HÖLDER CONTINUITY OF DIRICHLET SOLUTION FOR A GENERAL DOMAIN

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1. Introduction

Talk based on HA [Aik02].

Let ${}^{\forall}E \subset \mathbb{R}^n$, $n \geq 2$.

$\Lambda_\alpha(E)$: α-Hölder continuous functions on E, where $0 < \alpha \leq 1$.

Define the norm of $f \in \Lambda_\alpha(E)$:

$$\|f\|_{\Lambda_\alpha(E)} = \sup_{x\in E} |f(x)| + \sup_{\substack{x,y\in E\\ x\neq y\\ |x-y|\leq 1}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Note $f \in \Lambda_\alpha(E)$ extends to $\overline{E}$; $f \in \Lambda_\alpha(\overline{E})$.

Let D be a domain.

$\mathscr{H}(D)$: all harmonic functions on D.

Let $\mathscr{H}_\alpha(D) = \mathscr{H}(D) \cap \Lambda_\alpha(D)$.

$H_D f$: the Dirichlet solution of f over D, where f is a function on ∂D.

- If D is regular, then H_D maps $C(\partial D)$ to $\mathscr{H}(D) \cap C(\overline{D})$.

Investigate conditions for H_D to map $\Lambda_\alpha(\partial D)$ to $\mathscr{H}_\alpha(D)$. In other words, $\|H_D\|_\alpha < \infty$? where

$$\|H_D\|_\alpha = \sup_{\substack{f\in\Lambda_\alpha(\partial D)\\ \|f\|_{\Lambda_\alpha(\partial D)}\neq 0}} \frac{\|H_D f\|_{\Lambda_\alpha(D)}}{\|f\|_{\Lambda_\alpha(\partial D)}}.$$

- Hinkkanen [Hin88] considered this problem mainly for finitely connected planar regions.
- Sugawa [Sug99] also treated planar regions

$\omega(\cdot, E, U)$: harmonic measure over open U of $E \subset \partial U$.

Obs. $\omega(\cdot, E, U) = H_U(\chi_E)$.

Let $\delta_D(x) = \mathrm{dist}(x, \partial D)$.

$B(x, r)$: open ball with center x, radius r,

$S(x, r)$: sphere with center at x, radius r.

M stands for a positive constant.

2. Rule out a trivial boundary point

One might think $\|H_D\|_\alpha < \infty \implies D$:regular.

Let $B' = B(0,1) \setminus \{0\}$ be a punctured ball. 0 is irregular and yet $\|H_{B'}\|_\alpha < \infty$ for $0 < {}^{\forall}\alpha < 1$. This strange phenomenon is caused by an isolated boundary point.

Definition 1

$\xi \in \partial D$ is a trivial boundary point $\iff$ ${}^{\exists}r > 0$ s.t. $B(\xi, r) \setminus D$ is polar.

Proposition 1

Let E be the set of all trivial boundary points of D. Then

(i) E is polar.

(ii) $\widetilde{D} = D \cup E$ is a domain without trivial boundary points.

(iii) $\partial\widetilde{D} \subset \partial D$.

(iv) $H_D f = H_{\widetilde{D}} \widetilde{f}$ on D, where $\widetilde{f} = f|_{\partial\widetilde{D}}$.

Theorem 1

If D has no trivial boundary point, then $\|H_D\|_\alpha < \infty$ for ${}^{\exists}\alpha > 0$ $\implies D$ is regular.

3. Lipschitz continuity cannot be preserved by H_D

Let $D = B(0,1) \subset \mathbb{R}^2$ and $f(\xi) = f(\xi_1, \xi_2) = |\xi_2|$, a Lipschitz continuous function on ∂D.
However, the Dirichlet solution:

$$H_D f(x_1, 0) = \frac{1 - x_1^2}{\pi x_1} \log \frac{1 + x_1}{1 - x_1}$$

is not Lipschitz continuous.
Lipschitz continuity of $H_D f$ breaks down near $(\pm 1, 0)$. (Sugawa [Sug99, Example 6.1])
A similar example applies to the higher dimensional case.

