POTENTIAL THEORETIC CHARACTERIZATIONS OF NONSMOOTH DOMAINS

—Converse of the boundary Harnack principle—

Hiroaki Aikawa

1. Introduction

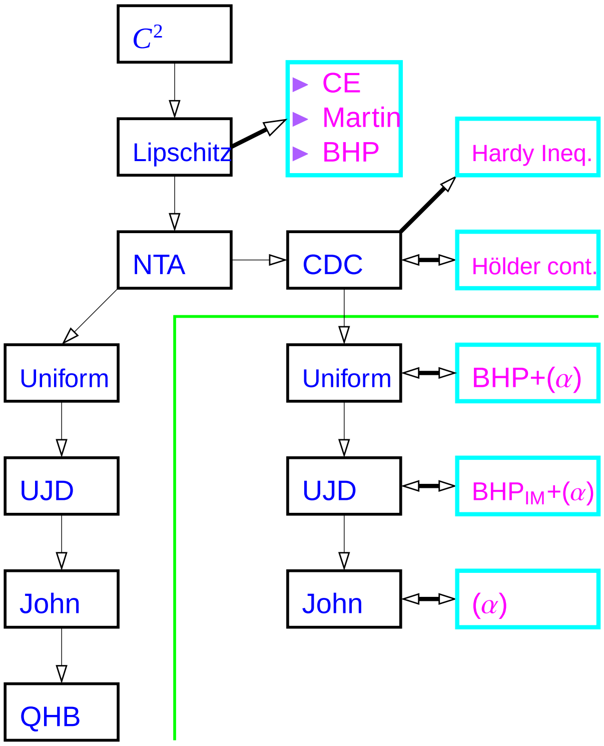
Potential theory for nonsmooth domains, e.g.

- Lipschitz domains,
- NTA domains,
- uniform domains,
- John domains,
- Hölder domains,

since Carleson (1962) and Hunt-Wheeden (1968) for Lipschitz domains.
Boundary Harnack principle (BHP) is crucial.

Geometry $\Longrightarrow$ Potential Properties

Domain	Topics	Authors
Half space	Carleson Estimates	Carleson (61)
Lipschitz	Martin boundary	Hunt-Wheeden (70)
Lipschitz	Harmonic analysis	Dahlberg (77)
Lipschitz	Uniform elliptic equation	Ancona (78)
Lipschitz	Harmonic measure	Wu (78)
NTA	Harmonic analysis	Jerison-Kenig (82)
Hölder, John	Probability, No exterior	Bass-Burdzy (91)
Uniformly John	CDC, Internal metric	Balogh-Volberg (96)
Uniform	No CDC, Uniform	HA (00)
Inner Uniform	CDC, Gromov hyperbolic	Bonk-Heinonen-Koskela (01)
Uniformly John	No CDC, Internal metric	HA-Lundh-Mizutani (03)



Consider the opposite direction.

Potential Properties $\Longrightarrow$ Geometry

Necessary and Sufficient condition,
provided the capacity density condition (CDC).

Notation: $D \subset \mathbb{R}^n$ with $n \geq 2$, $\delta_D(x) = \text{dist}(x, \partial D)$.
$B(x,r)$: open ball center at x and radius r;
$S(x,r)$: sphere center at x and radius r.

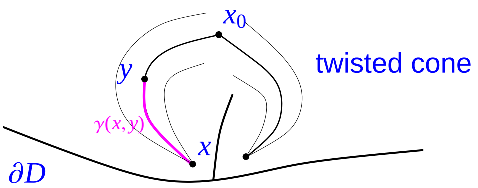
John domain

$^\exists c_J > 0$: John constant and
$^\exists x_0 \in D$: John center s.t.
$^\forall x \in D$ can be joined to x_0 by γ with

(1) $\delta_D(y) \geq c_J \ell(\gamma(x,y))$ for all $y \in \gamma$;

$\gamma(x,y)$: subarc of γ connecting x and y;
$\ell(\gamma(x,y))$: length.

