Topological triviality of linear deformations with constant Lê numbers

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Consider an analytic function \( f(t, z) \) in a neighbourhood of the origin in \( \mathbb{C} \times \mathbb{C}^n \). We are interested in the topology of the hypersurfaces 

\[
V(f_t) := f_t^{-1}(0) \subseteq \mathbb{C}^n \text{ as } t \text{ varies}
\]

(as usual, \( f_t(z) := f(t, z) \) and \( f_t(0) = 0 \))

**Lê-Ramanujam**

\[\{f_t\} \mu\text{-constant family of isolated singularities, } n \neq 3 \Rightarrow \{V(f_t)\} \text{ topologically trivial}\]

\[\Downarrow\]

\[\text{i.e., } \exists \text{ family } \{\varphi_t\} \text{ of local, ambient homeo. of } \mathbb{C}^n, \text{ depending continuously on } t, \text{ such that } \varphi_t(V(f_0)) = V(f_t)\]

**Timourian**

Same assumption \(\Rightarrow\) \(\{f_t\}\) topologically trivial

\[\Downarrow\]

\[\text{i.e., } \exists \{\varphi_t\} \text{ as above such that } f_0 = f_t \circ \varphi_t\]
Massey

Is the following true for non-isolated singularities?

\[
\text{Constancy of the Lê numbers of the functions } \ f_t \ (\text{as } t \text{ varies}) \quad \Rightarrow \quad \{ V(f_t) \} \text{ topologically trivial}
\]

Fernández de Bobadilla

- Yes, if \( n \geq 5 \) and \( \dim_0 \Sigma f_t = 1 \)
- No, in general — even for families which are linear in \( t \), i.e., of the form \( f(t, z) = f_0(z) + tg(z) \)
Parusiński

\{f_t\} \mu\text{-constant linear family of isolated singularities} \Rightarrow \{f_t\} \text{ topologically trivial}

For the proof:

- **Theorem** (Parusiński)
  Suppose \(f(t, z) = f_0(z) + tg(z)\). If, in a neighbourhood of the origin,
  \[|g(z)| \ll \|\text{grad}f(t, z)\|_\infty \text{ as } (t, z) \to \mathbb{C} \times (f_0^{-1}(0) \cap g^{-1}(0)),\]
  then the family \(\{f_t\}\) is topologically trivial.

- **Theorem** (Lê-Saito)
  \(\{f_t\} \mu\text{-constant family of isolated singularities} \Rightarrow t\text{-axis satisfies Thom’s } a_f \text{ condition at the origin w.r.t. ambient stratum}\)
Today

Constancy of the Lê numbers of the function \( f \) itself along the strata of an analytic stratification of \( \mathbb{C} \times (f_0^{-1}(0) \cap g^{-1}(0)) \)

\( \downarrow \)

\( \{f_t\} \) topologically trivial
Gap sheaves

\((X, \mathcal{O}_X)\) complex analytic space
\(W \subseteq X\) analytic subset
\(\mathcal{I}\) coherent sheaf of ideals in \(\mathcal{O}_X\)
\(V(\mathcal{I})\) the analytic space defined by the vanishing of \(\mathcal{I}\)

Take a point \(x \in V(\mathcal{I})\)

\[
\mathcal{I}_x := \bigcap_{1 \leq i \leq s} Q_i \quad \text{(minimal primary decomposition)}
\]

\(\mathcal{I}_x \cap W := \bigcap_{1 \leq i \leq s} Q_i \quad \text{if} \quad V(Q_i) \cap W = \emptyset
\]

Performing this operation simultaneously at all points of \(V(\mathcal{I})\) leads to a coherent sheaf of ideals called gap sheaf and denoted by \(\mathcal{I} \cap W\)

**Notation**

\(V(\mathcal{I}) \cap W := \text{the analytic space } V(\mathcal{I} \cap W)\)
 Lê numbers

Consider an analytic function $h(z)$ in a neighbourhood of $0 \in \mathbb{C}^n$.

$z = (z_1, \ldots, z_n)$ linear coordinates for $\mathbb{C}^n$.