More strongly we have

Theorem 2
There is no domain for which $\|H_D\|_1 < \infty$.

4. Characterization in terms of harmonic measure

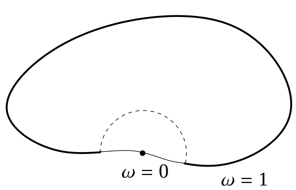
According to Sugawa [Sug99], we introduce Local Harmonic Measure Decay property (LHMD) and Global Harmonic Measure Decay property (GHMD).
LHMD with α: $^\exists M$ and $^\exists r_0$ depending only on D and α s.t.

$$\omega(x, D \cap S(a,r), D \cap B(a,r)) \le M \left(\frac{|x-a|}{r} \right)^\alpha \quad \text{for } x \in D \cap B(a,r),$$

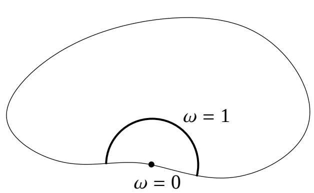
whenever $a \in \partial D$ and $0 < r < r_0$.
GHMD with α:

$$\omega(x, \partial D \setminus B(a,r), D) \le M \left(\frac{|x-a|}{r} \right)^\alpha \quad \text{for } x \in D \cap B(a,r),$$

whenever $a \in \partial D$ and $0 < r < r_0$.



GHMD.



LHMD.

Obviously, LHMD $\Longrightarrow$ GHMD by the maximum principle.
LHMD and GHMD are related to $\|H_D\|_\alpha$.
Define $\varphi_{a,\alpha}$ for $a \in \partial D$ by

$$\varphi_{a,\alpha}(\xi) = \min\{|\xi - a|^\alpha, 1\} \quad \text{for } \xi \in \partial D.$$

Theorem 3
Let D be regular, $0 < \alpha < 1$.
Consider the following conditions:

(i) $\|H_D\|_\alpha < \infty$.
(ii) $H_D \varphi_{a,\alpha}(x) \le M|x-a|^\alpha$ for $^\forall a \in \partial D$.
(iii) D satisfies LHMD with α.
(iv) D satisfies GHMD with α.

Then

(i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv).

Moreover, if (iii) holds with $^\exists \alpha' > \alpha$, then (i) holds.

5. More geometric characterization

Define the Green capacity $\mathrm{Cap}_U(E)$ for $E \subset U$ by

$$\mathrm{Cap}_U(E) = \sup\{\mu(E) : G_U\mu \le 1 \text{ on } U,\ \mu \text{ is on } E\}.$$

Definition 2
∂D satisfies the capacity density condition (CDC) $\iff$ ${}^\exists\varepsilon$ and ${}^\exists r_0$ s.t.

$$\frac{\mathrm{Cap}_{B(a,2r)}(B(a,r)\cap\partial D)}{\mathrm{Cap}_{B(a,2r)}(B(a,r))} \ge \varepsilon \quad \text{for} \quad {}^\forall a \in \partial D,\ 0 < {}^\forall r < r_0.$$

Definition 3
∂D is uniformly perfect (UP) $\iff$ $0 < {}^\exists c < 1$, ${}^\exists r_1 > 0$ s.t.

$$\partial D \cap A(a, cr, r) \ne \emptyset \quad \text{for } {}^\forall a \in \partial D,\ 0 < {}^\forall r < r_1,$$

where $A(x, r, R) = B(x, R) \setminus B(x, r)$.

- If $n = 2$, then CDC $\iff$ UP.
- CDC $\implies$ UP; the converse is not true for $n \ge 3$.
- GHMD with ${}^\exists\alpha > 0 \implies$ UP.
- LHMD with ${}^\exists\alpha > 0 \iff$ CDC (Ancona [Anc86]).