In general, $0 < c_J < 1$.
Visualized as a twisted cone condition.

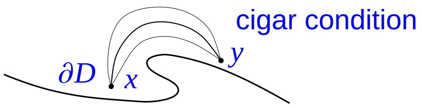


Uniform domain

$^\forall x, y \in D$ can be joined by $\gamma \subset D$ s.t. $\ell(\gamma) \le A|x-y|$ (Bounded Turning) and

(2) $\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \le A\delta_D(z)$

for $^\forall z \in \gamma$ (Cigar condition).

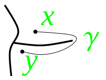


Remark 1

Uniform + corkscrew for $D^c \implies$ NTA (Jerison-Kenig (1982)).

Define the internal metric $\rho_D(x,y)$ by

$$\rho_D(x,y) = \inf\{\mathrm{diam}(\gamma)\};$$



- inf is over γ connecting x and y in D.
- $\mathrm{diam}(\gamma)$ is the diameter of γ.
- $|x-y| \le \rho_D(x,y)$.

Uniformly John domain (UJD)

$^\forall x, y \in D$ can be connected by $\gamma \subset D$ with $\ell(\gamma) \le A\rho_D(x,y)$ (Weak BT) and Cigar.

Bonk-Heinonen-Koskela (2001),
Väisälä (1998)

- Lipschitz $\subsetneq$ NTA $\subsetneq$ uniform $\subsetneq$ UJD $\subsetneq$ John.
- UJD = inner uniform.

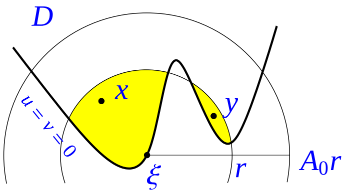
(uniform) BHP

$^\exists A_0,\ A_1 > 1,\ r_0 > 0$ depending only on D:

- $\xi \in \partial D$, $0 < r < r_0$,
- $u, v > 0$ are harmonic on $D \cap B(\xi, A_0 r)$,
- $u = v = 0$ on $\partial D \cap B(\xi, A_0 r)$,

then

(3) $\dfrac{u(x)/u(y)}{v(x)/v(y)} \le A_1$ for $x, y \in D \cap B(\xi, r)$.



Theorem A

Uniform domain $\implies$ BHP. HA (2001).

Sensitive about the boundary. Additional assumption on the boundary, capacity density condition (CDC). Define

$$\mathrm{Cap}_U(E) = \sup\{\mu(E) : G_U\mu \le 1 \text{ on } U,\ \mu \text{ is on } E\}.$$

Capacity density condition (CDC)

$^\exists A > 1,\ r_0 > 0$ s.t.

$$\frac{\mathrm{Cap}_{B(\xi,2r)}(B(\xi,r)\setminus D)}{\mathrm{Cap}_{B(\xi,2r)}(B(\xi,r))} \ge \frac{1}{A},$$

whenever $\xi \in \partial D$ and $0 < r < r_0$.

CDC $\iff$ Harmonic Measure Decay:

$$\omega(x, D\cap S(\xi,r), D\cap B(\xi,r)) \le A\left(\frac{\delta_D(x)}{r}\right)^{\beta}$$

for $x \in D \cap B(\xi, r/3)$, $\xi \in \partial D$, $0 < r < r_0$.
Hardy's inequality. Ancona (1986).
CDC $\iff$ Dirichlet solutions are Hölder continuous. HA (2002).

Theorem 1

Assume CDC. Then
John $\iff$ $^\exists\alpha > 0$ s.t.

(α) $\omega(x, D \cap S(\xi, r), D \cap B(\xi, r)) \ge \frac{1}{A}\left(\frac{\delta_D(x)}{r}\right)^\alpha$

for $x \in D \cap B(\xi, r/A)$, $\xi \in \partial D$ and $0 < r < r_0$.

