Recent progress of homogeneous Einstein metrics on generalized flag manifolds

Yusuke Sakane
Kawanishi
March 6, 2013

Outline

Homogeneous Einstein metrics on generalized flag manifolds

- introduction
  - generalized flag manifolds
  - Ricci tensor of a compact homogeneous space
  - structures of generalized flag manifolds
  - $t$-roots of generalized flag manifolds
  - decomposition associated to generalized flag manifolds
    ( $t$-roots and decompositions )
  - invariant Einstein metrics on a generalized flag manifold

Introduction

$(M, g)$: Riemannian manifold

- $(M, g)$ is called Einstein if the Ricci tensor $r(g)$ of the metric $g$ satisfies $r(g) = cg$ for some constant $c$.

We consider $G$-invariant Einstein metrics on a homogeneous space $G/K$.

- General Problem: Find $G$-invariant Einstein metrics on a homogeneous space $G/K$ and classify them if it is not unique.

- Einstein homogeneous spaces can be divided into three cases depending on Einstein constant $c$.
  - Here we consider the case $c > 0$.

1. Examples of the case $c > 0$
   - $(G/K)$ is compact and $\pi_1(G/K)$ is finite.
   - Sphere ($S^n = SO(n + 1)/SO(n), g_0$), Complex Projective space ($\mathbb{C}P^n = SU(n + 1)/(S(U(1) \times U(n)))$), Symmetric spaces of compact type, isotropy irreducible spaces (in these cases $G$-invariant Einstein metrics is unique)
   - Compact semi-simple Lie groups (bi-invariant metric (negative of Killing form))
   - Generalized flag manifolds (Kähler C-spaces) (if we fix a complex structure, it admits a unique Kähler-Einstein metric, but complex structure may not be unique)
Introduction


Example. Let $G = SU(4)$, $L = Sp(2)$, $K = SU(2)$ ($SU(2)$ is a maximal subgroup of $Sp(2)$). Then $G/K$ has no $(G)$-invariant Einstein metrics. Note that $\dim G/K = 12$.

(Bohm-Kerr (2006) ) For a simply connected compact homogeneous space $G/K$ of $\dim G/K \leq 11$, there exists at least one $G$-invariant Einstein metric on $G/K$.

Problem: Find all $G$-invariant Einstein metrics on a compact homogeneous space $G/K$.

( Nikonorov, Rodionov (2003) ) For a simply connected compact homogeneous space $G/K$ of $\dim G/K \leq 7$, all $G$-invariant Einstein metrics has been determined on $G/K$, except for $SU(2) \times SU(2)$.

For $SU(2) \times SU(2)$, there exist at least two left-invariant Einstein metrics. The first is the standard metric, and the other was found by Jensen.

In 2003 Nikonorov and Rodionov computed the scalar curvature of left-invariant metrics on $SU(2) \times SU(2)$, but these depend on 14 parameters and it is difficult to find critical points (Einstein metrics).

Open problem : How many left-invariant Einstein metrics are there on compact simple Lie groups $G$ ( $\dim G \geq 4$ )? (finite or infinite?)

(Wang-Ziller (1990) ) The principal $S^1$-bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ are all diffeomorphic to $S^2 \times S^3$, but as homogeneous spaces $(SU(2) \times SU(2))/S^1$ they are quite different. There are infinitely many ways to embed the group $S^1$ in $SU(2) \times SU(2)$. On $S^2 \times S^3$ the moduli space of Einstein metrics has infinitely many components.

Generalized flag manifolds

A generalized flag manifold $M$ is an adjoint orbit of a compact connected semi-simple Lie group $G$, and is a homogeneous space of the form $M = G/C(S)$, where $C(S)$ is the centralizer of a torus $S$ in $G$.

Generalized flag manifolds exhaust compact simply connected homogeneous Kähler manifolds.

A generalized flag manifold admits a finite number of $G$-invariant complex structures. For each $G$-invariant complex structure there is a compatible Kähler-Einstein metric.

Generalized flag manifolds can be classified by use of painted Dynkin diagrams.