Corollary 1
Let D be a bounded regular domain. Then

$\|H_D\|_\alpha < \infty\ {}^\exists\alpha > 0 \iff \partial D$ satisfies CDC;

if furthermore $n = 2$, then

$\iff \partial D$ is UP.

The sufficiency is more or less well-known; the necessity part is new.

6. GHMD $\implies$ LHMD

Lemma 1
GHMD $\implies$ UP of ∂D.

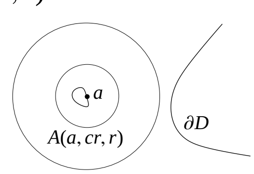
Proof. Suppose $n \ge 3$. Find $0 < {}^\exists\kappa < 1$ with

$$\omega(x, \partial D \setminus B(a, r), D) \le \frac{1}{3} \quad \text{for } x \in D \cap B(a, \kappa r).$$

Let $c = 2^{1/(2-n)}\kappa < \kappa < 1$. Suppose $\partial D \cap A(a, cr, r) = \emptyset$ for ${}^\exists a \in \partial D$ and $0 < {}^\exists r < r_1$. Then $A(a, cr, r) \subset D$ and

$$\omega(x, \partial D \setminus B(a, r), D) \ge 1 - \left(\frac{|x - a|}{cr}\right)^{2-n}$$

for $x \in A(a, cr, r) \subset D$.



If we let $|x-a| \uparrow \kappa r$, then GHMD yields

$$\frac{1}{3} \geq 1 - \left(\frac{\kappa r}{cr}\right)^{2-n} = \frac{1}{2},$$

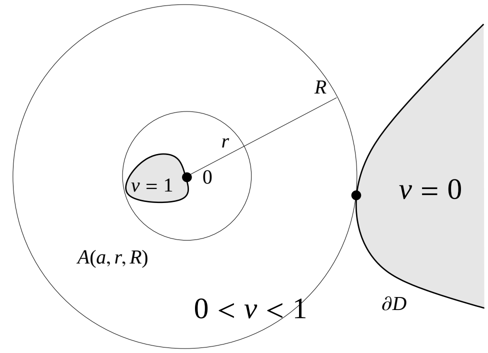
a contradiction. Thus the lemma follows for $n \geq 3$. □

Lemma 2
GHMD with $\alpha \implies$ LHMD with α.

7. GHMD $\implies$ UP of ∂D for $n = 2$

Proof of GHMD $\implies$ UP of ∂D for $n = 2$. We may assume $\overline{D} \subset B = B(0,1)$. Suppose $0 \in \partial D$ and $A(0,r,R) \subset D$ with $0 < r < R < 1$. It suffices to show $R/r \leq M$. Let

$$v = \begin{cases} \omega(\cdot, \partial D \cap B(0,r), D) & \text{on } D, \\ 1 & \text{on } B(0,r) \setminus D, \\ 0 & \text{on } B \setminus (B(0,r) \cup D). \end{cases}$$



Then

$$v = G_B\mu - G_B\nu$$

with measures $\mu \geq 0$ on $\partial D \cap \overline{B(0,r)}$ and $\nu \geq 0$ on $\partial D \setminus B(0,R)$. Let σ_ρ be the unit measure uniformly distributed on $S(0,\rho)$ for $0 < \rho < 1$. Then

(1) $\quad G_B\sigma_\rho(x) = \min\{\log \frac{1}{|x|}, \log \frac{1}{\rho}\} \quad$ on B

(See [AG01, p.104]). Let ρ be close to 1, $\overline{D} \subset B(0,\rho) \subset B$. Then $v = G_B\mu - G_B\nu = 0$ on $S(0,\rho)$. Fubini's theorem implies

