Resonant bands, Aomoto complex and real 4-nets

Michele Torielli
Joint work with M. Yoshinaga (ArXiv: 1404.5014)
Department of Mathematics, Hokkaido University, JSPS
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Introduction
The resonant band is a useful notion for the computation of the nontrivial monodromy eigenspaces of the Milnor fiber of a real line arrangement. We develop the resonant band description for the cohomology of the Aomoto complex. As an application, we prove that real 4-nets do not exist. Let us fix some notation:
- \( k \in \mathbb{Z} \), \( k \geq 3 \);
- \( \mathbb{K} \) a field (generally, \( \mathbb{R} \) or \( \mathbb{C} \)) and \( \mathbb{K}^2 \) the projective plane;
- \( \mathcal{A} = \{ H_1, \ldots, H_n \} \) a line arrangement in \( \mathbb{K}^2 \);
- \( \mathcal{A} = \{ \mathcal{H}_1, \ldots, \mathcal{H}_6 \} \) the affine line arrangement in \( \mathbb{K}^2 = \mathbb{K}^2 \setminus \mathcal{H}_0 \) obtained from \( \mathcal{A} \);
- \( \mathcal{A}_0(\mathcal{A}) \) the Orlik-Solomon algebra of \( \mathcal{A} \) over \( \mathbb{F}_2 \) generated by the symbols \( e_1, \ldots, e_n \);
- For \( S \subseteq \mathcal{A} \), consider \( e(S) := \sum_{i \in S} e_i \in \mathcal{A}_0^1(\mathcal{A}) \) and \( \eta_0 := e(\mathcal{A}) = \sum_{i=1}^n e_i \).

Definition (k-nets)
\( \mathcal{A} \) supports a \( k \)-net structure if and only if there exist a partition \( \mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k \) and a finite set of points \( \mathcal{X} \subseteq \mathbb{K}^2 \) such that:
- For all \( i \neq j \), if \( H \in \mathcal{A}_i \) and \( H' \in \mathcal{A}_j \), then \( H \cap H' \neq \emptyset \);
- For all \( p \in \mathcal{X} \) and for all \( i = 1, \ldots, k \), there exists a unique \( H \in \mathcal{A}_i \) such that \( p \in H \).

Known facts
- If \( k \geq 5 \) there does not exist any \( k \)-net;
- There exist infinitely many 3-nets;
- The Hesse arrangement is the only known 4-net.

Theorem 2 (Papadima-Suciu)
Consider \( S \subseteq \mathcal{A} \). Then \( e(S) \land \eta_0 = 0 \) if and only if \( \forall p \in \mathbb{K}^2 \) one of the following is satisfied:
- if \( |A_p| \) is odd, then \( |A_p| = |S_p| \);
- if \( |A_p| \) is even then \( |S_p| \) is even, where \( A_p := \{ H \in \mathcal{A} \mid p \in H \} \).

From now on we consider the case \( \mathbb{K} = \mathbb{R} \) and \( n = \text{odd} \).
- The connected components of \( \mathbb{R}^2 \setminus \bigcup_{i \in \mathcal{A}} H \) are called chambers. The set of all chambers is denoted by \( \text{ch}(\mathcal{A}) \).
- Given \( C_1, C_2 \in \text{ch}(\mathcal{A}) \), \( d(C_1, C_2) \) is the number of line that separate the chambers.
- A band is a region bounded by two consecutive parallel lines.
- Each band \( B \) has two unbounded chambers \( U_1(B) \) and \( U_2(B) \).
- A band \( B \) is called resonant if \( d(U_1(B), U_2(B)) \) is even. The set of all resonant bands is denoted by \( \text{RB}(\mathcal{A}) \).

Theorem A (T.-Yoshinaga)
\( \text{Ker}(\nabla) \cong H^0(A^{\text{res}}_1(\mathcal{A}), \eta_0) \).

Proposition (T.-Yoshinaga)
When \( |A_p| = 4 \), then there are four cases:
- \( S_p = \emptyset \);
- \( S_p = A_p \);
- \( |S_p| = 2 \) and lines in \( S_p \) are adjacent.
- \( |S_p| = 2 \) and lines in \( S_p \) are separated by lines in \( A_p \setminus S_p \).
Moreover, if \( e(S) \land \eta_0 = 0 \), then (4) cannot happen.

Theorem B (T.-Yoshinaga)
There does not exist a real arrangement \( \mathcal{A} \) that supports a 4-net structure.

Proof
Suppose \( \mathcal{A} \) supports a 4-net structure with partition \( \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4 \). There exists a multiple point \( p \in \mathbb{R}^2 \) of \( \mathcal{A} \) with multiplicity 4 such that \( p \) is the intersection point of 4 lines \( H_i \in \mathcal{A}_i \). The lines are ordered like:

\[
\begin{align*}
1 \rightarrow & \ 2 \rightarrow \ 3 \rightarrow \ 4
\end{align*}
\]

We can now define \( S = \mathcal{A}_1 \cup \mathcal{A}_3 \). Then we have \( \eta_0 \land e(S) = 0 \). By definition, \( S_p = \{ H_1, H_3 \} \) consists of two lines and separated by the other two lines \( H_2 \) and \( H_4 \). Therefore (4) in the previous Proposition happens. This contradicts the statement of the last Proposition.

Bibliography