1 \circ F = F \circ J_2 \]

\[(g, J_1) : \text{Hermitian structure} \implies (F^*g, J_2) : \text{Hermitian structure}\]

Proposition 17. [Ugarte '07]
\[(g, J_1) : \text{LCK structure} \implies (F^*g, J_2) : \text{LCK structure}\]
3. In the case of $n = 1$


$\{\theta, \alpha, \beta, \gamma\}$ is a dual base of $\{A, X, Y, Z\}$:

\[
\begin{align*}
    d\theta &= 0, \\
    d\alpha &= -\theta \wedge \alpha, \\
    d\beta &= \theta \wedge \beta, \\
    d\gamma &= -\alpha \wedge \beta
\end{align*}
\]

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**LCK structure on $\Gamma_1 \setminus G_1$ (Inoue surface $S^+$)**

(Tricerri '82, Fernández et al '89, Kamishima '01)

- $\langle \ , \, \rangle$ : left-inv. metric s.t $\{A, X, Y, Z\}$ is an orthonormal frame.
- $J$ : left-inv. complex structure s.t $JA = Y, JZ = X$.

$\implies (\langle \ , \, \rangle, J) : \text{LCK structure with Lee form } \theta$.

- $\nabla_Y \theta(Y) = -\theta(\nabla_Y Y) = -\langle A, \nabla_Y Y \rangle = -\langle [A, Y], Y \rangle = \langle Y, Y \rangle \neq 0$. 
Proposition 18. [cf. Belgun, ’00] 
\( \Gamma_1 \backslash G_1 \) has no Vaisman structures.

Proof. • A complex structure on a 4-dimensional solvmanifold is left-invariant [Hasegawa, ’05].

• A complex structure on \( \Gamma_1 \backslash G_1 \) is equivalent to

\[
J_0 : J_0 A = Y, J_0 Y = -A, J_0 Z = X, J_0 X = -Z \\
\text{or} \\
J_1 : J_1 A = Y + Z, J_1 Y = -A - X, J_1 Z = X, J_1 X = -Z \quad [Ovando, ’04].
\]

• \( (\Gamma_1 \backslash G_1, J) \) has a Vaisman structure : \( \Omega_0 = d_{-k\theta}\eta \) 

\[ \implies (\Gamma_1 \backslash G_1, J_q) \text{ has a LCK structure : } \Omega^q_0 = d_{-k_q\theta}\eta_q \quad (q = 0 \text{ or } 1) \]

\[ \langle Z, Z \rangle = d_{-k_q\theta}\eta_q(J_q Z, Z) = (-k_q\theta \wedge \eta_q + d\eta_q)(X, Z) = 0 \]

because \( X, Z \in [g_1, g_1] \) and \( Z \) is in the center of \( g_1 \).
Remark 19.
$(\Gamma_1 \backslash G_1, J_1)$ has no LCK structures.

Remark 20. [Vaisman, ’82]
The first Betti number of a Vaisman manifold is odd.

Proposition 21. [Kasuya ’12]
If $g = \mathbb{R}^n \ltimes \mathbb{R}^m$ and $\dim[g, g] > \frac{1}{2} \dim g$, then $(\Gamma \backslash G, J)$ has no Vaisman structures.

$\Rightarrow$ Inoue surface $S^0$, O-T manifolds have no Vaisman structures.
4. In the case of $n \geq 2$

$$g_n = \text{span}\{A, X_i, Y_i, Z : [A, X_i] = a_iX_i, [A, Y_i] = -a_iY_i, [X_i, Y_i] = Z\}$$

$$\{\theta, \alpha_i, \beta_i, \gamma\}$$ is a dual base of $\{A, X, Y, Z\}$:

$$d\theta = 0,$$
$$d\alpha_i = -a_i\theta \wedge \alpha_i, d\beta_i = a_i\theta \wedge \beta_i,$$
$$d\gamma = -\sum_i \alpha_i \wedge \beta_i$$

Note that

$$[g_n, g_n] = \text{span}\{X_i, Y_i, Z\}, \text{span}\{Z\}$$ is the center of $g_n$.