Generalized flag manifolds are also referred to as Kähler C-spaces.
Examples of Generalized flag manifolds

- Set \( G = SU(n+1), K = S(U(n) \times U(1)) \). Then \( G/K \) is a complex projective space \( \mathbb{C}P^n \).
- Set \( G = SU(n+m), K = S(U(n) \times U(m)) \). Then \( G/K \) is a Grassmann manifold \( G_{m+n,m}(\mathbb{C}) \).
- Set \( G = SU(n+m+\ell), K = S(U(n) \times U(m) \times U(\ell)) \). Then \( G/K \) is a generalized flag manifold.
- Set \( G = Sp(n+1), K = Sp(n) \times U(1) \). Then \( G/K \) is a complex projective space \( \mathbb{C}P^{2n-1} \).

Ricci tensor of a compact homogeneous space \( G/K \)

- Let \( G \) be a compact semi-simple Lie group and \( K \) a connected closed subgroup of \( G \).
- Let \( \mathfrak{m} \) be the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g} \) with respect to \( B = - \text{Killing form of } \mathfrak{g} \). Then we have \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}, [\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m} \) and a decomposition of \( \mathfrak{m} \) into irreducible \( \text{Ad}(K) \)-modules:
  \[
  \mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q .
  \]
- We assume that \( \text{Ad}(K) \)-modules \( \mathfrak{m}_j \) (\( j = 1, \cdots, q \)) are mutually non-equivalent.
- Then a \( G \)-invariant metric on \( G/K \) can be written as
  \[
  < , > = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} ,
  \]
  for positive real numbers \( x_1, \cdots, x_q \).

Ricci tensor of a compact homogeneous space \( G/K \)

- Note that \( G \)-invariant symmetric covariant 2-tensors on \( G/K \) are the same form as the metrics.
- In particular, the Ricci tensor \( r \) of a \( G \)-invariant Riemannian metric on \( G/K \) is of the same form as (1).
- Let \( \{ e_i \} \) be a \( B \)-orthonormal basis adapted to the decomposition of \( \mathfrak{m}_i \), i.e., \( e_a \in \mathfrak{m}_i \) for some \( i \), and \( \alpha < \beta \) if \( i < j \) (with \( e_a \in \mathfrak{m}_i \) and \( e_\beta \in \mathfrak{m}_j \)).
- We put \( A_{\alpha\beta}^y = B(\{ e_\alpha, e_\beta \}, e_y) \), so that \( [ e_\alpha, e_\beta ] = \sum_{y} A_{\alpha\beta}^y e_y \), and
  \[
  \sum_{k} ^{k}_{ij} = \sum_y (A_{\alpha\beta}^y)^2 ,
  \]
  where the sum is taken over all indices \( \alpha, \beta, y \) with \( e_\alpha \in \mathfrak{m}_i \), \( e_\beta \in \mathfrak{m}_j \), \( e_y \in \mathfrak{m}_k \).
- Notations \( \left[ ^{k}_{ij} \right] \) are introduced by Wang and Ziller [17].

Lemma

The components \( r_1, \cdots, r_q \) of Ricci tensor \( r \) of the metric \( < , > = x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \) on \( G/K \) are given by

\[
 r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{ij} \frac{x_i}{x_j x_i} \left[ ^{k}_{ij} \right] - \frac{1}{2d_k} \sum_{ij} \frac{x_j}{x_k x_i} \left[ ^{j}_{ki} \right] \quad (k = 1, \cdots, q)
\]

where the sum is taken over \( i, j = 1, \cdots, q \).
Let $G$ be a compact semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of $G$ and $\mathfrak{h}$ a maximal abelian subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{g}^\mathbb{C}$ and $\mathfrak{h}^\mathbb{C}$ the complexification of $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

We identify an element of the root system $\Delta$ of $\mathfrak{g}^\mathbb{C}$ relative to the Cartan subalgebra $\mathfrak{h}^\mathbb{C}$ with an element of $\mathfrak{h}_0 = \sqrt{-1}\mathfrak{h}$ by the duality defined by the Killing form of $\mathfrak{g}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system of $\Delta$ and $\{\Lambda_1, \ldots, \Lambda_\ell\}$ the fundamental weights of $\mathfrak{g}^\mathbb{C}$ corresponding to $\Pi$, that is

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (1 \leq i, j \leq \ell).$$

Let $\Pi_\mathbb{C}$ be a subset of $\Pi$ and $\Pi = \Pi_0 \cup \{\alpha_i\}$

$(1 \leq \alpha_i < \cdots < \alpha_{i_{\ell}} \leq \ell)$. We put $[\Pi_0^\mathbb{C}] = \Pi_0 \cap \{\Pi_0\}_{\mathbb{C}}$, where $\{\Pi_0\}_{\mathbb{C}}$ denotes the subspace of $\mathfrak{h}_0$ generated by $\Pi_0$. We put $\Pi_0 = \{\alpha_{i_1}, \ldots, \alpha_{i_{\ell}}\}$

