GENERALIZED CRANSTON-MCCONNELL INEQUALITIES AND MARTIN BOUNDARIES OF UNBOUNDED DOMAINS

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1. Cranston-McConnell inequalities

Joint work with M. Murata.
$D \subset \mathbf{R}^2$, Green function: $G(x,y)$,
Area of D: $|D|$.

Theorem (Cranston–McConnell(83))

$$\frac{1}{h(x)}\int_D G(x,y)h(y)dy \le c|D|$$

$h > 0$: harmonic

D: No assumption on smoothness

Generalized Cranston–McConnell inequalities
$\Phi(t_1,\dots,t_n) \ge 0$, $t_j \ge 0$, $j = 1,\dots,n$.
$\eta > 1$

$$\Psi(t_1,\dots,t_n) = \sup_{\eta^{-2}<c_1,\dots,c_n<\eta^2} \Phi(c_1t_1,\dots,c_nt_n).$$

Theorem

$$\frac{1}{u(x)}\int_D G(x,y)u(y)\Phi(v_1(y),\dots,v_n(y))dy$$
$$\le c_n \int_D \Psi(v_1(y),\dots,v_n(y))dy$$

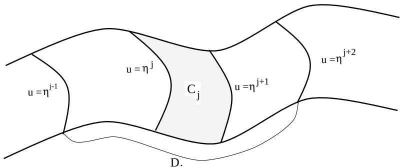
$u, v_1,\dots,v_n > 0$ are superharmonic.

Let $\Phi(t_1,\dots,t_n) = t_1\cdots t_n$.

Corollary

$$\frac{1}{u(x)}\int_D G(x,y)u(y)\prod_{i=1}^n v_i(y)dy$$
$$\le c_n \int_D \prod_{i=1}^n v_i(y)dy.$$

Basic Estimates
$D_j = \{x \in D : \eta^{j-1} < u(x) < \eta^{j+2}\}$,
$C_j = \{x \in D : \eta^j \le u(x) \le \eta^{j+1}\}$,



Lemma
$f \ge 0$

$$\sup_D \frac{1}{u}\int_D G(\cdot,y)f(y)dy$$
$$\le c\sum_j \sup_{D_j}\frac{1}{u}\int_{C_j} G_{D_j}(\cdot,y)f(y)dy,$$

$$\sup_D \int_D G(\cdot,y)f(y)dy$$
$$\le c\sum_j \sup_{D_j}\int_{C_j} G_{D_j}(\cdot,y)f(y)dy.$$

Proof of Theorem. Case: $u \equiv 1$.

$$\int_D G(x,y)\Phi(v_1(y),\dots,v_n(y))dy$$
$$\le c_n \int_D \Psi(v_1(y),\dots,v_n(y))dy.$$

$f = \Phi(v_1,\dots,v_n)$,
$C_j^i = \{x \in D : \eta^j \le v_i(x) \le \eta^{j+1}\}$,
$D_j^i = \{x \in D : \eta^{j-1} < v_i(x) < \eta^{j+2}\}$
Apply Lemma to $u = v_1$.

$$\int_D G(\cdot,y)f(y)dy$$
$$\le c\sum_{j_1} \sup_{D_{j_1}^1}\int_{C_{j_1}^1} G_{D_{j_1}^1}(\cdot,y)f(y)dy.$$

Repeat.

$$\int_D G(\cdot,y)f(y)dy$$
$$\le \sum \sup_{D_{j_1}^1\cap\cdots\cap D_{j_n}^n}\int_{C_{j_1}^1\cap\cdots\cap C_{j_n}^n} G_{D_{j_1}^1\cap\cdots\cap D_{j_n}^n}(\cdot,y)f(y)dy.$$

$$\psi(t_1,\dots,t_n)=\sup_{1\le c_1,\dots,c_n\le\eta}\Phi(c_1t_1,\dots,c_nt_n)$$

Then

$$\Phi(v_1,\dots,v_n)\le\psi(\eta^{j_1},\dots,\eta^{j_n})\quad\text{on } C^1_{j_1}\cap\cdots\cap C^n_{j_n},$$
$$\psi(\eta^{j_1},\dots,\eta^{j_n})\le\Psi(v_1,\dots,v_n)\quad\text{on } D^1_{j_1}\cap\cdots\cap D^n_{j_n}.$$

