Splitting Singular Fibers for Degenerations of Elliptic Curves

A degeneration of complex curves is a proper surjective holomorphic map from a complex surface to a disk such that its central fiber is singular and the other fibers are smooth complex curves. In order to classify "atomic fibers", that is, unsplittable singular fibers, S. Takamura (Kyoto Univ.) introduced the concept of "barking deformation", which splits a single singular fiber of a degeneration into simpler ones. What types of singular fibers appear under such a deformation? The author will give an almost complete answer to this question in the case of degenerations of elliptic curves. In the poster, the author will present the result together with some explicit examples, after reviewing the Takamura Theory.

\[ M : \text{complex surface} \quad \Delta := \{ s \in \mathbb{C} : |s| < 1 \} : \text{unit disk} \]
\[ \pi : M \to \Delta : \text{proper surjective holomorphic map such that} \]
\[ (1) \text{its central fiber over } 0 \in \Delta \text{ is singular} \]
\[ (2) \text{the other fibers are complex curves of genus } g \]
We call \( \pi : M \to \Delta \) a degeneration of complex curves of genus \( g \)

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Supposed that
\[ (1) \ \exists_0 M_0 \to \Delta_0 \text{ is a degeneration } \pi : M \to \Delta, \]
\[ (2) \pi_* : M_0 \to \Delta_0 \text{ has at least 2 singular fibers.} \]
We call \( \Psi : M \to \Delta \times \Delta' \)

A singular fiber \( \pi^{-1}(s) \) of \( \pi_t : M_t \to \Delta_t \) is called the main fiber if \( s = 0 \) a subordinate fiber if \( s \neq 0 \)

Toy model ~ What is a barking?

\[ \pi : \mathbb{C}^2 \to \mathbb{C} \text{ defined by } (w, \eta) = w^3 \eta^4 \quad (w, \eta) \in \mathbb{C}^2 \]
\[ \pi_t : \mathbb{C}^2 \to \mathbb{C} \text{ defined by } (w, \eta) = w^3 \eta^2(\eta w - t)^2 \quad (w, \eta) \in \mathbb{C}^2 \]
\[ \pi^{-1}(0) : w^3 \eta^4 = 0 \quad \pi_t^{-1}(0) : w^3 \eta^2(\eta w - t)^2 = 0 \quad (t \neq 0) \]

... Patch such deforming hypersurfaces ...

What subordinate fibers appear?

The singular fiber \( X \) splits into \( X(1), X(2), \ldots, X(j) \),
\[ \Rightarrow \epsilon(X) - 2(1 - g) = \sum (\epsilon(X_i) - 2(1 - g)) \]
In particular, if \( g = 1 \), then \( \epsilon(X) = \sum_{i=0}^{j} \epsilon(X_i) \).

Assume that
(1) The core is a projective line,
(2) \( \frac{m_1^{(1)} + m_2^{(1)} + \cdots + m_n^{(1)}}{m_1^{(0)} + m_2^{(0)} + \cdots + m_n^{(0)}} = \frac{m_1^{(2)} + m_2^{(2)} + \cdots + m_n^{(2)}}{m_1^{(0)} + m_2^{(0)} + \cdots + m_n^{(0)}} \]
(3) \( X \) has 3 branches.

Then we can use the following criteria.

Example 1

If there are proportional sequences,
\[ \# \{ \text{singularities on a subordinate fiber} \} = \gcd(m_1^{(0)}, n_1^{(0)}) \]
\[ \# \{ \text{subordinate fibers} \} = m_1^{(0)} / \gcd(m_1^{(0)}, n_1^{(0)}) \]
where \( m_1^{(0)}, n_1^{(0)} \) are the respective last terms.

Example 2

If there are no proportional sequences,
\[ \# \{ \text{singularities on a subordinate fiber} \} = \gcd(m_0, n_0) \]
\[ \# \{ \text{subordinate fibers} \} = m_1^{(0)} / \gcd(m_0, n_0) \]
where \( m_0, n_0 \) are the cores' terms.

Singular fibers appearing under BARKING deformations for degenerations of elliptic curves

Reference