(2) $\quad \|\mu\| = \|\nu\|.$

By definition $0 \leq v = G_B\mu - G_B\nu \leq 1$ on B. In particular, $G_B\mu(0) - G_B\nu(0) \leq 1$. Since μ and ν are measures on $\partial D \cap B(0,r)$ and $\partial D \setminus B(0,R)$ respectively,

$$G_B\mu(0) \geq \|\mu\| \log \frac{1}{r}, \quad G_B\nu(0) \leq \|\nu\| \log \frac{1}{R}.$$

Hence by (2)

(3) $\quad \log \frac{R}{r} \leq \frac{1}{\|\mu\|}.$

Give a lower bound of $\|\mu\|$ in terms of the Dirichlet integral of v. Since $\Delta G_B(\cdot, y) = -2\pi\delta_y$, $v = 1$

q.e. on supp μ and $v = 0$ q.e. on supp ν,

(4) $\quad \int_B |\nabla v|^2 dx = 2\pi \int_B v d(\mu - \nu) = 2\pi\|\mu\|.$

We may assume that $^\exists a \in S(0,R) \cap \partial D$. Then

$$v \leq \omega(\cdot, \partial D \setminus B(a, R-r), D) \quad \text{on } D.$$

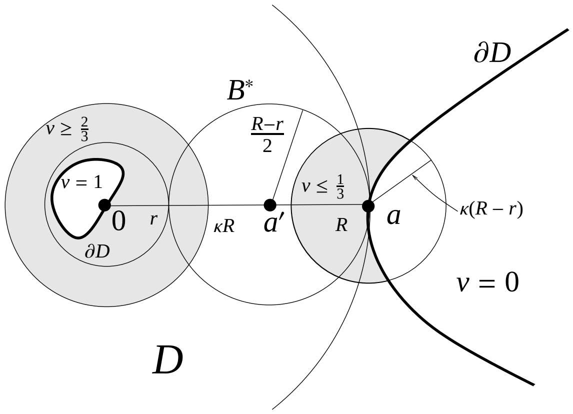
GHMD implies

$$v \leq \frac{1}{3} \quad \text{on } D \cap B(a, \kappa(R-r)).$$

Note $\omega(\cdot, \partial D \cap B(0,r), D) = \omega(\cdot, \partial D \cap B(0,R), D)$ from $A(0,r,R) \subset D$. Hence

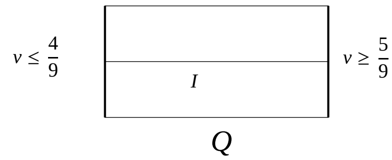
$$v = 1 - \omega(\cdot, \partial D \setminus B(0,R), D) \geq \frac{2}{3} \quad \text{on } D \cap B(0, \kappa R).$$

Let $a' = \frac{R+r}{2R}a$, $B^* = B(a', \frac{1}{2}(R-r))$. Then $B^* \subset A(0,r,R) \subset D$ and $^\exists x, {}^\exists y \in B(a', \frac{1-\kappa}{2}(R-r))$ s.t. $v(x) \leq 1/3$, $v(y) \geq 2/3$.



By the Harnack inequality and a suitable coordinate we find a square $Q = I \times J \subset B^*$ s.t. $|I| \approx |J| \approx R$,

$$v \le \frac{4}{9} \quad \text{on the left side of } Q,$$
$$v \ge \frac{5}{9} \quad \text{on the right side of } Q.$$



Then

$$\frac{1}{9} \le \int_I |\nabla v(x_1, x_2)| dx_1 \le \left(\int_I |\nabla v|^2 dx_1 \right)^{\frac{1}{2}} |I|^{\frac{1}{2}}$$

for $^\forall x_2 \in J$. Hence

$$\int_Q |\nabla v|^2 dx_1 dx_2 = \int_J dx_2 \int_I |\nabla v|^2 dx_1 \ge M \frac{|J|}{|I|} \ge M'.$$

Hence (3) and (4) yield

$$\log \frac{R}{r} \le \frac{1}{\|\mu\|} = \frac{2\pi}{\int_B |\nabla v|^2 dx} \le \frac{2\pi}{M'}.$$

Thus R/r is bounded. □

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