We assume that $g_n$ has a LCK structure $(\langle \ , \rangle, J_n)$ with $\Omega_0$ ($-k\theta$-closed 2-from).
**Lemma 22.** $\Omega_0(Z, X_i) = \Omega_0(Z, Y_i) = 0$ for each $i$.

**Proof.**

\[
\Omega_0(Z, X_i) = \Omega_0([X_j, Y_j], X_i) \\
= -d\Omega_0(X_j, Y_j, X_i) + \Omega_0([X_j, X_i], Y_j) - \Omega_0([Y_j, X_i], X_j) \\
= k\theta \wedge \Omega_0(X_j, Y_j, X_i) = 0.
\]

$\gamma_0 : g^* \to g$ isomorphism induced by $\langle , \rangle$.

**Corollary 23.** $J_n \circ \gamma_0(\theta) \in \text{span}\{Z\}$.

**Proof.** Since $\langle J_n Z, X_i \rangle = -\Omega_0(Z, X_i) = 0, \langle J_n Z, Y_i \rangle = -\Omega_0(Z, Y_i) = 0$, we have $J_n Z \in \text{span}\{\gamma_0(\theta)\}$, that is, $J_n \circ \gamma_0(\theta) \in \text{span}\{Z\}$. 

\[\square\]
(g, ⟨ , ⟩, J) LCK solvable Lie algebra with Lee form ω₀.

**Proposition 24. [S. ’15]**

If \langle [γ₀(ω₀), J ◦ γ₀(ω₀)], J ◦ γ₀(ω₀) \rangle = 0,
then (⟨ , ⟩, J) is a Vaisman structure.

⇒ LCK structure on (⟨ , ⟩, Jₙ) on gₙ is Vaisman,
because J ◦ γ₀(θ) ∈ span{Z}.

**Proposition 25. [S. ’17]**

If g is completely solvable and (⟨ , ⟩, J) is a Vaisman structure,
then g = ℝ × ℱ, where ℱ is a Heisenberg Lie algebra.

⇒ This is a contradiction,
because gₙ is completely solvable.
Remark 26. A non-degenerate $-\omega$-closed 2-form is called LCS. The solvmanifold $\Gamma_n \backslash G_n$ has LCS structures:

\[
\Omega_1 = d_{-\theta} \gamma = -\theta \wedge \gamma - \sum_i \alpha_i \wedge \beta_i \text{(exact type)}
\]

\[
\Omega_2 = \alpha_k \wedge \alpha_l + d(a_k + a_l) \theta \gamma = \alpha_k \wedge \alpha_l + (a_k + a_l) \theta \wedge \gamma - \sum_i \alpha_i \wedge \beta_i \text{(non-exact type)}
\]

Remark 27. If $a_i = 0$ for each $i$, then the solvmanifold $\Gamma_n \backslash G_n$ has a LCK structure (Vaisman).
4. Future Work

- Structure Theorem for Vaisman solvmanifolds?

- Classification of low dimensional LCK solvmanifolds

**Proposition 28.** [S. ’12]
Classification of 4-dimensional LCK solvmanifolds: Kodaira-Thurston manifold, Inoue surfaces

**Proposition 29.** [Bock, ’16]
Classification of 6-dimensional solvable Lie algebras
Proposition 30. [S. ’15]
If $\Omega_0$ is $-\omega_0$-exact, then $(\Omega, J)$ is Vaisman.

Since $g$ is solvable, $n := [g, g]$ is nilpotent:

$$n = [g, g] \supset n^{(1)} = [n, n] \supset n^{(2)} = [n, n^{(1)}] \supset \cdots \supset n^{(r)} \supset n^{(r+1)} = 0,$$

where $n^{(i+1)} = [n, n^i]$.

Proposition 31.
If $Jn^{(r)} \subset [g, g] \perp$, then $(\Omega, J)$ is Vaisman,
where $[g, g] \perp$ is the orthogonal component.

$\Rightarrow$ We can construct 6-dimensional solvmanifolds without LCK structures.