$(1 \leq \alpha_{i_1} < \cdots < \alpha_{i_{\ell}} \leq \ell)$. We put $[\Pi_0^\mathbb{C}] = \Pi_0 \cap \{\Pi_0\}_{\mathbb{C}}$, where $\{\Pi_0\}_{\mathbb{C}}$ denotes the subspace of $\mathfrak{h}_0$ generated by $\Pi_0$.

Let $\Pi_0$ be a subset of $\Pi$ and $\Pi = \Pi_0 \cup \{\alpha_i\}$

$(1 \leq \alpha_i < \cdots < \alpha_{i_{\ell}} \leq \ell)$. We put $[\Pi_0^\mathbb{C}] = \Pi_0 \cap \{\Pi_0\}_{\mathbb{C}}$, where $\{\Pi_0\}_{\mathbb{C}}$ denotes the subspace of $\mathfrak{h}_0$ generated by $\Pi_0$. We put $\Pi_0 = \{\alpha_{i_1}, \ldots, \alpha_{i_{\ell}}\}$

$(1 \leq \alpha_{i_1} < \cdots < \alpha_{i_{\ell}} \leq \ell)$. We put $[\Pi_0^\mathbb{C}] = \Pi_0 \cap \{\Pi_0\}_{\mathbb{C}}$, where $\{\Pi_0\}_{\mathbb{C}}$ denotes the subspace of $\mathfrak{h}_0$ generated by $\Pi_0$.

**Structures of generalized flag manifolds**

- Consider the root space decomposition of $\mathfrak{g}^\mathbb{C}$ relative to $\mathfrak{h}^\mathbb{C}$:

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} + \sum_{\alpha \in \Delta^+, \alpha \neq \alpha_0} \mathfrak{g}^\mathbb{C}_\alpha.$$

For a subset $\Pi_\mathbb{C}$ of $\Pi$, we define a parabolic subalgebra $\mathfrak{u}$ of $\mathfrak{g}^\mathbb{C}$ by

$$\mathfrak{u} = \mathfrak{h}^\mathbb{C} + \sum_{\alpha \in \{\Pi_0\}_{\mathbb{C}} \cup \Delta^+} \mathfrak{g}^\mathbb{C}_\alpha,$$

where $\Delta^+$ is the set of all positive roots relative to $\Pi$. We put $\Delta^+_m = \Delta^+ - [\Pi_0^\mathbb{C}]$.

**i-roots of generalized flag manifolds**

- We consider the restriction map

$$\kappa : \mathfrak{h}_0^\mathbb{C} \to \mathfrak{t}^\mathbb{C} \quad \alpha \mapsto \alpha|_\mathbb{C}$$

and set $\Delta_i = \kappa(\Delta_i)$. The elements of $\Delta_i$ are called i-roots.

( The notion of i-roots is introduced by Alekseevky and Perelomov [2] around 1985 to study invariant Kähler-Einstein metrics of generalized flag manifolds. )

- There exists a 1-1 correspondence between i-roots $\xi$ and irreducible submodules $m_\xi$ of the $\text{Ad}_G(K)$-module $m^\mathbb{C}$ that is given by

$$\Delta_i \ni \xi \mapsto m_\xi = \sum_{\alpha(\mathfrak{u}) = \xi} \mathfrak{g}^\mathbb{C}_\alpha.$$

- Thus we have a decomposition of the $\text{Ad}_G(K)$-module $m^\mathbb{C}$:

$$m^\mathbb{C} = \sum_{\xi \in \Delta_i} m_\xi.$$
Decomposition associated to generalized flag manifolds

Denote by $\Delta_1^+$ the set of all positive $t$-roots, that is, the restricton of the system $\Delta^+$. Then $\Delta = \sum_{\xi \in \Delta_1^+} m_\xi$. 