Remark 2

In general, $0 < \beta < 1 < \alpha$. If D is smooth, then we may let $\alpha = \beta = 1$.

Theorem 2

Assume CDC. Then
uniform $\iff$ BHP + (α).

A ball wrt the internal metric is a connected component of $D \cap B(\xi, r)$.

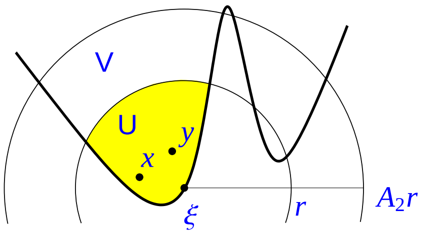
BHP wrt the internal metric

$^\exists A_2$, $^\exists A_3 > 1$ such that

- $\xi \in \partial D$, $0 < r < r_0$,
- U: connected compogent $D \cap B(\xi, r)$,
- $V \supset U$: connected comp. $D \cap B(\xi, A_2 r)$,
- $u, v > 0$ are harmonic on V, $u = v = 0$ on $\partial D \cap \partial V$,

then

$$\frac{u(x)/u(y)}{v(x)/v(y)} \le A_3 \quad \text{for } x, y \in U.$$



Theorem 3

Assume CDC. Then
UJD $\iff$ BHP wrt I. M. + (α).

Remark 3

UJD $\implies$ BHP wrt I. M. (ALM (2002)).

Remark 4

Finitely connected (FC) planar domains without singleton boundary satisfy CDC.

Corollary 1

Let $D \subset \mathbb{R}^2$ be bdd. FC. Then
(i) John $\iff$ (α).
(ii) uniform $\iff$ BHP + (α).
(iii) UJD $\iff$ BHP wrt IM + (α).

Remark 5

Jones (1980): BMO, Sobolev extensions.
Smith-Stegenga-Ullrich (1995): Hölder domain, satisfying the quasihyperbolic boundary condition.

2. Proof of Theorem 1

Define the quasihyperbolic metric $k_D(x, y)$ by

$$k_D(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\delta_D(z)},$$

where $\gamma \subset D$ connects x to y. Observe

Length of Harnack chain $\approx k_D(x, y)$.

Hence $^\exists A_4$ depending only on n such that

$$(4) \quad \exp(-A_4 k_D(x, y)) \le \frac{h(x)}{h(y)} \le \exp(A_4 k_D(x, y))$$

for harmonic function $^\forall h > 0$ on D.

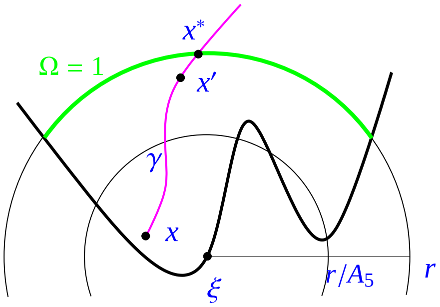
Proof of Theorem 1. Necessity. John $\implies$ (α). Let $\xi \in \partial D$, $0 < r < r_0$, and

$$\Omega = \omega(\cdot, D \cap S(\xi, r), D \cap B(\xi, r)).$$

Take $x \in D \cap B(\xi, r/A_5)$ with $A_5 > 1$. Then $^\exists\gamma$ joining x, x_0 s.t. $\delta_D(y) \ge c_J \ell(\gamma(x, y))$ for $^\forall y \in \gamma$.