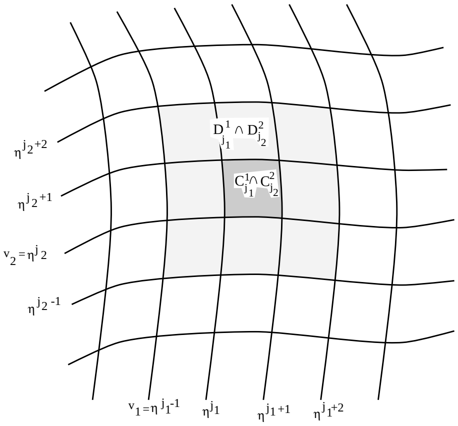
Hence

$$\int_{C^1_{j_1}\cap\cdots\cap C^n_{jn}}G_{D^1_{j_1}\cap\cdots\cap D^n_{jn}}(\cdot,y)f(y)dy$$
$$\le\psi(\eta^{j_1},\dots,\eta^{j_n})\int_{C^1_{j_1}\cap\cdots\cap C^n_{jn}}G_{D^1_{j_1}\cap\cdots\cap D^n_{jn}}(\cdot,y)dy$$
$$\le c_0\psi(\eta^{j_1},\dots,\eta^{j_n})|D^1_{j_1}\cap\cdots\cap D^n_{j_n}|$$
$$\le c_0\int_{D^1_{j_1}\cap\cdots\cap D^n_{jn}}\Psi(v_1(y),\dots,v_n(y))dy.$$

Therefore

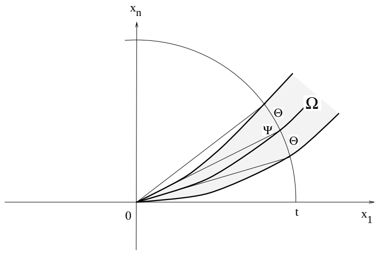
$$\int_D G(\cdot,y)f(y)dy$$
$$\le c\sum_{j_1,\dots,j_n}\int_{D^1_{j_1}\cap\cdots\cap D^n_{jn}}\Psi(v_1(y),\dots,v_n(y))dy$$
$$\le 3^n c\int_D\Psi(v_1(y),\dots,v_n(y))dy.$$

□



2. Application to Martin boundary

Maz'ya (77): Martin boundary at 0



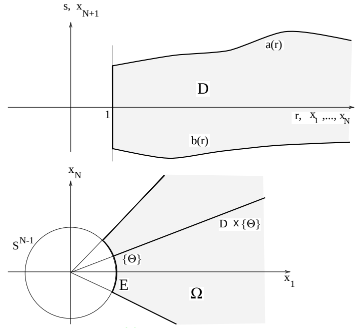
Θ, Φ: smooth + conditions.

$$\int_0\frac{\Theta(t)dt}{(\pi-2|\Psi(t)|)^2t}\begin{cases}=\infty & \Longrightarrow\ \text{1 point}\\ <\infty & \Longrightarrow\ S^{n-2}.\end{cases}$$

Ioffe-Pinski (94): Martin boundary at ∞
Probabilistic proof.

Theorem

Consider



Then $\displaystyle\int_1^\infty\frac{a(r)-b(r)}{r^2}dr<\infty$, in other words $\displaystyle\int_D\frac{drds}{r^2}<\infty\Longrightarrow$

$$\Omega^*\cong(\overline{D}\cup\{\infty\})\times\overline{E}.$$

Transform $\Delta_{N+1}u = \left(\Delta_N + \dfrac{\partial^2}{\partial s^2}\right)u = 0$. Write

$$\Delta_N = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{\Lambda}{r^2},$$

Λ: Laplace-Beltrami operator.
By $u \to r^{(N-1)/2}v$, $-\Delta_{N+1}u = 0$ becomes

$$Pv = \left(-\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial s^2} - \frac{\Lambda}{r^2} + \frac{(N-1)(N-3)}{4r^2}\right)v = 0$$

The Green function for $(-\Delta_{N+1}, \Omega)$ is

$$\left(\frac{|x|}{|\widetilde{x}|}\right)^{\frac{1-N}{2}} \mathcal{G}(|x|, s, \frac{x}{|x|}; |\widetilde{x}|, \widetilde{s}, \frac{\widetilde{x}}{|\widetilde{x}|}),$$

$\mathcal{G}$: the Green function of $(P, D \times E)$.

$$(-\Delta_{N+1}, \Omega) \iff (P, D \times E).$$

$\therefore$

$$\Omega^* \cong (D \times E)^*.$$

Murata (Publ. RIMS 1990) deals with Skew Product

$$P = -\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial s^2}\right) + \frac{1}{r^2}\left(-\Lambda + \frac{(N-1)(N-3)}{4}\right)$$
$$= -\Delta_2 + \frac{1}{r^2}A$$

$\lambda_0 < \lambda_1 \le \cdots$: eigenvalues of A w.r.t. $L^2(E)$.
φ_j: corresponding eigenfunctions, CONS.
H_j: Green function for $L_j = -\Delta_2 + \frac{\lambda_j}{r^2}$.