Denote by $\tau$ the complex conjugation of $g^\mathbb{C}$ with respect to $g$ (note that $\tau$ interchanges $g^\mathbb{C}$ and $g^\mathbb{C}_{\alpha_i}$) and by $v^\perp$ the set of fixed points of $\tau$ in a (complex) vector subspace $v$ of $g^\mathbb{C}$. Thus we have a decomposition of $\text{Ad}_G(K)$-module $m$ into irreducible submodules:

$$m = \sum_{\xi \in \Delta_1^+} (m_\xi + m_{-\xi})^\tau.$$

Decomposition associated to generalized flag manifolds

For integers $j_1, \ldots, j_r$ with $(j_1, \ldots, j_r) \neq (0, \ldots, 0)$, we put $\Delta(j_1, \ldots, j_r) = \left\{ \sum_{i=1}^r m_i \alpha_i \in \Delta^+ \mid m_{j_1} = j_1, \ldots, m_{j_r} = j_r \right\}$. There exists a natural 1-1 correspondence between $\Delta_1^+$ and the set $\Delta(j_1, \ldots, j_r) \neq \emptyset$

For a generalized flag manifold $G/K$, we have a decomposition of $m$ into mutually non-equivalent irreducible $\text{Ad}_G(H)$-modules:

$$m = \sum_{\xi \in \Delta_1^+} (m_\xi + m_{-\xi})^\tau = \sum_{j_1, \ldots, j_r} m(j_1, \ldots, j_r).$$

Thus a $G$-invariant metric $g$ on $G/K$ can be written as

$$g = \sum_{\xi \in \Delta_1^+} x_\xi B_{\xi}(m_\xi + m_{-\xi})^\tau = \sum_{j_1, \ldots, j_r} x_{j_1, \ldots, j_r} B_{\text{Ad}(j_1, \ldots, j_r)}$$

for positive real numbers $x_\xi, x_{j_1, \ldots, j_r}$.

Recent progress of homogeneous Einstein metrics on generalized flag manifolds

March 6, 2013 18 / 44

From now on we assume that the Lie group $G$ is simple.

We denote by $q$ the number of elements of $\Delta_1^+$ for a generalized flag manifold $G/K$, that is, the number of irreducible components of $\text{Ad}_G(K)$-module $m$.

If $q = 1$, then $\Delta_1^+ = \{ \xi \}$ and $G/K$ is an irreducible Hermitian symmetric space with the symmetric pair $(g, \tau)$.

If $q = 2$, then we see that $r = b_2(G/K) = 1$ and $m = m(1) \oplus m(2)$, that is, $\Delta_1^+ = \{ \xi, 2\xi \}$. We say this case that $t$-roots system is of type $A_1(2)$.

Example. $\mathbb{C}P^{2n-1} = Sp(n)/(Sp(n-1) \times U(1))$

$$\array{& \alpha_1 & \alpha_2 & \cdots & \alpha_p & \alpha_{p-1} & \alpha_n \\
2 & 2 & \cdots & 2 & \cdots & \cdots & 2 \\
1 & 1 & \cdots & \cdots & \cdots & \cdots & 1}$$

Ricci tensor for case $q = 2$

Note that only $\begin{bmatrix} 2 \\
11 \end{bmatrix}$ is non-zero.

Put $d_1 = \dim m(1)$ and $d_2 = \dim m(2)$.

For a $G$-invariant metric $<, > = x_1 \cdot B_{\text{Ad}(1)} + x_2 \cdot B_{\text{Ad}(2)}$, components $r_1, r_2$ of Ricci tensor $\tau$ of the metric $<, >$ are given by

$$\array{r_1 = \frac{1}{2x_1} - \frac{x_2}{2d_1 x_1^2} \begin{bmatrix} 2 \\
11 \end{bmatrix} \\
r_2 = \frac{1}{2x_2} - \frac{1}{2d_2 x_2} \begin{bmatrix} 1 \\
21 \end{bmatrix} + \frac{x_2}{4d_2 x_2^2} \begin{bmatrix} 2 \\
11 \end{bmatrix}.}$$

Note that Kähler-Einstein metric is given by

$$<, > = 1 \cdot B_{\text{Ad}(1)} + 2 \cdot B_{\text{Ad}(2)}$$

and thus we can determine the value $\begin{bmatrix} 2 \\
11 \end{bmatrix}$ and find 2 Einstein metrics.
The case $q = 3$

- If $q = 3$, then we see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$.
- We say the case of $r = b_2(G/K) = 1$ and $q = 3$ that $t$-roots system is of type $A_1(3)$, that is, $\Delta^+_t = \{\xi, 2\xi, 3\xi\}$. There are 7 cases and the Lie group $G$ is always exceptional, that is, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ (for $E_7$, $E_8$, there are 2 cases.)
- We say the case of $r = b_2(G/K) = 2$ and $q = 3$ that $t$-roots system is of type $A_2$, that is, $\Delta^+_t = \{\xi_1, \xi_2, \xi_1 + \xi_2\}$. There are 3 cases.