Let $x^* \in \gamma \cap S(\xi, r)$. Then

$$\delta_D(x^*) \geq c_J(1 - A_5^{-1})r,$$
$$k_D(x, x^*) \leq A \log(r/\delta_D(x)) + A.$$



Let $x' \in \gamma(x, x^*) \cap B(x^*, \delta_D(x^*)/2) \cap B(\xi, (1 - A^{-1})r)$. Then $\Omega(x') \approx 1$, and (4) gives

$$\frac{\Omega(x')}{\Omega(x)} \leq \exp(A_4 k_{D \cap B(\xi, r)}(x, x')) \leq A\left(\frac{r}{\delta_D(x)}\right)^{\alpha}$$

with $\alpha > 0$ depending on D. Hence (α) follows. $\square$

For the sufficiency we prepare

Lemma 1

Assume CDC. If $\xi \in \partial D$, $0 < r < r_0$, $\Omega = \omega(\cdot, D \cap S(\xi, r), D \cap B(\xi, r))$, then

$$\Omega(x) \leq 3^{\beta} A \left(\frac{\delta_D(x)}{r}\right)^{\beta} \quad \text{for } x \in D \cap B(\xi, r/3).$$

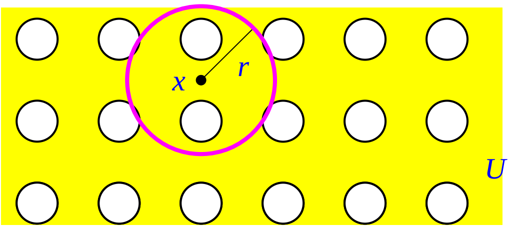
Lemma A

Define the *capacitary width* $w_\eta(U)$ by

$$\inf\{r > 0 : \frac{\mathrm{Cap}_{B(x,2r)}(B(x,r) \setminus U)}{\mathrm{Cap}_{B(x,2r)}(B(x,r))} \geq \eta \ {}^\forall x \in U\}$$

for $0 < \eta < 1$. Then, for $x \in U$, $R > 0$,

$$\omega(x, U \cap S(x, R), U \cap B(x, R)) \leq \exp(2 - \frac{AR}{w_\eta(U)}).$$



Proof of Theorem 1. Sufficiency. $(\alpha) \Longrightarrow$ John. Sufficient to show ${}^\exists A_6 \geq 2$: Let $x \in D$ and $\xi \in \partial D$ be s.t. $|x - \xi| = \delta_D(x) < r_0/2$. Then

- ${}^\exists y \in D \cap B(\xi, A_6\delta_D(x))$ with $\delta_D(y) \geq 2\delta_D(x)$,
- ${}^\exists \widetilde{xy}$ connecting x and y in $D \cap B(\xi, A_6\delta_D(x))$ s.t. $\delta_D(z) \geq \delta_D(x)/A_6$ for ${}^\forall z \in \widetilde{xy}$.

Let $\Omega = \omega(\cdot, D \cap S(\xi, 3A_6\delta_D(x)), D \cap B(\xi, 3A_6\delta_D(x)))$ with $A_6 \geq 2$. Then (α) gives

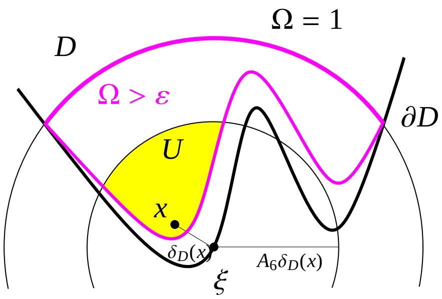
$$\Omega(x) \geq \frac{1}{A}\left(\frac{\delta_D(x)}{3A_6\delta_D(x)}\right)^{\alpha} = \frac{1}{A}\left(\frac{1}{3A_6}\right)^{\alpha} = 2\varepsilon.$$

Let U be the connected component of

$$\{z \in D \cap B(\xi, A_6\delta_D(x)) : \Omega(z) > \varepsilon\}$$

containing x. Observe $h = (\Omega - \varepsilon)/(1 - \varepsilon)$ is

- harmonic on U,
- $0 \leq h \leq 1$ on U,
- $h = 0$ on $\partial U \cap B(\xi, A_6\delta_D(x))$.



