1 What is a Lefschetz fibration?

A Lefschetz fibration is a fibering structure on a 4-dimensional manifold, which resembles a surface bundle over a surface, but which allows finitely many singularities, called Lefschetz type critical points.

Definition 4.1 $f$ is a Lefschetz fibration over $\Sigma_g$ with regular fiber $\Sigma_b$ (for short, genus-$g$ LF/$\Sigma_g$) if the following conditions are satisfied:

1. $b_1, \ldots, b_6 \in \Sigma_b$; critical values of $f$ is a $\Sigma_b$-bundle over $\Sigma_g \setminus \{b_1, \ldots, b_6\}$.
2. Each singular fiber $f^{-1}(b_i)$ has a unique critical point $q_i \in f^{-1}(b_i)$.

Around $q_i$ and $b_i = f(q_i)$, $f$ is expressed as $(z_1, z_2) \mapsto z_1^2 + z_2^2$ by local complex coordinates, which are compatible with the orientations.

3. (relative minimality) Each fiber does not contain a $(-1)$ sphere.

4. (non-triviality) $f$ contains at least one singular fiber.

Each singular fiber $f^{-1}(b_i)$ is obtained by crushing a simple closed curve $\epsilon_i$ on a regular fiber $f^{-1}(b)$ to a point.

The curve $\epsilon_i$ is called the vanishing cycle of the singular fiber $f^{-1}(b_i)$.

2 Question

The minimal number of singular fibers in a LF is an interesting object.

$N(g,h) = \min \{\text{the number of singular fibers in } f | f \text{ is a genus-}g \text{ LF/}\Sigma_g\}$.

There have been a lot of studies about $N(g,h)$ [KOO1], etc.

In the case of LF’s over the torus, however, very few constraints have been known. So we will study $N(g,1)$, which means the case of LF’s over the torus.

3 Mapping class group

$M_g = \{f : \Sigma_g \rightarrow \Sigma_g | f \text{ is an orientation-preserving diffeomorphism}/\text{isotopy}\}$ is a group w.r.t. the product induced by composition as maps and called the mapping class group of $\Sigma_g$. $M_g$ has fundamental elements, called Dehn twists which generate $M_g$.

$\tau_c : A \text{ right hand Dehn twist} along a simple closed curve } c.

Let $\gamma_i$ be a loop on $\Sigma_g$ based at $b_i$ which surrounds exclusively $b_i$. The local monodromy around $b_i$ is the right hand Dehn twist $\tau_{\gamma_i}$ along the corresponding vanishing cycle $\gamma_i$. That is, $f^{-1}(\gamma_i) \rightarrow \Sigma_g \times \{0, 1\}$ identifying $\Sigma_g \times 0$ and $\Sigma_g \times 1$ via the diffeomorphism $\tau_{\gamma_i}$: $f^{-1}(\gamma_i) \cong \Sigma_g \times \{0, 1\}$, $(x,1) \sim (\tau_{\gamma_i}(x),0)$.

4 Monodromy representation

There is a good relation between Lefschetz fibrations and the mapping class groups of surfaces.

Theorem 4.1 (Classification of LF) [M].

If $g \geq 2$, then

$$\underset{\text{isomorphism}}{g : \text{LF/}\Sigma_g} \rightarrow M_g$$

$\Phi : \tau_c(\Sigma_g \setminus \{b_1, \ldots, b_6\}) \rightarrow M_g$; anti-homomorphism with $\Phi(\gamma_i)(\text{right hand Dehn twist})/\text{conjugacy}$

$\{\tau_{\gamma_1}, \ldots, \tau_{\gamma_6} \text{ = a product of } h \text{ commutators in } M_g\}$

$\text{elementary transformation & global conjugacy}$

In the case of LF’s over the torus, if we find a relation in the form a product of $h$ right hand Dehn twists $\tau_{\gamma_i}$ is a single commutator in $M_g$, then we can construct a genus-$g$ LF/\Sigma_g with $h$ singular fibers. So, for our purpose, all we have to do is to find a relation as above with a small number $n$ of the right hand Dehn twists in the left side.

In the rest of this poster, we will denote a right hand Dehn twist along a curve $\alpha$ simply by $\alpha$ (the curve itself), and the inverse of a right hand Dehn twist $\alpha$ by $\bar{\alpha}$.

5 $k$-holed torus relation

$k$-holed torus relations which are relations in the mapping class group of $k$-holed torus (torus with $k$ boundary components), have been constructed by Korkmaz & Ozbagci [KOO2]. These relations will be used to construct LF’s over the torus. (In the following figures, $\alpha_i$’s represent the boundary components of $k$-holed torus.)

6 Relations representing a commutator

By embedding the $k$-holed torus in $\Sigma_g$ and using the $k$-holed torus relations, the author found the following relations.

7 Conclusion

The relations above can be used to construct a genus-$\ell$ LF/T$^2$ with $n$ singular fibers, a genus-$\ell$ LF/$T^2$ with $7$ singular fibers, a genus-$\ell$ LF/T$^2$ with $6$ singular fibers, and a genus-$g$ LF/T$^2$ with $5$ singular fibers ($g \geq 7$). As a result of these, we obtain:

Theorem 7.1 [H]

References


[N] Hamada, Upper bounds for the minimal number of singular fibers in a Lefschetz fibration over the torus, in preparation.