### The case $q = 3$ and $b_2(G/K) = 1$

<table>
<thead>
<tr>
<th>Flag manifold</th>
<th>Painted Dynkin diagram</th>
<th>number of Einstein metrics up to isometry</th>
</tr>
</thead>
</table>
| $SU(n)$/ | $SO(2n)/(U(n-1)\times U(1))$ | Kähler 3$\gamma$
| $S(U(\ell)\times U(m)\times U(k))$ | $E_6/\left( SO(8) \times U(1) \times U(1) \right)$ | non-Kähler 1 |
| $(n = \ell + m + k)$ | $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7$ | |
| $2\alpha_8$ | $3\alpha_8$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |
| $2\alpha_7$ | $2\alpha_7$ | |

### The case $q = 3$ and $b_2(G/K) = 2$

- The case $q = 4$ has started to study by A. Arvanitoyeorgos and I. Chrysikos around 2009 [4].
- We see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$ also occur in this case and we divide into 2 cases.
- We call the case of $r = b_2(G/K) = 1$ that $t$-roots system is of type $A_1(4)$, that is, $\Delta^+_t = \{\xi, 2\xi, 3\xi, 4\xi\}$. There are 4 cases and $G$ is always exceptional Lie group.
- We call the case of $r = b_2(G/K) = 2$ that $t$-roots system is of type $B_2$, that is, $\Delta^+_t = \{\xi_1, \xi_2, \xi_1 + \xi_2, \xi_1 + 2\xi_2\}$. There are 6 cases.

### The case $q = 4$ and $b_2(G/K) = 1$

- The system of equations $r_1 = r_2 = r_3$ reduces to a polynomial equation of degree 5.
### The case $q = 4$ and $b_2(G/K) = 1$

<table>
<thead>
<tr>
<th>Flag manifold</th>
<th>Painted Dynkin diagram</th>
<th>number of Einstein metrics up to isometry</th>
</tr>
</thead>
</table>
| $F_4/(SU(3) \times SU(2) \times U(1))$ | $\begin{array}{c}
\begin{array}{c}
\alpha_1 
\alpha_2
\alpha_3
\alpha_4
\end{array}
\end{array}$
|                                       | $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{2}{2}
\frac{3}{4}
\frac{4}{2}
\end{array}
\end{array}
\end{array}$ | Kähler 1 non-Kähler 2 |
| $E_7/(SU(4) \times SU(3)$ $\times SU(2) \times U(1))$ | $\begin{array}{c}
\begin{array}{c}
\alpha_1 
\alpha_2
\alpha_3
\alpha_4
\alpha_5
\alpha_6
\alpha_7
\end{array}
\end{array}$
|                                       | $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{2}{2}
\frac{3}{4}
\frac{4}{2}
\frac{6}{4}
\frac{2}{3}
\frac{3}{4}
\end{array}
\end{array}
\end{array}$ | Kähler 1 non-Kähler 2 |
| $E_6/(SO(10) \times SU(3) \times U(1))$ | $\begin{array}{c}
\begin{array}{c}
\alpha_1 
\alpha_2
\alpha_3
\alpha_4
\alpha_5
\alpha_6
\alpha_7
\end{array}
\end{array}$
|                                       | $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{2}{2}
\frac{3}{4}
\frac{4}{2}
\frac{6}{4}
\frac{2}{3}
\frac{3}{4}
\end{array}
\end{array}
\end{array}$ | Kähler 1 non-Kähler 4 |

### The case $q = 4$  $B_2$

| $SO(2n)/(U(p) \times U(n-p))$ | $\begin{array}{c}
\begin{array}{c}
\alpha_1 
\alpha_2
\alpha_p
\alpha_n
\end{array}
\end{array}$
|                                       | $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{1}{1}
\frac{2}{2}
\frac{p}{n-p}
\frac{n}{n-p}
\end{array}
\end{array}
\end{array}$ | Kähler 2 non-Kähler 2 |
| $Sp(n)/(U(p) \times U(n-p))$ | $\begin{array}{c}
\begin{array}{c}
\alpha_1 
\alpha_2
\alpha_p
\alpha_n
\end{array}
\end{array}$
|                                       | $\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\frac{2}{2}
\frac{2}{2}
\frac{p}{n-p}
\frac{n}{n-p}
\end{array}
\end{array}
\end{array}$ | Kähler 2 non-Kähler 1 |