Hence, the maximum principle yields

$$h(x) \leq \omega(x, U \cap S(x, (A_6 - 1)\delta_D(x)), \quad U \cap B(x, (A_6 - 1)\delta_D(x))).$$

Claim ${}^\exists y \in U$ with $\delta_D(y) \geq 2\delta_D(x)$, provided A_6 is large. Suppose, to the contrary,

$$\delta_D(y) < 2\delta_D(x) \quad \text{for } {}^\forall y \in U.$$

Then CDC implies $w_\eta(U) \leq 4\delta_D(x)$ for ${}^\exists \eta > 0$, so that Lemma A yields that

$$h(x) \leq \exp(2 - A\frac{(A_6 - 1)\cancel{\delta_D(x)}}{4\cancel{\delta_D(x)}}).$$

Hence

$$h(x) \leq \exp(2 - A\frac{(A_6 - 1)}{4}).$$

On the other hand

$$h(x) = \frac{\Omega(x) - \varepsilon}{1 - \varepsilon} \geq \frac{\varepsilon}{1 - \varepsilon} \geq \frac{1}{2A}\left(\frac{1}{3A_6}\right)^{\alpha}.$$

Hence, A_6 cannot be so large. Claim follows.

- $^{\exists}y \in D \cap B(\xi, A_6\delta_D(x))$ with $\delta_D(y) \geq 2\delta_D(x)$,
- $^{\exists}\widetilde{xy}$ connecting x and y in $D \cap B(\xi, A_6\delta_D(x))$ s.t. $\delta_D(z) \geq \delta_D(x)/A_6$ for $^{\forall}z \in \widetilde{xy}$.

Since U is connected, $^{\exists}\widetilde{xy} \subset U$ connecting x and y. Let $z \in \widetilde{xy}$. Then by Lemma 1

$$\varepsilon < \Omega(z) \leq A\left(\frac{\delta_D(z)}{3A_6\delta_D(x))}\right)^{\beta},$$

so that $\delta_D(z) \geq \delta_D(x)/A$ for $^{\exists}A > 1$. □

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3. Proof of Theorem 2

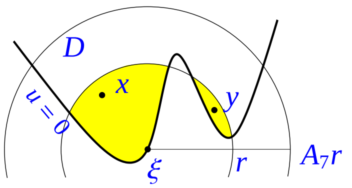
Carleson estimate (CE)

$^{\exists}A_7$ such that

- $\xi \in \partial D$, $0 < r < r_0$,
- $y \in D \cap B(\xi, r)$, $\delta_D(y) \geq \varepsilon r$ with $0 < \varepsilon < 1$,
- $u > 0$ is harmonic in $D \cap B(\xi, A_7 r)$,
- $u = 0$ on $\partial D \cap B(\xi, A_7 r)$,

then

$$u(x) \leq A_\varepsilon u(y) \quad \text{for } x \in D \cap B(\xi, r).$$



Lemma 2

BHP + (α) $\Longrightarrow$ CE.

Repeated application of the mean value property $\Longrightarrow$ upper estimate of the Green function.

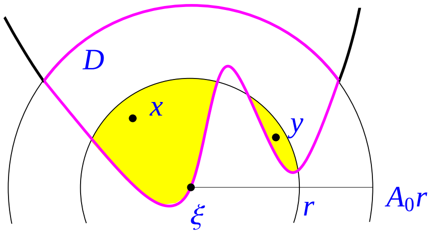
Lemma 3

CE + CDC $\Longrightarrow$

$$G(x, y)$$
$$\leq A \min\{\delta_D(x), \delta_D(y)\}^{2-n} \exp(-\tau k_D(x, y))$$

for $|x-y| \geq \frac{1}{2}\min\{\delta_D(x), \delta_D(y)\}$, where $\tau > 0$.

$D \cap B(\xi, r)$ may be disconnected; yet it is included in one connected component of $D \cap B(\xi, A_0 r)$ with large A_0 by BHP.