For $0 \leq i \leq n - 1$,

- $\Gamma^i_{h,z} := \mathcal{V}\left(\frac{\partial h}{\partial z_{i+1}}, \ldots, \frac{\partial h}{\partial z_n}\right) - \Sigma h$

  i-th polar variety of $h$ with respect to $z$.

- $[\Lambda^i_{h,z}] := \left[\Gamma^i_{h,z} \cap \mathcal{V}\left(\frac{\partial h}{\partial z_{i+1}}\right)\right] - \left[\Gamma^i_{h,z}\right]$

  i-th Lê cycle of $h$ with respect to $z$.

The i-th Lê number of $h$ at $p = (p_1, \ldots, p_n)$ with respect to $z$ is the intersection number

$$\lambda^i_{h,z}(p) := \left([\Lambda^i_{h,z}] \cdot [\mathcal{V}(z_1 - p_1, \ldots, z_i - p_i)]\right)_p$$

provided this intersection is 0-dimensional or empty at $p$; otherwise, we say that $\lambda^i_{h,z}(p)$ is undefined.
\( \lambda^i_{h,z}(p) = 0 \) for \( i > \dim_p \Sigma h \). For this reason, we usually only consider

\[ \lambda^0_{h,z}(p), \ldots, \lambda^{\dim_p \Sigma h}_{h,z}(p) \]

- If \( p \) is an isolated singularity of \( h \), then

\[ \lambda^0_{h,z}(p) = \text{Milnor number of } h \text{ at } p \]
Let \( f(t, z) = f_0(z) + tg(z) \) be a linear deformation
\((t, z) := (t, z_1, \ldots, z_n) \) linear coordinates for \( \mathbb{C} \times \mathbb{C}^n \)

We may assume \( \Sigma f \subseteq V(f), \) and hence, \( \Sigma f \subseteq \mathbb{C} \times (f_0^{-1}(0) \cap g^{-1}(0)) \)

Pick an analytic stratification \( \mathcal{S} \) of \( f_0^{-1}(0) \cap g^{-1}(0) \)

- \( q_0 \in \Sigma f \Rightarrow q_0 = (a_0, p_0) \in \mathbb{C} \times S(p_0) \)

Definition

The deformation \( f \) is \( \lambda_{(t,z)} \)-constant with respect to \( \mathcal{S} \) if for any point \( q_0 = (a_0, p_0) \in \Sigma f \) and any integer \( 0 \leq i \leq \dim_{q_0} \Sigma f \):

\[
\lambda_{f, (t,z)}^i(q) \text{ is defined and independent of } q,
\]

for any \( q = (a, p) \in \mathbb{C} \times S(p_0) \) near \( q_0 \)

Theorem (with M. Ruas)

\( f \) is \( \lambda_{(t,z)} \)-constant with respect to \( \mathcal{S} \) \( \Rightarrow \) \( \{f_t\} \) topologically trivial
Example

- Consider the Briançon-Speder example

  \[ F(t, z) := z_3^5 + z_2 z_1 + z_1^{15} + tz_2^6 z_3 \]

Then \( \{F_t\} \) is a \( \mu \)-constant family of isolated surface singularities in \( \mathbb{C}^3 \) in which the \( t \)-axis cannot be chosen as a Whitney stratum of \( V(F) \).

- Now, consider

  \[ f(t, w, z) := F(t, z) \]

Then \( \{f_t\} \) is a family of line singularities in \( \mathbb{C}^4 = \mathbb{C}_w \times \mathbb{C}_z^3 \)

  - \( f \) is \( \lambda_{(t, w, z)} \)-constant with respect to the stratification

    \[ \mathcal{S} := \left\{ \mathbb{C}_w \times \{0\}, (f^{-1}_0(0) \cap g^{-1}(0)) \setminus \mathbb{C}_w \times \{0\} \right\} \]

  - The \( t \)-axis in \( \mathbb{C}_t \times \mathbb{C}_w \times \mathbb{C}_z^3 \) cannot be chosen as a Whitney stratum of \( V(f) \).