- Einstein metrics for the case of $r = b_2(G/K) = 1$ has been studied by A. Arvanitoyeorgos and I. Chrysikos [4]. Einstein metrics for the case of $r = b_2(G/K) = 2$, that is, t-roots system is of type $B_2$, has been studied by A. Arvanitoyeorgos and I. Chrysikos [4] and A. Arvanitoyeorgos, I. Chrysikos and Y. S. [5], [6], [7].

### The case $q = 5$

- For the case $q = 5$ we also see that either $r = b_2(G/K) = 1$ or $r = b_2(G/K) = 2$.
- We call the case of $r = b_2(G/K) = 1$ that t-roots system is of type $A_1(5)$, that is, $\Delta^+_1 = \{ \xi_1, 2\xi_2, 3\xi_3, 4\xi_4, 5\xi_5 \}$. There is only one case, $G = E_8$ and $K = SU(4) \times SU(5) \times U(1)$ is the case.
- We call the cases of $r = b_2(G/K) = 2$ that t-roots system is of “extended” type $B_2$, that is,
  - type A: $\Delta^+_2 = \{ \xi_1, \xi_2, 2\xi_3, 3\xi_4, 4\xi_5, 5\xi_6, 6\xi_7, 7\xi_8 \}$, or
  - type B: $\Delta^+_2 = \{ \xi_1, \xi_2, 2\xi_3, 3\xi_4, 4\xi_5, 5\xi_6, 6\xi_7, 7\xi_8 \}$.
- There are 4 cases for each. We can show there is an isometry between homogeneous spaces of type A and of type B.
- Einstein metrics for the case of $r = b_2(G/K) = 1$ is studied by I. Chrysikos and Y. S. [11], and for the case of $r = b_2(G/K) = 2$ is studied by A. Arvanitoyeorgos, I. Chrysikos and Y. S. [10] recently.
The case $q = 6$

- For the case $q = 6$ we also see that either $r = b_2(G/K) = 1$, $r = b_2(G/K) = 2$ or $r = b_2(G/K) = 3$.
- We call the case of $r = b_2(G/K) = 1$ that t-roots system is of type $A_1(6)$, that is, $\Delta^+_1 = \{ \xi, 2\xi, 3\xi, 4\xi, 5\xi, 6\xi \}$. There is only one case, $G = E_6$ and $K = SU(5) \times SU(3) \times SU(2) \times U(1)$.
- For $r = b_2(G/K) = 2$, we have 4 cases:
  $\Delta^+_1 = \{ \xi_1, \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2 \}$, of type $BC_2$, $\Delta^+_1 = \{ \xi_1, \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2 \}$, of type $G_2$, $\Delta^+_1 = \{ \xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2 \lfloor 3\xi_1 + \xi_2 \}$, of type $G_2$, $\Delta^+_1 = \{ \xi_1, \xi_2, \xi_1 + \xi_2, 2\xi_1 + \xi_2, 2\xi_1 + \xi_2 \lfloor 3\xi_1 + \xi_2 \}$, of type $G_2$.
- For $r = b_2(G/K) = 3$, we have only one case of t-roots system with $q = 6$, that is, of type $A_3$.

The case of $G_2/T$

- We first consider the case of full flag manifold $G_2/T$. Note that the highest root $\alpha$ of $G_2$ is given by $\alpha = 3\alpha_1 + 2\alpha_2$ and $G_2/T$ has a t-roots system of type $G_2$. Note that $G_2/T$ has only one complex structure and thus, up to isometry, there exist only one Kähler-Einstein metric. There exits exactly two non-Kähler Einstein metrics up to isometry. These are obtained from solutions of polynomial of degree 14. (A. Arvanitoyeorgos, I. Chrysikos and Y. S. [9])
- There are four other generalized flag manifolds (all exceptional Lie groups, $F_4$, $E_6$, $E_7$, $E_8$) with t-roots of type $G_2$. There are only one Kähler-Einstein metric and 6 non-Kähler Einstein metrics up to isometry. Recently M. Graev [12] has studied also these cases and he obtained one non-Kähler Einstein metric by a different method.