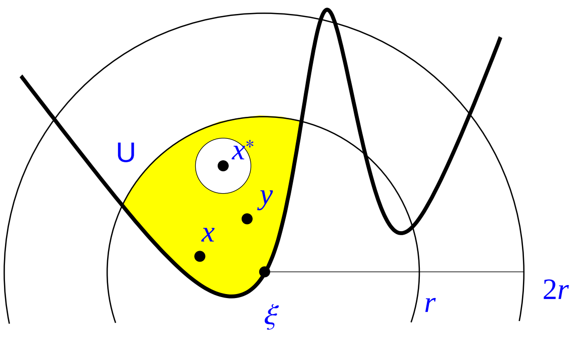


Lemma 4 (modified BHP)

- $0 < \varepsilon < 1/2$, $\xi \in \partial D$, $0 < r < r_0$,
- $^{\exists}x^* \in D \cap B(\xi, r)$ with $\delta_D(x^*) \geq 2\varepsilon r$,
- U is a connected component of $D \cap B(\xi, r)$ containing x^*,
- $u, v > 0$ are harmonic on $D \cap B(\xi, 2r) \setminus \{x^*\}$,
- $u = v = 0$ on $\partial D \cap B(\xi, 2r)$,

then

$$(5) \quad \frac{u(x)/u(y)}{v(x)/v(y)} \leq A_\varepsilon \quad x, y \in U \setminus B(x^*, \varepsilon r).$$



Uniform $\iff$ (6)

$$k_D(x,y) \le A \log \frac{\delta_D(x)+\delta_D(y)+|x-y|}{\min\{\delta_D(x),\delta_D(y)\}} + A'$$

whenever $x, y \in D$.

Proof of Theorem 2. BHP+(α) $\Longrightarrow$ uniform. Let us prove (6). May assume $\delta_D(y) \le \delta_D(x)$ small.

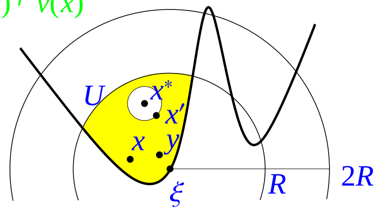
$|x-y| \le \frac{1}{2}\delta_D(x)$

$\overline{xy} \subset B(x, \frac{1}{2}\delta_D(x))$ connects x and y. So, $k_D(x,y) \le A$ and (6) follows.

Suppose then $r = |x-y| \ge \frac{1}{2}\delta_D(x)$.
Take $\xi \in \partial D$ with $\delta_D(x) = |x-\xi|$.
Note $x, y \in D \cap B(\xi, 3r)$. By BHP, x, y are in the same conn. comp. of $D \cap B(\xi, 3A_0 r)$.
John domain is OK (Theorem 1).
Hence $^{\exists}x^* \in \gamma \cap B(\xi, 4A_0 r)$ s.t. $\delta_D(x^*) \ge c_J A_0 r$ and x, y, x^* altogether belong to the same conn. comp. U of $D \cap B(\xi, 4A_0 r)$.

Apply Lemma 4 to $u = G(x^*, \cdot)$ and $v = \omega(\cdot, D \cap S(\xi, 2R), D \cap B(\xi, 2R))$ with $\varepsilon = c_J/8$ and $R = 4A_0 r$ in place of r. We obtain

$$\frac{G(x^*,x')}{G(x^*,x)} \Big/ \frac{v(x')}{v(x)} \le A \quad \text{for } x' \in S(x^*, c_J A_0 r/2).$$



Hence, (α), Lemmas 2 and 3 yield

$$r^{2-n} \approx G(x^*,x') \le A\frac{v(x')}{v(x)}G(x^*,x)$$
$$\le A\left(\frac{2R}{\delta_D(x)}\right)^{\alpha} \delta_D(x)^{2-n} \exp(-\tau k_D(x,x^*)).$$

Since $R \approx r$, it follows that

$$k_D(x,x^*) \le \frac{n-2+\alpha}{\tau} \log \frac{r}{\delta_D(x)} + A.$$

The same inequality holds for y in place of x, and hence (6) follows. □

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