$$\mathcal{G}(x, \xi; y, \eta) = \sum_j H_j(x, y)\varphi_j(\xi)\varphi_j(\eta).$$

Convergence of Martin kernel.

$$\frac{\mathcal{G}(x, \xi; y, \eta)}{\mathcal{G}(x_0, \xi_0; y, \eta)} = \frac{\sum_j H_j(x, y)\varphi_j(\xi)\varphi_j(\eta)}{\sum_j H_j(x_0, y)\varphi_j(\xi_0)\varphi_j(\eta)}$$

$$= \frac{\sum_j \dfrac{H_j(x, y)}{H_0(x_0, y)}\varphi_j(\xi)\dfrac{\varphi_j(\eta)}{\varphi_0(\eta)}}{\sum_j \dfrac{H_j(x_0, y)}{H_0(x_0, y)}\varphi_j(\xi_0)\dfrac{\varphi_j(\eta)}{\varphi_0(\eta)}}.$$

Let D^{*j} be the Martin compact. of (L_j, D).

Theorem

For $j = 1, 2, \ldots$

(A) $\quad \exists \lim \dfrac{H_j(x, y)}{H_0(x_0, y)} > 0, \quad y \to y^*$ in D^{*0}

$\Longrightarrow$

$$(D \times E)^* \cong D^{*0} \times \overline{E}.$$

Reduction of (A).

(B1) $\quad \exists \lim \dfrac{H_j(x, y)}{G(x_0, y)} > 0, \quad y \to y^*$ in D^*,

(B2) $\quad \exists \lim \dfrac{G(x, y)}{H_j(x_0, y)} > 0, \quad y \to y^*$ in D^{*j}.

By (B1) and (B2)

$$\frac{H_j(x, y)}{H_0(x_0, y)} = \frac{H_j(x, y)}{G(x_0, y)} \cdot \frac{G(x_0, y)}{H_0(x_0, y)} \implies \text{(A)}$$

By (B1) and (B2) again

$$D^{*j} \cong D^*.$$

Therefore

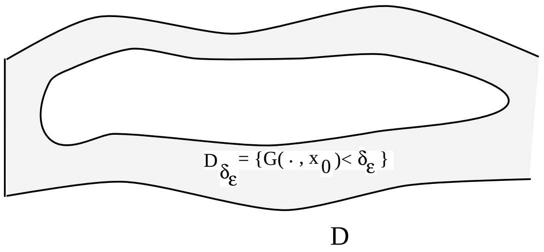
$$\Omega^* \cong (D \times E)^* \cong D^{*0} \times \overline{E} \cong D^* \times \overline{E} \cong (\overline{D} \cup \{\infty\}) \times \overline{E}.$$

Proof of (B1). Resolvent equation

$$H_j(x, y) = G(x, y) - \int_D H_j(x, z)\frac{\lambda_j}{z_1^2}G(z, y)dz.$$

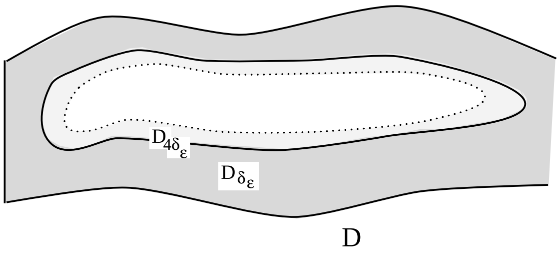
Divide by $G(x_0, y)$.

$$\frac{H_j(x, y)}{G(x_0, y)} = K(x, y) - \int_{D\setminus D_{\delta_\varepsilon}} H_j(x, z)\frac{\lambda_j}{z_1^2}K(z, y)dz$$
$$- \int_{D_{\delta_\varepsilon}} H_j(x, z)\frac{\lambda_j}{z_1^2}K(z, y)dz.$$



First two terms converge. We know $H_j(x,y) \approx G(x,y)$. The Generalized Cranston-McConnell inequality yields

$$\begin{aligned} &|\int_{D_{\delta_\varepsilon}} H_j(x,z)\frac{\lambda_j}{z_1^2}K(z,y)dz| \\ &\qquad \le c\int_{D_{\delta_\varepsilon}} G(x,z)\frac{1}{z_1^2}K(z,y)dz \\ &\qquad \le cK(x,y)\int_{D_{4\delta_\varepsilon}} \frac{1}{z_1^2}dz \\ &\qquad \le c\varepsilon K(x,y). \end{aligned}$$



Thus (B1)

$$\exists \lim \frac{H_j(x,y)}{G(x_0,y)} > 0 \quad y \to y^* \text{ in } D^*$$

follows. □