The case of $t$-roots of type $G_2$

<table>
<thead>
<tr>
<th>Flag manifold</th>
<th>Painted Dynkin diagram</th>
<th>number of Einstein metrics up to isometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4/(U(3) \times U(1))$</td>
<td>$\alpha_1 \alpha_2 \alpha_3 \alpha_4$</td>
<td>Kähler 1 non-Kähler 6</td>
</tr>
<tr>
<td>$E_6/(U(3) \times U(3))$</td>
<td>$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6$</td>
<td>Kähler 1 non-Kähler 6</td>
</tr>
<tr>
<td>$E_7/(U(6) \times U(1))$</td>
<td>$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6$</td>
<td>Kähler 1 non-Kähler 6</td>
</tr>
<tr>
<td>$E_8/(E_6 \times U(1) \times U(1))$</td>
<td>$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8$</td>
<td>Kähler 1 non-Kähler 6</td>
</tr>
</tbody>
</table>

The case of flag manifold $SU(4)/T$ and $SU(10)/S(U(1) \times U(2) \times U(3) \times U(4))$

- Note that for these cases $q = 6$ and the system of t-roots is of type $A_3$.
- For the case $SU(4)/T$, there is only one complex structure and thus, up to isometry, there exist only one Kähler-Einstein metric. There exits 3 non-Kähler Einstein metrics up to isometry, one of them is normal. (cf. Sakane [16] Lobachevskii J. Math. 4 (1999))
- For the case $SU(10)/S(U(1) \times U(2) \times U(3) \times U(4))$, There are 12 complex structure and thus, up to isometry, there exist 12 Kähler-Einstein metrics. There exits 12 non-Kähler Einstein metrics up to isometry. These are obtained from solutions of polynomial of degree 68.
Kähler-Einstein metric of a generalized flag manifold

- Put \( Z_t = \left\{ \Lambda \in t \mid \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for each } \alpha \in \Delta \right\} \).
- Then \( Z_t \) is a lattice of \( t \) generated by \( \{ \Lambda_1, \ldots, \Lambda_n \} \).
- Put \( Z_t^\perp = \{ \alpha \in Z_t \mid (\alpha, \alpha) > 0 \text{ for } \alpha \in \Pi - \Pi_0 \} \).
- Then we have \( Z_t^\perp = \sum_{\alpha \in \Pi - \Pi_0} \mathbb{Z} \Lambda_\alpha \).

We define an element \( \delta_m \in h_0 = \sqrt{-1} h \) by

\[
\delta_m = \frac{1}{2} \sum_{\alpha \in \Pi - \Pi_0} \alpha.
\]

Let \( c_1(M) \) be the first Chern class of \( M \). Then \( 2\delta_m \in Z_t^\perp \) corresponds to \( c_1(M) \).

Note that \( 2\text{nd} \) Betti number \( b_2(M) \) of \( M \) is given by

\[
b_2(M) = \dim t = \text{the cardinality of } \Pi - \Pi_0 = r.
\]

Riemannian submersion

- Let \( G \) be a compact semi-simple Lie group and \( K, L \) two closed subgroups of \( G \) with \( K \subset L \). Then we have a natural fibration \( \pi : G/K \to G/L \) with fiber \( L/K \).
- With respect to \( B \) (Killing form of \( g \)), \( \mathfrak{p} = \mathfrak{l}^+ \) in \( g \); the orthogonal complement of \( \mathfrak{l} \) in \( g \), \( \mathfrak{n} = \mathfrak{t}^+ \) in \( \mathfrak{l} \); the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{l} \). Then \( g = \mathfrak{l} \oplus \mathfrak{p} = \mathfrak{t} \oplus \mathfrak{n} \oplus \mathfrak{p} \).
- Denote a \( G \)-invariant metric \( \hat{g} \) on \( G/L \) defined by an \( \text{Ad}_G(L) \)-invariant scalar product on \( \mathfrak{p} \), an \( L \)-invariant metric \( \hat{g} \) on \( L/K \) defined by an \( \text{Ad}_L(K) \)-invariant scalar product on \( \mathfrak{n} \) and a \( G \)-invariant metric \( g \) on \( G/K \) defined by the orthogonal direct sum for these scalar products on \( \mathfrak{n} \oplus \mathfrak{p} \).

