Introduction
The famous Sard’s Theorem asserts that the set of critical values of a $C^m$ map $f : \mathbb{R}^n \to \mathbb{R}^k$ is of Lebesgue measure zero. Here, we focus on semi-algebraic functions on real closed fields. The notion of Lebesgue measure does not really make sense on a real closed field. However, the dimension of a semi-algebraic set can always be defined. We introduce a more general notion of critical values and show a Sard type theorem for these critical values, estimating the size of the set of our generalized critical values. There is no nice geometric measure theory on real closed field. We therefore make use of the notion of $\nu$-thin sets introduced in [V].

Strategy of the proof
We argue by induction on $k$.
Given $z \in \mathbb{R}$ and $Y \subset \mathbb{R}^n$, we define the $\nu$-neighborhood of $Y$ as the set
$$
\nu(Y) := \{x \in \mathbb{R}^n : d(x, Y) < z\}.
$$
We prove:

**Lemma 1**
If $Y \subset \mathbb{R}^n$ is a semi-algebraic subset of dimension less than $n$, then for any $z \in \nu$, the set $\nu(Y) \cap \nu = \nu$-thin.

We then give a characterization of $\nu$-thin sets:

**Lemma 2**
Let $X \subset \mathbb{R}^n$ be a semi-algebraic set of dimension $k$. The set $X$ is $\nu$-thin if there exists a semi-algebraic set $Y \subset \mathbb{R}^n$ of dimension less than $k$ such that $d_H(X, Y) \in \nu$, where $d_H$ stands for the Hausdorff metric.

We finally prove the following key lemma:

**Lemma 3**
Let $X \subset \mathbb{R}^{m+n}$ be a semi-algebraic set. If $X_t$ is $\nu$-thin for all $t \in \mathbb{R}$ then $X$ is a $\nu$-thin set.

This lemma makes it possible to derive the result from the induction hypothesis, applied to $X_t$.

Application
Let $f : X \to \mathbb{R}^k$ be a semi-algebraic mapping, with $X$ (not necessarily bounded) manifold. Recall that the set of critical values of $f$ is
$$
K_0(f) := \{y \in \mathbb{R}^k \mid \exists x \in f^{-1}(y), \nu(f, d_x, f) = 0\}.
$$
We prove:

**Definition 4**
The set
$$
K_\mu(f) := \{y \in \mathbb{R}^k \mid \exists x \in X, f(x) \to y \text{ and } (1 + |x|)\nu(f, d_x, f) \to 0\}
$$
is called the set of generalized critical values.

This set of course contains all the critical values of $f$ but also other elements that are sometimes called asymptotic critical values. In particular, $K(f)$ contains the asymptotic critical values at infinity studied in [KOS] and defined as:
$$
K_\infty(f) := \{y \in \mathbb{R}^k \mid \exists x \in X, |x| \to +\infty, f(x) \to y, \text{ and } |x|\nu(f, d_x, f) \to 0\}.
$$
This is proved in [KOS] that the set $K_\infty(f)$ is of dimension less than $k$.

Theorem 2
Let $f : X \to \mathbb{R}^k$ be a $C^1$ semi-algebraic mapping, with $X$ a $C^1$-submanifold of $\mathbb{R}^n$. The set of generalized critical values has dimension less than $k$.

Bibliography