Kähler-Einstein metric of a generalized flag manifold

- Put \( k_\alpha = \frac{2(2\delta_m, \alpha)}{(\alpha, \alpha)} \) for \( \alpha \in \Pi - \Pi_0 \). Then

\[
2\delta_m = \sum_{\alpha \in \Pi - \Pi_0} k_\alpha \Lambda_\alpha = k_{\alpha_1} \Lambda_{\alpha_1} + \cdots + k_{\alpha_r} \Lambda_{\alpha_r}
\]

and each \( k_{\alpha_i} \) is a positive integer.

- The \( G \)-invariant metric \( g_{2\delta_m} \) on \( G/K \) corresponding to \( 2\delta_m \), which is a Kähler-Einstein metric, is given by

\[
g_{2\delta_m} = \sum_{\xi \in \Lambda_t^*} (2\delta_m, \xi) B_{(m_1 + \cdots + m_r)} = \sum_{j_1, \ldots, j_r} \left( \sum_{i=1}^r k_{i} j_{i} (\alpha_{i_1}, \alpha_{i_2}) \right) B_{|j_1, \ldots, j_r|},
\]

where \( \Lambda_t^* \) denotes the dual lattice of \( \Lambda_t \).

Riemannian submersion

Theorem

The map \( \pi \) is a Riemannian submersion from \((G/K, g)\) to \((G/L, \hat{g})\) with totally geodesic fibers isometric to \((L/K, \hat{g})\).

Note that \( \mathfrak{n} \) is the vertical subspace of the submersion and \( \mathfrak{p} \) is the horizontal subspace.

For a Riemannian submersion, O’Neill [14] has introduced two tensors \( A \) and \( T \). In our case we have \( T = 0 \), because the fibers are totally geodesic. We also have

\[
A_X Y = \frac{1}{2} [X, Y]_n \text{ for } X, Y \in \mathfrak{p}.
\]
Let \( \{X_i\} \) be an orthonormal basis of \( \mathfrak{p} \) and \( \{U_j\} \) an orthonormal basis of \( \mathfrak{n} \). We put for \( X, Y \in \mathfrak{p}, \ g(A_X, A_Y) = \sum_i g(A_X X_i, A_Y X_i) \). Then we have

\[
g(A_X, A_Y) = \frac{1}{4} \sum_i \hat{g}([X, X_i]_\mathfrak{n}, [Y, X_i]_\mathfrak{n}).
\]

Let \( r, \hat{r} \) be the Ricci tensor of the metric \( g, \hat{g} \) respectively. Then we have

\[
r(X, Y) = \hat{r}(X, Y) - 2g(A_X, A_Y) \quad \text{for} \ X, Y \in \mathfrak{p}.
\]

We decompose each irreducible component \( \mathfrak{p}_j \) into irreducible \( \text{Ad}(K) \)-modules:

\[
\mathfrak{p}_j = m_{j,1} \oplus \cdots \oplus m_{j,k_j}.
\]

As before we assume that \( \text{Ad}(K) \)-modules \( m_{j,t} \)

\((j = 1, \cdots, \ell, \ t = 1, \cdots, k_j)\) are mutually non-equivalent. Note that the metric of the form (5) can be written as

\[
g = y_1 \sum_{j=1}^{k_1} B_{l_{m_{1,j}}} + \cdots + y_{\ell} \sum_{j=1}^{k_\ell} B_{l_{m_{\ell,j}}} + z_1 B_{l_{n_1}} + \cdots + z_s B_{l_{n_s}} \quad (6)
\]

and this is a special case of the metric of the form (1).

---

\textbf{Lemma}

Let \( d_{j,t} = \dim m_{j,t} \). The components \( r_{j,t} \) \((j = 1, \cdots, \ell, \ t = 1, \cdots, k_j)\) of Ricci tensor \( r \) for the metric (6) on \( G/K \) are given by

\[
r_{j,t} = \hat{r}_j - \frac{1}{2d_{j,t}} \sum_i \sum_{j', t'} \frac{z_i}{y_j y_{j'}} \left[ i \right]_{j,t} \left[ j', t' \right].
\]

where \( \hat{r}_j \) are the components of Ricci tensor \( \hat{r} \) for the metric \( \hat{g} \) on \( G/L \).